Journal of Integer Sequences, Vol. 10 (2007),
Article 07.7.6

Semiorders and Riordan Numbers

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Abstract

In this paper, we define a class of semiorders (or unit interval orders) that arose in the context of polyhedral combinatorics. In the first section of the paper, we will present a pure counting argument equating the number of these interesting (connected and irredundant) semiorders on $n+1$ elements with the nth Riordan number. In the second section, we will make explicit the relationship between the interesting semiorders and a special class of Motzkin paths, namely, those Motzkin paths without horizontal steps of height 0, which are known to be counted by the Riordan numbers.

1 Counting Interesting Semiorders

We begin with some basic definitions.

Definition 1. *A partially ordered set* $(X, \prec)$ *is a* semiorder *if it satisfies the following two properties for any* $a, b, c, d \in X$.

- *If* $a \prec b$ *and* $c \prec d$, $a \prec d$ *or* $c \prec b$.
- *If* $a \prec b \prec c$, *then* $d \prec c$ *or* $a \prec d$.

Semiorders are also known as *unit interval orders* in the literature. This name comes from the fact that each element $x \in X$ can be identified with an interval on the real line. All intervals are the same length, and two intervals intersect if and only if their corresponding elements are incomparable. If the intervals for a and b do not intersect, and the interval for a lies to the left of the interval for b, then $a \prec b$. We may assume without loss of generality that the intervals in our representation have different endpoints. We define the predecessor (pred) and successor (succ) sets intuitively: $\text{pred}(x) = \{a \in X \mid a \prec x\}$ and $\text{succ}(x) = \{a \in X \mid x \prec a\}$. For a semiorder, the predecessor and successor sets are weakly ordered (for different elements x and y, either $\text{pred}(x) \subseteq \text{pred}(y)$, $\text{pred}(y) \subseteq \text{pred}(x)$, or both, with the same criterion for successor sets). These impose two weak orderings on the set X, and their intersection is a weak order known as the *trace*.

A semiorder is *interesting* if it satisfies the following two properties.

- (*Connectedness*) Each element is incomparable with its immediate predecessors in the trace.
- (*Irredundancy*) No two elements have both the same predecessor sets and the same successor sets.

We use $\mathcal{I}_n$ to denote the set of all interesting semiorders on n elements.

The criterion of connectedness guarantees that the semiorder is represented by a topologically connected set of distinct intervals on the real line. The criterion of irredundancy guarantees that the semiorder has a linear order as its trace (i.e., no two elements are incomparable in the trace). These criteria came about from the study of the polyhedral set of all representations of a semiorder. This polyhedron has one dimension for each element of the semiorder, save for the smallest element in the trace, and one dimension for the length of the intervals (which we denote the r-value). The connectedness criterion guarantees that the polyhedron is bounded along each hyperplane corresponding to a fixed r value (that is, if the interval length is bounded, then so is any numeric representation of the semiorder). The irredundancy criterion guarantees that each dimension will take on a different set of values for the polyhedron (that is, no two elements may be represented by the same interval in any representation of the semiorder). Though the motivation comes from the study of this polyhedron, the remainder of the paper will focus on the properties of the semiorders and an exploration of the following theorem.

Theorem 1. *The number of interesting semiorders on $n+1$ elements is the nth Riordan number, r_n. That is, $|\mathcal{I}_n| = r_n$.*

The Riordan numbers (1,0,1,1,3,6,15,36,91,...) are found at [5] (A005043) and explored in depth by Bernhart in [2]. This theorem is an order-theoretic analog of a graph-theoretic result of Hanlon [3].

We prove the result by showing that the number of interesting semiorders on $n+1$ elements, which we will denote ρ_n, satisfies the same recurrence relation as do the Riordan numbers, namely that

$$c_n = \sum_{k=0}^{n} \binom{n}{k} \rho_k \qquad \text{with} \qquad \rho_0 = 1$$

where $c_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number (sequence [A000108]). The above recurrence for the Riordan numbers was proven in detail by Chen et. al. in [4].

Proof. We first check that the base case holds, namely that there is 1 interesting semiorder on 1 element ($\rho_0 = 1$), which is the only semiorder on one element.

