



Using Bonse's Inequality to Find Upper Bounds on Prime Gaps

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Abstract

One can apply Bonse's inequality to points on the real line to find an upper bound on any prime gap, including both for first occurrences and for maximal prime gaps. However, such a result is neither as fine as the upper bound found by Mozzochi, nor as fine as the lower bound obtained by Rankin for maximal prime gaps. Without deep sieve methods, such as those used by Maier and Pomerance to compute a lower bound for maximal prime gaps, we show one can use Bonse's inequality to arrive at an upper bound for any given prime gap without intricate derivations for any real constants.

1 Introduction and a new upper bound $\Gamma(p_k)$.

Iwaniec, Pintz, and later Mozzochi [5] found good upper bounds on the difference between two consecutive primes, namely,

$$p_{k+1} - p_k \ll p_k^\theta,$$

where either $\theta = \frac{11}{20} - \frac{1}{406}$ or $\theta = \frac{11}{20} - \frac{1}{384}$ [5]. Maier and Pomerance improved a lower bound for prime gaps found previously by Rankin [4]. All of them used deep methods beyond the scope of this paper. Here instead we demonstrate the relevance of Bonse's inequality to the topic of prime gaps.

We use Bonse's inequality to find an upper bound on the number $g(p_k)$, $k = 1, 2, \dots$, of composite integers between two consecutive primes, p_k and $p_{k+1} > p_k$. The result can be used to find, for each large integer $k \gg 4$, an open interval on \mathbb{R}^1 within which one will find

the real value for the prime gap. We also remark in Section 2, that computer work can be done on comparing the rate of growth of the upper bound on $g(p_k)$ we found and the rate of growth of $(\log(p_k))^2$ as $k \rightarrow \infty$, which by Cramér's conjecture, is the asymptotic limit for a maximal prime gap whenever k is a very large integer. The sequence of prime gaps is related to the sequence of prime differences [9].

Let

$$\Delta p_k = p_{k+1} - p_k \tag{1}$$

denote the prime difference function, and

$$g(p_k) = p_{k+1} - p_k - 1 \tag{2}$$

the prime gap, meaning the number of consecutive composites between the consecutive primes p_{k+1} and p_k . We show that for large integer $k \gg 4$, such that the consecutive primes p_k, p_{k+1} are very large, there exists an infinite sequence $\{\Gamma(p_k)\}_{k=1}^{\infty}$, such that each term

$$\Gamma(p_k) := \frac{p_k \left(\prod_{i=1}^{k-1} p_i - p_k \right)}{\log((k+1)^{k+1} k^k)} - 1 \tag{3}$$

$$> g(p_k),$$

is an upper bound for each term $g(p_k)$ that appears in the infinite sequence $\{g(p_k)\}_{k=1}^{\infty}$ of prime gaps. However the result is not for a least upper bound on prime gaps $g(p_k)$. The values $\Gamma(p_k)$ also are an upper bound on the prime differences Δp_k . In addition one should be able to show that each term in the sequence $\{\Gamma(p_k)\}_{k=1}^{\infty}$ is an upper bound on each corresponding term (i.e., for each $k \gg 4$) in the sequence $\{\log(p_k)^2\}_{k=1}^{\infty}$, where Cramér's conjecture says that, for large k, K , maximal prime gaps $G(p_K)$ — meaning maximal if $G(p_K)$ is a prime gap such that $G(p_K) > g(p_k)$ for all $k < K$ — approach the asymptotic limit $(\log(p_k))^2$.

The graph for the prime difference function Δp_k resembles — at the very least — frequency vs. time graphs in the field of signal processing [8] and Gaussian noise [6]. Our theorem below rests on the foundations of two previously proved propositions, which immediately follow. One can prove Bonse's inequality as an exercise [3]. Uspensky and Heaslet [7] also discuss Bonse's inequality.