A semiorder on $n+1$ elements can be represented by an $n+1$ by $n+1$ 0-1 matrix, A, with the rows and columns labeled by the semiorder elements. The entry $A_{i,j}$ is 1 if element i is above element j in the semiorder, and 0 otherwise. If the labels on the rows and columns are in accordance with the trace, then the matrix will be in echelon form with each row beginning with a string of zeroes and ending with a (possibly empty) string of ones. The set of leading ones in each row will form a 'staircase' pattern, as in Figure 2. We can move along the steps to obtain a path from the upper right to the lower left corner of the matrix. Since this path never goes below the diagonal, we have as many semiorders on n elements as we do lattice paths from $(0,0)$ to $(n+1,n+1)$ that don't go below the line $y=x$, the number of which is well known to be the $n+1$st Catalan number, c_{n+1}.

If we restrict our attention to connected semiorders, the matrix has no non-zero entries in the sup-diagonal $(a_{i,i+1})$ or below. Hence, the lattice paths in question can be viewed as going from $(1,0)$ to $(n,n+1)$ without crossing the line $y=x+1$, which are just counted by the nth Catalan number, c_n.

Suppose that there are ρ_k interesting semiorders on $k+1$ elements, and take an arbitrary connected semiorder on $n+1$ elements. Either it is irredundant, or for some pairs of elements x_i and x_{i+1} which are adjacent in the trace, x_i and x_{i+1} have the same predecessor sets and successor sets. There are n such adjacent pairs which might be 'the same' in this way. For each pair that is the same, iteratively replace them with a single element. If there are $n-k$ such pairs the same, then what results is an interesting semiorder on $n+1-(n-k)=k+1$ elements. There are ρ_k interesting semiorders on $k+1$ elements, and hence there are $\binom{n}{n-k}\rho_k = \binom{n}{k}\rho_k$ orders on $n+1$ elements which reduce to an interesting semiorder on $k+1$ elements. Summing over all possible values of k gives the desired recursion. Therefore $\rho_n = r_n$, the nth Riordan number.

□

2 Semiorder/Motzkin Path Bijection

It is known that the Riordan numbers count the number of Motzkin paths without horizontal steps of height zero. In the remainder of the paper, we make explicit a bijection between semiorders and these Motzkin paths. We begin with some more terminology about semiorders, and we follow definitions similar to those of Pirlot [6] for two other relations on the elements of a semiorder, nose relations and hollow relations.

2.1 Definitions and Terminology

In terms of the matrix A, we have that aNb if $A_{ab}=1$, and changing that 1 to a 0 leaves a staircase matrix pattern. We have that cHd if $A_{cd}=0$, and changing that 0 to a 1 leaves a staircase matrix pattern. We make another definition in terms of predecessor and successor sets.

Definition 2. *The relations N and H are defined as follows, with $x_j \prec_T x_i$*

- $x_i N x_j$ *if* $x_j \in pred(x_i)$, $x_{j+1} \notin pred(x_i)$, *and* $x_j \notin pred(x_{i-1})$

- $x_i H x_j$ *if* $x_j \notin pred(x_i)$, $x_{j-1} \in pred(x_i)$, *and* $x_j \in pred(x_{i+1})$ [1]

As shown in [6], and in explicit detail in [7], the semiorder can be reconstructed in its entirety by its nose and hollow relations. We also say that x_i *noses* x_j if $x_i N x_j$, and similarly, x_i *hollows* x_j if $x_i H x_j$.

Definition 3. *A* **Motzkin path** *of order* $n > 0$ *is a walk from the point* $(0, 0)$ *to the point* $(n, 0)$ *along integer lattice points, none of which lie below the* x*-axis, consisting of three types of steps.*

- **Up-steps** *from a point* (i, j) *to* $(i + 1, j + 1)$
- **Horizontal steps** *from a point* (i, j) *to a point* $(i + 1, j)$
- **Down-steps** *from a point* (i, j) *to a point* $(i + 1, j - 1)$.

We denote by P_i *the point on the path with* x*-coordinate* i $(0 \leq i \leq n)$.

Motzkin paths are a generalization of the classic Catalan paths. The Catalan paths are often viewed as going from (0,0) to (k, k), and consisting of k horizontal and k vertical steps (that stay above the 'baseline' $y = x$). By rotating 45° and using the described up-steps and down-steps, a Catalan path can be viewed as going from $(0, 0)$ to $(2k, 0)$ (and staying above the 'baseline' $y = 0$). The primary difference between Catalan and Motzkin paths is that Motzkin paths allow for horizontal steps. As such, we can construct such paths from $(0, 0)$ to $(n, 0)$ for n even or odd. We further narrow our focus to the set of paths that will correspond to our set of semiorders.

Definition 4. *A* **Riordan path** *of order* n *is a Motzkin path of order* n *in which no horizontal steps occur on the horizontal axis of the plane. We will denote the set of Riordan paths of order* n *by* $\mathcal{R}_n$.

We will make an explicit bijection between these Riordan paths and the set of interesting semiorders. We begin by closely examining the Riordan paths.

Definition 5. *Let* P_i *represent the* i*th point in a Riordan path of order* n*, where* $1 < i < n$*. There are nine possibilities for any given point* P_i.

(i) P_i *is a* **hard peak** *if* P_i *lies vertically above both of the points* P_{i-1}, P_{i+1}.

(ii-iii) P_i *is a* **positive (negative) soft peak** *if* P_i *lies vertically above the point* P_{i-1} (P_{i+1})*, and level with the point* P_{i+1} (P_{i-1}).

(iv) P_i *is a* **hard dip** *if* P_i *lies vertically below both of the points* P_{i-1}, P_{i+1}.

(v-vi) P_i *is a* **positive (negative) soft dip** *if* P_i *lies vertically below the point* P_{i+1} (P_{i-1})*, and level with the point* P_{i-1} (P_{i+1}).

[1]The definition given here is the inverse of the relation defined in [6], but we use this for the ease in discussing the bijection in the remainder

(vii-viii) P_i *is a* **positive (negative) slope** *if it lies vertically above (below)* P_{i-1} *and vertically below (above)* P_{i+1}.

(ix) P_i *is a* **level point** *if it lies vertically level with* P_{i-1}, P_{i+1}

The nine types of points are shown graphically in Figure 1. Thus, each Riordan path is uniquely defined by its list of point types. We now define several operations on the Riordan paths and some useful terminology associated with those operations.

Definition 6. *The* **sum** *of two Riordan paths* M_m, M_n *of order* m, n*, respectively, is the Riordan path* M *in which points* P_i *for* $(1 \leq i < m)$ *are identical to those of* M_m*, and points* $P_{(m-1)+i}$ *in* M *are identical to points* P_i *in* M_n *for* $1 \leq i \leq n$*. This sum is denoted* $M_m + M_n$*.*

Definition 7. *A Riordan path* M *is* **splittable** *if it can be expressed as the sum of two Riordan paths of lesser order.*

Definition 8. *Let* $P_1, P_2, \ldots, P_n$ *be the ordered set of points of* M*, where* $M \in \mathcal{I}_n$*, and* P_i *lies adjacently to the right of* P_{i-1} *for* $i > 1$*. The* **reverse** *of* M*, denoted* M^r*, is the ordered set of points* $Q_n, Q_{n-1}, \ldots, Q_1$ *with the following property: the steps of* M^r *are defined by* $(Q_{k-1}, Q_k) = (P_{n-k+2}, P_{n-k+1})$ *and* $(Q_k, Q_{k+1}) = (P_{n-k}, P_{n-k+1})$*. Equivalently, if* P_{n-k+1} *is a positive (negative) soft peak or dip, then* Q_k *is a negative (positive) soft peak or dip. Similarly, if* P_{n-k+1} *is positive (negative) slope, then* Q_k *is a negative (positive) slope. Finally, if* P_{n-k+1} *is either a hard peak, a hard dip, or a level point,* Q_k *is the same type of point. Note that this results in* M^r *appearing graphically as the mirror image of* M*.*

We will furthermore let I^r *denote the* **reverse** *of a semiorder* I*, which we will define as being equivalent to its dual; that is,* $I^r = I^\partial$ *for any* $I \in \mathcal{I}_n$*, as defined in [1].*

2.2 Construction of the Bijection

We will now define a mapping $F_n : \mathcal{I}_n \to \mathcal{R}_n$. Our bijection uses the nose and hollow relations on the elements of the semiorder to explicitly construct a Riordan path. We prove the bijectiveness of the mapping inductively, using our operations to construct larger semiorders from smaller ones.

Let $I \in \mathcal{I}_n$, and let x_i denote the ith element of I where $1 < i < n$. We will now define the nine possible cases for x_i, and, for each, the type of point that x_i will correspond to in the Riordan path.

- 2 Hollows - There exist elements a, b such that aHx_i, x_iHb, and there exists no element c such that cNx_i or x_iNc. In this case, we set the ith point in the path to a hard peak.