Proposition 1 (Bonse's inequality). *For $k > 4$,*

$$p_{k+1}^2 < \prod_{i=1}^k p_i.$$

Proposition 2. *Let p_k be the k th prime. Then*

$$p_k \sim k \log k.$$

For a proof, see Gioia [1]. Now we are ready to prove a theorem that establishes the upper bound on $g(p_k)$ through Bonse's inequality.

Theorem 3. For $k \gg 4$,

$$\begin{aligned} g(p_k) &< \Gamma(p_k) \\ &= \frac{p_k(\prod_{i=1}^{k-1} p_i - p_k)}{\log((k+1)^{k+1}k^k)} - 1. \end{aligned}$$

Proof. By Bonse's inequality (Proposition 1), when $k \geq 4$,

$$\begin{aligned} p_k^2 &< p_1 p_2 \cdots p_{k-1} \\ \implies p_{k+1}^2 - p_k^2 &< (p_1 p_2 \cdots p_k) - p_k^2 \\ \implies p_{k+1} - p_k &< \frac{p_1 \cdots p_k - p_k^2}{p_{k+1} + p_k} \\ &= \frac{p_k(\prod_{i=1}^{k-1} p_i - p_k)}{p_{k+1} + p_k}. \end{aligned}$$

Now when $k \gg 4$ grows ever larger, it is true that $p_k \sim k \log k$ while by Proposition 2, it remains true that $\varepsilon = o(1)$, where $\varepsilon = \left| \frac{p_k - k \log k}{p_k} \right|$ (See Table 1, and the proof by Gioia [1] of Proposition 2). Furthermore, when $k \gg 4$ grows larger it also follows from Proposition 2 and from the properties of logarithms that

$$p_{k+1} + p_k \sim (k+1) \log(k+1) + k \log k = \log((k+1)^{k+1}k^k),$$

where as $k \rightarrow \infty$,

$$\begin{aligned} \left| \frac{(p_{k+1} + p_k) - \log((k+1)^{k+1}k^k)}{p_{k+1} + p_k} \right| &= \varepsilon' \\ &\rightarrow 0. \end{aligned}$$

Letting $0 \leq \varepsilon' \ll 1$, that is, so that $\varepsilon' \approx 0$, it follows that, as $k \rightarrow \infty$,

$$\frac{p_k(\prod_{i=1}^{k-1} p_i - p_k)}{p_{k+1} + p_k} \sim \frac{p_k(\prod_{i=1}^{k-1} p_i - p_k)}{\log((k+1)^{k+1}k^k)}. \quad (4)$$

Then as $k \rightarrow \infty$,

$$p_{k+1} - p_k < \frac{p_k(\prod_{i=1}^{k-1} p_i - p_k)}{p_{k+1} + p_k} \sim \frac{p_k(\prod_{i=1}^{k-1} p_i - p_k)}{\log((k+1)^{k+1}k^k)}, \quad (5)$$

where we have used Propositions 1 and 2. Then it follows that for large integer $k \gg 4$ such that both p_k, p_{k+1} are very large,

$$\begin{aligned} g(p_k) + 1 &= p_{k+1} - p_k \\ &= \Delta p_k \\ &< \frac{p_k(\prod_{i=1}^{k-1} p_i - p_k)}{\log((k+1)^{k+1}k^k)}, \\ \implies g(p_k) &< \Gamma(p_k) = \frac{p_k(\prod_{i=1}^{k-1} p_i - p_k)}{\log((k+1)^{k+1}k^k)} - 1. \end{aligned}$$

2 The relationship $\log(\Gamma(p_k))$ has to the upper bound $p_k^{\frac{11}{20}-\delta}$ and other conclusions.

Table 1 suggests that our formula for $\Gamma(p_k)$ indeed is an upper bound for $g(p_k)$, as k gets large. Also from Table 1 one can see that the fast rate of increase of $\Gamma(p_k)$ by far exceeds the slower rate of increase of $(\log(p_k))^2$. Hence it could be interesting to investigate by computer the behavior of the three respective functions $\Gamma(p_k)$, $g(p_k)$ and $(\log(p_k))^2$, where $(\log(p_k))^2$ by Cramér's conjecture is such that for large k , $\frac{p_{k+1}-p_k}{(\log(p_k))^2} = O(1)$ is true for maximal prime gaps.