- 2 Noses - There exist elements a, b such that aNx_i, x_iNb, and there exists no element c such that cHx_i or x_iHc. In this case, we set the ith point in the path to a hard dip.

- 2 Hollows, 1 Nose - There exist elements a, b such that aHx_i, x_iHb, and there exists an element c such that x_iNc (cNx_i). In this case, we set the ith point in the path to a positive (negative) soft peak.

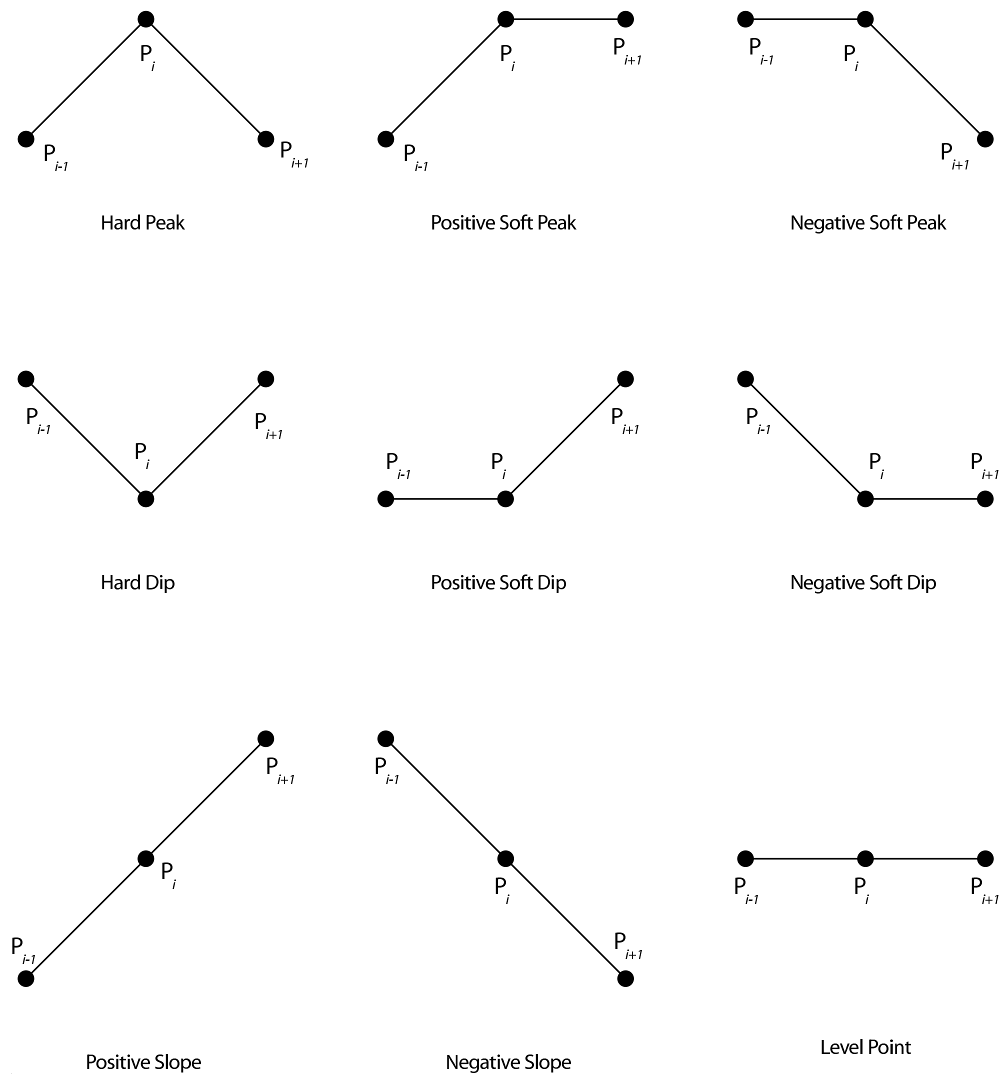

Figure 1: The nine possible types of points in a Riordan path.

- 2 Noses, 1 Hollow - There exist elements a, b such that aNx_i, x_iNb, and there exists an element c such that x_iHc (cHx_i). In this case, we set the ith point in the path to a positive (negative) soft dip.

- 1 Nose, 1 Hollow - There exist elements a, b such that x_iNa and x_iHb (aNx_i and bHx_i), and there exists no elements c such that cNx_i or cHx_i (x_iNc or x_iHc). In this case, we set the ith point in the path to a positive (negative) slope.

- 2 Noses, 2 Hollows - There exist elements a, b, c, d such that aNx_i,bHx_i,x_iNc,x_iHd. In this case, we set the ith point in the path to be a level point.

Since any two semiorders with different nose and hollow relations are different (as shown by Pirlot in [6]), as are any two paths with a different list of point types, this gives a well-defined map from $\mathcal{I}_n$ to $\mathcal{R}_n$. Since we showed in part (i) that $\mathcal{I}_n$ and $\mathcal{R}_n$ have the same size, we need only show that the map is onto for each value of n to prove each map F_n is a bijection. We formalize this here.

Theorem 2. *Let $n > 2$. If $F_n : \mathcal{I}_n \to \mathcal{R}_n$ is onto, then F_n is a bijection.*

Definition 9. *Let x_i be the ith element in the trace of an interesting semiorder $I \in \mathcal{I}_n$.*

- *We say that x_i* **initiates** *a nose (hollow) if there exists an element $b \in I$ such that x_iNb (x_iHb) and no element $a \in I$ such that aNx_i (aHx_i).*

- *We say that x_i* **terminates** *a nose (hollow) if there exists an element $a \in I$ such that aNx_i (aHx_i) and no element $b \in I$ such that x_iNb (x_iHb).*

- *We say that x_i* **continues** *a nose (hollow) if there exist elements $a, b \in I$ such that aNx_i (aHx_i) and x_iNb (x_iHb).*

Given that I is an interesting semiorder, we see that there are *a priori* 16 types of elements in our semiorder (Each point may be the starting and/or ending point of a nose and/or a hollow, or not), yet we've claimed there are only nine types of elements. We now show that there are no other possible cases for the element x_i given that I is an interesting semiorder. The following theorem provides a set of restrictions on x_i that follow from the properties of I.

Theorem 3. *Let I be a semiorder of order n and let x_i be the ith element of I. If I is interesting, then x_i has the following properties:*

(i) If x_i initiates a hollow, then x_i does not terminate a nose.

(ii) If x_i terminates a hollow, then x_i does not initiate a nose.

(iii) x_i shares a nose/hollow relationship with at least two distinct elements $a, b \in I$.

Proof. Let $I \in \mathcal{I}_n$ and let x_i be the ith element of I.