Rankin [2] showed that there exists a real constant c such that

$$\frac{c(\log k)(\log \log k)(\log \log \log \log k)}{(\log \log \log k)^2} < p_{k+1} - p_k. \quad (6)$$

Combined with the result from Theorem 3, this indicates that for large integer $k \gg 4$, taken large enough so that p_k and p_{k+1} both are very large, each term of the prime difference sequence Δp_k is bounded, for each such k , as

$$\begin{aligned} \frac{c(\log k)(\log \log k)(\log \log \log \log k)}{(\log \log \log k)^2} &< p_{k+1} - p_k \\ &< \frac{p_k \left(\prod_{i=1}^{k-1} p_i - p_k \right)}{\log((k+1)^{k+1} k^k)} = \Gamma(p_k) + 1. \end{aligned}$$

Let, for some real constant c and for any fixed integer $k = k_0 \gg 4$,

$$a(k_0) = \frac{c \log(k_0) \log \log(k_0) \log \log \log \log(k_0)}{(\log \log \log(k_0))^2} - 1, \quad (7)$$

$$b(k_0) = \frac{p_{k_0} \left(\prod_{i=1}^{k_0-1} p_i - p_{k_0} \right)}{\log((k_0+1)^{k_0+1} k_0^{k_0})} - 1 = \Gamma(p_{k_0}). \quad (8)$$

Then it follows from Theorem 3 that, for each such positive integer $k = k_0 \gg 4$, the value for the prime gap $g(p_{k_0})$ lies inside an open interval $(a(k_0), b(k_0))$ on the real line.

Remarks: The upper bound $\Gamma(p_k)$ found in Section 1 and which appears on the real line in Eq. 8 as $b(k_0)$ for any large fixed integer $k_0 \gg 4$ such that $g(p_{k_0}) \in (a(k_0), b(k_0)) \subseteq \mathbb{R}^1$, admittedly, is a large upper bound. This actually can be an advantage. We now can compare the values of $g(p_k)$ to those for $\log(\Gamma(p_k))$, $\log \log(\Gamma(p_k))$ and $\log \log \log(\Gamma(p_k))$. In fact an inspection of Tables 2 and 3 shows it might be profitable to compare the values of $\log(\Gamma(p_k))$, $\log \log(\Gamma(p_k))$ and $\log \log \log(\Gamma(p_k))$ to those for $g(p_k)$, Δp_k and $(\log(p_k))^2$, whenever $k > 4$ is large enough so that both p_k and p_{k+1} are two very large consecutive primes. This is because one should be able to find by computer that $\log(\Gamma(p_k)) > g(p_k)$, $\log(\Gamma(p_k)) > \Delta p_k$ and $\log(\Gamma(p_k)) > (\log(p_k))^2$ as k grows large, after which one even might be able to prove

that $\log(\Gamma(p_k))$ is a better upper bound on $g(p_k)$ than is $\Gamma(p_k)$, as $k \rightarrow \infty$. In Table 3 the reader can compare the upper bound found by Iwaniec and Pintz [5],

$$p_k^{\frac{11}{20}-\delta} \tag{9}$$

with $\Gamma(p_k)$, $\log(\Gamma(p_k))$ and $\log \log(\Gamma(p_k))$, where in Table 3 we have allowed $\delta = \frac{1}{406}$. As an alternative one could use the refinement found by Mozzochi, who found that, for real $\delta = \frac{1}{384}$, $p_{k+1} - p_k \ll p_k^{\frac{11}{20} - \frac{1}{384}}$. However a comparison of column 3 and column 5 with column 9 in Table 3 elicits the following question:

Does there exist, as $k \rightarrow \infty$, a real constant $\alpha > 0$, such that

$$g(p_k) = p_{k+1} - p_k - 1 < \log((\log(\Gamma(p_k)))^\alpha) < p_k^{\frac{11}{20}-\delta} - 1?$$

Finally we can use the fact that the field of real numbers on the real line \mathbb{R}^1 is closed for the two binary operations $(+, \times)$, to express the upper bound for prime gaps found by Iwaniec and Pintz [5] for real $\delta = \frac{1}{406}$, by using $\log(\Gamma(p_k))$. Let

$$a_1, a_2, \dots, \tag{10}$$

be an infinite sequence $\{a_k\}_{k=1}^\infty$ of real numbers such that

$$p_k = a_k \log(\Gamma(p_k)) = \log((\Gamma(p_k))^{a_k}). \tag{11}$$

Then we have at once

$$p_k^{\frac{11}{20}-\delta} = (\log((\Gamma(p_k))^{a_k}))^{\frac{11}{20}-\delta} \tag{12}$$

$$\implies g(p_k) \in (a(k), b'(k)) \subseteq \mathbb{R}^1, \tag{13}$$

where in Eq. (13) we have replaced $a(k_0)$ in Eq. (7) with $a(k)$ and, by utilizing the upper bound result found by Iwaniec and Pintz [5],

$$b'(k) = p_k^{\frac{11}{20}-\delta} \tag{14}$$

$$= (\log((\Gamma(p_k))^{a_k}))^{\frac{11}{20}-\delta}. \tag{15}$$

In the following Tables 1–3, we have chosen $\delta = \frac{1}{406}$, Iwaniec and Pintz's [5] result) and relative error (see [1, Theorem 45.3]).

$$\varepsilon = \left| \frac{p_k - k \log k}{p_k} \right| \rightarrow 0. \tag{16}$$

k	p_k	$g(p_k)$	Δp_k	$(\log(p_k))^2$	$k \log k$	$\Gamma(p_k)$	ε
3	5	1	2	2.59 ...	3.27 ...	-0.44 ...	0.34 ...
4	7	3	4	3.79 ...	5.56 ...	10.85 ...	0.21 ...
5	11	1	2	5.75 ...	8.00 ...	115.45 ...	0.27 ...
6	13	3	4	6.58 ...	10.74 ...	1224.22 ...	0.17 ...
7	17	1	2	8.03 ...	13.65 ...	16860.24 ...	0.19 ...

Table 1

k	p_k	$g(p_k)$	Δp_k	$(\log(p_k))^2$	$\Gamma(p_k)$	$\log(\Gamma(p_k))$	$\log \log(\Gamma(p_k))$
3	5	1	2	2.59 ...	-0.44 ...	—	—
4	7	3	4	3.79 ...	10.85 ...	2.38 ...	0.87 ...
5	11	1	2	5.75 ...	115.45 ...	4.74 ...	1.55 ...
6	13	3	4	6.58 ...	1224.32 ...	7.11 ...	1.96 ...
7	17	1	2	8.03 ...	16860.24 ...	9.73 ...	2.28 ...

Table 2

k	p_k	$g(p_k)$	Δp_k	$p_k^{\frac{11}{20}-\delta}$	$\Gamma(p_k)$	$\log(\Gamma(p_k))$	a_k	$\log \log(\Gamma(p_k))$
3	5	1	2	2.42 ...	-0.44 ...	—	—	—
4	7	3	4	2.90 ...	10.85 ...	2.38 ...	2.94 ...	0.87 ...
5	11	1	2	3.72 ...	115.45 ...	4.74 ...	2.32 ...	1.55 ...
6	13	3	4	4.07 ...	1224.32 ...	7.11 ...	1.83 ...	1.96 ...
7	17	1	2	4.72 ...	16860.24 ...	9.73 ...	1.74 ...	2.28 ...

Table 3

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(Concerned with sequence [A001223](#).)

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