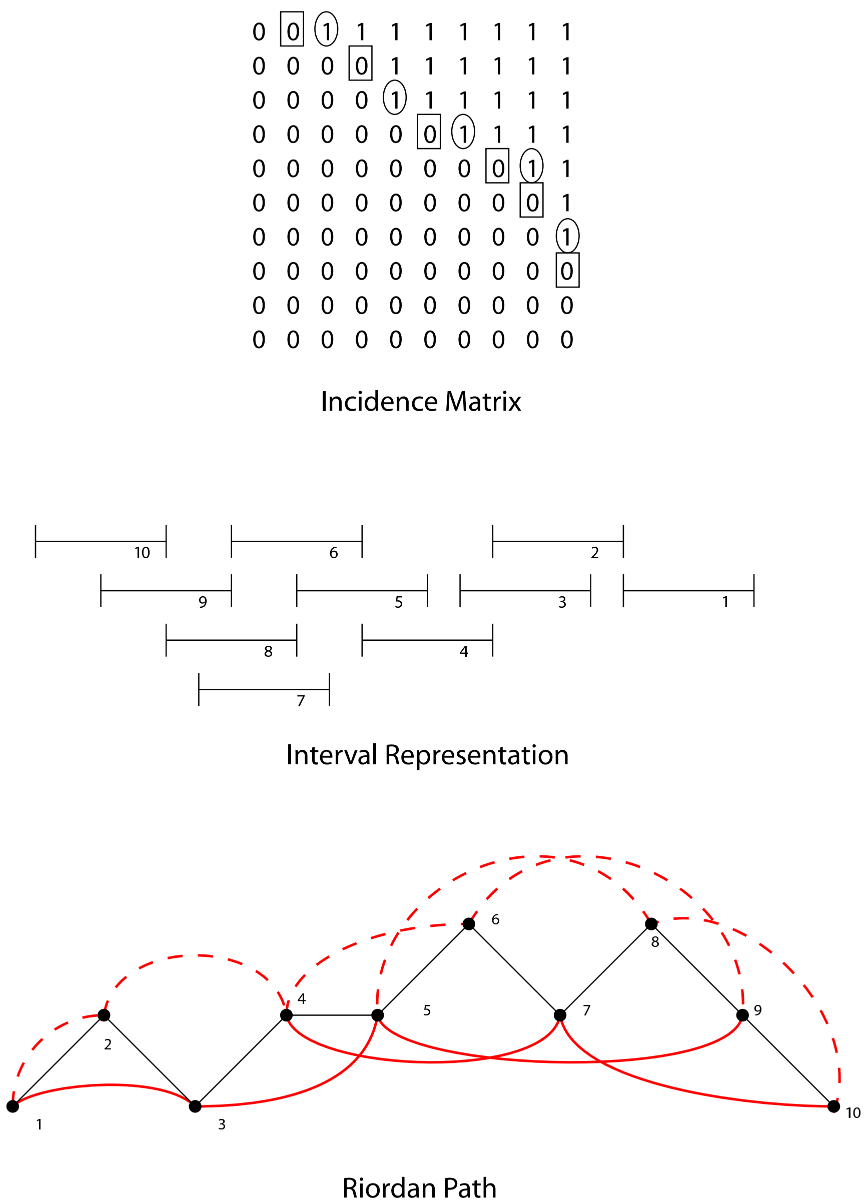

Figure 2: An example of an interesting semiorder I, given in matrix and interval representations, and its corresponding path.

Case i: Suppose x_i initiates a hollow. Thus we have that no element hollows x_i, which implies that $\text{succ}(x_i) = \text{succ}(x_{i+1})$. Now suppose x_i terminates a nose, implying that x_i does not nose any element of I. Thus we have that $\text{pred}(x_i) = \text{pred}(x_{i+1})$. We have therefore shown that x_i and x_{i+1} are redundant elements, a contradiction. We conclude that x_i does not terminate a nose.

Case ii: Suppose x_i terminates a hollow. Thus we have that x_i hollows no element of I, which implies that $\text{pred}(x_i) = \text{pred}(x_{i-1})$. Now suppose x_i initiates a nose, implying that no element of I noses x_i. Thus we have that $\text{succ}(x_i) = \text{succ}(x_{i-1})$. Again, this shows redundancy between elements x_i and x_{i-1}, contradicting the fact that I is interesting. We conclude that x_i does not initiate a nose.

Case iii: This case follows directly from the arguments in the previous cases.

□

The following theorems will provide the necessary procedures for building interesting semiorders from those of smaller order based on the differences between their corresponding Riordan paths. Given two Riordan paths of orders m and n, respectively, we may use the machinery provided in Theorem 4 to "glue" these two Riordan paths together at the ends, as well as construct the semiorder associated with our result. Theorem 5 will allow us to take the mirror image of a Riordan path and find its associated semiorder, based on the semiorder of the original. Using Theorem 5 in conjunction with the different cases of Theorem 6 we will be able to take any Riordan path and add a horizontal step anywhere above the axis, and find the semiorder associated with our result. Finally, Theorem 7 will allow us to add an up-step and a down-step to the beginning and end of a Riordan path (respectively) and, again, generate the corresponding semiorder

By generating semiorders from paths of smaller order, we will be able to prove inductively that F_n is onto for all $n > 2$.

Theorem 4. *If two Riordan paths $M_m \in \mathcal{R}_m, M_n \in \mathcal{R}_n$ have preimages under F_m and F_n, respectively, the Riordan path $M_m + M_n$ has a preimage under F_{m+n-1}.*

Proof. Let M_m and M_n be the images of two interesting semiorders I_m, I_n under F_m, F_n, respectively. We will now construct a semiorder $I \in \mathcal{I}_{m+n-1}$ and show that its image under F_{m+n-1} is $M_m + M_n$ by constructing its incidence matrix, which we will denote A. We will denote the incidence matrices of I_m and I_n as B, C, respectively.

Let x_m denote the last element in the trace of I_m and let y_1 denote the first element in the trace of I_n. In adding our two Riordan paths together we are essentially combining the points x_m and y_1 into one, with their corresponding element in I being denoted x. Since I_m and I_n are interesting semiorders, let $\alpha H x_m$ and $x_m N \beta$, where α is the pth element in the trace of I_m and β is the qth element in the trace of I_n. We have the entries of A as follows:

$$\begin{array}{c|c} I_m & 1 \\ \hline 0 & I_n \end{array} \quad \rightarrow$$

	1	2	$\cdots$	α	$\cdots$	x	$\cdots$	β	q	$\cdots$
1										
2										
$\vdots$										
p	0	0	$\cdots$	0	1	1	$\cdots$	1	1	$\cdots$
α	0	0	$\cdots$	0	0	0	$\cdots$	0	1	$\cdots$
$\vdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	1	$\cdots$
x	0	0	$\cdots$	0	0	0	$\cdots$	0	1	$\cdots$
$\vdots$						0	$\cdots$	0	0	$\cdots$
β										

Figure 3: Constructing the Preimage of $M_m + M_n$

$$A_{i,j} = \begin{cases} B_{(i,j)}, & \text{if } i < p, j \leq m; \\ C_{(i-m+1,j-m+1)} & \text{if } i > m, j \geq m; \\ 0, & \text{if } i > m, j < m; \\ 0, & \text{if } p < i \leq m, j < m+q-1; \\ 1, & \text{if } i < p, j > m; \\ 1, & \text{if } p < i \leq m, j \geq m+q-1 \end{cases}$$

This procedure simply adds all elements of I_n to the predecessor sets of the first $p-1$ elements in I_m, and adds the last $n-q+1$ elements of I_n to the predecessor sets of the last $m-p+1$ elements of I_m. This is demonstrated in Figure 3. Since we are simply adding elements to the predecessor sets of elements from I_m, we have that the irredundancy and connectedness of I follow necessarily from those of I_m and I_n. Thus our resultant semiorder I is interesting.

Since $I \in \mathcal{I}_{m+n-1}$, we will now show that F_{m+n-1} maps I to $M_m + M_n$. Note that when we restrict A to rows and columns $1, \ldots, m$, we obtain the matrix A, and thus the only relationship difference between the elements of I_m and their corresponding elements of I is that $\alpha H x_m$ in I_m and $\alpha H \beta$ in I. Similarly, when restricting A to rows and columns $m, \ldots, m+n-1$, we obtain the matrix B, and our only difference is that $x_n H \beta$ in I_n and $\alpha H \beta$ in I. Note that α and β are therefore mapped to the same type of point (with reference to Definition 5) as they were in F_m and F_n, respectively, since α still hollows another element and an element still hollows β. Applying our function F_{m+n-1} to I, we therefore find that the shape of the first m points of our path is equivalent to that of M_m. Since noses are preserved by our procedure, x is a hard dip. This implies that x is followed by an up-step, just as is x_n in M_n. Finally, we find that the shape of the last n points of the path is equivalent to that of M_n. We have therefore shown that $F_{m+n-1}(I) = M_m + M_n$.

□

Theorem 5. *If M is a Riordan path of order n and has a preimage I under F_n, then M^r has a preimage under F_n, namely I^r.*

Proof. Let $M \in \mathcal{R}_n$, let $\{P_k\}_{k=1}^n$ be the ordered set of points of M, and suppose $F_n(I) = M$, where $I \in \mathcal{I}_n$. Let $x \in I$. If aHx for some element $a \in I$, then xHa in I^r, since pred(x) in I

Reassigning Noses and Hollows: Specific Cases

Hard Peak	if $\{p_i\}_{i=1}^k, \{q_i\}_{i=1}^k \neq \emptyset$	$\rightarrow$	Shift Summary Table
	if $\{p_i\}_{i=1}^k = \{q_i\}_{i=1}^k = \emptyset$	$\rightarrow$	Set x_0Hx_1.
Positive Soft Peak	if $\{p_i\}_{i=1}^k, \{q_i\}_{i=1}^k \neq \emptyset$	$\rightarrow$	Shift Summary Table
	if $\{p_i\}_{i=1}^k = \{q_i\}_{i=1}^k = \emptyset$	$\rightarrow$	Set x_0Hx_1.
Positive Soft Dip	$\{p_i\}_{i=1}^k, \{q_i\}_{i=1}^k \neq \emptyset$	$\rightarrow$	Shift Summary Table
Positive Slope	$\{p_i\}_{i=1}^k, \{q_i\}_{i=1}^k \neq \emptyset$	$\rightarrow$	Shift Summary Table
Hard Dip	Either Case	$\rightarrow$	Shift Summary Table

is equal to succ(x) in I^r. This applies analogously to the cases of aNx, xNa, and xHa in I. Thus if $x \in I$ is mapped to a positive (negative) soft peak or dip, then $x \in I^r$ is mapped to a negative (positive) soft peak or dip. Similarly, if $x \in I$ is mapped to a positive (negative) slope, then $x \in I^r$ is mapped to a negative (positive) slope. If $x \in I$ is mapped to either a hard peak, a hard dip, or a level point, $x \in I^r$ is mapped to the same type of point. Let $F_n(I^r) = M'$ and let $\{Q_k\}_{k=1}^n$ represent the ordered set of points of M'.

Since the ordering of elements in the trace of I^r is reversed with respect to their ordering in the trace of I, we now have that each point Q_k of M' corresponds to that of P_{n-k+1} with respect to the element they represent in I^r and I, respectively. Thus we have shown that $M' = M^r$, and that the preimage of M^r under F_n is I^r. □

Theorem 6. *Let P_j denote the jth point in a Riordan path $M \in \mathcal{R}_n$ where $1 < j < n$. Furthermore, let M_j^h denote the path $P_1, \ldots, P_{j-1}, P_a, P_b, P_{j+1}, \ldots, P_n$ where (P_{j-1}, P_a) and (P_b, P_{j+1}) are the same types of steps as (P_{j-1}, P_j) and (P_j, P_{j+1}) in M, respectively, and (P_a, P_b) is a horizontal step. We describe this process as adding a horizontal step to point P_j, which is shown graphically in Figures 5 and 6. If M has a preimage under F_n, and M_j^h is a Riordan path, then M_j^h has a preimage under F_{n+1}.*

Shift Summary Table

I	I_j^h
aHb ($a < b < x$ or $x < a < b$) aNb ($a < b < x$ or $x < a < b$)	aHb aNb
p_iHq_i ($p_i < x < q_i$)	p_1Hx_0 p_iHq_{i-1} x_1Hq_k
r_iNs_i ($r_i < x < s_i$)	r_1Nx_0 r_iNs_{i-1} x_1Ns_l

Proof. Let $M \in \mathcal{R}_n$, P_j, and M_j^h be as defined above, and suppose M has a preimage I under F_n. The following notation will be used when applicable in each case: Let x denote the element of I corresponding to P_j. Let $\{p_i\}_{i=1}^k$ be the possibly empty set of elements preceding x in the trace of I such that $p_i H q_i$, where $\{q_i\}_{i=1}^k$ is a set of elements succeeding x in the trace of I. Let $\{r_i\}_{i=1}^l$ be the set of elements preceding x in the trace of I such that $r_i N s_i$, where $\{s_i\}_{i=1}^l$ is a set of elements succeeding x in the trace of I. Note that in our first four cases, this set is trivially nonempty. Let $\alpha H x$, $x H \beta$, $\gamma N x$, $x N \delta$, if such elements exist.

Now, let I_j^h denote the semiorder in which the first $j-1$ elements and the last $n-j$ elements are obtained from their corresponding elements in I. These elements will inherit all nose/hollow relationships from I with the exception of those involving $\alpha, \beta, \gamma, \delta$, and x, those of the form $p_i H q_i$, and those of the form $r_i N s_i$. Between these elements we will place two elements x_0, x_1, in that order.

In each following case we set $p_i H q_{i-1}$, $r_i N s_{i-1}$ for $i > 1$, essentially "shifting" our noses and hollows to make room for those of the newly-added point. We will then construct the remaining nose/hollow relationships of I_j^h, and show that $I_j^h \in \mathcal{I}_n$ and $F_n(I_j^h) = M_j^h$.

Case 1: P_j is a hard peak. Let I_j^h have the following properties: $p_1 H x_1$, $x_1 H \beta$, $\alpha H x_0$, $x_0 H q_k$, $r_1 N x_1$, and $x_0 N s_l$. If our sets $\{p_i\}_{i=1}^k$ is empty, we set $x_0 H x_1$. By this construction every element in I_j^h, x_0 and x_1, has the same number and type of nose/hollow relationships as their corresponding elements of I. Furthermore, with the definition of the nose/hollow relationships of x_0 and x_1, (x_0, x_1) is now a horizontal step, with x_0 mapping to a positive soft peak and x_1 mapping to a negative soft peak. Thus we have shown that if $I \in \mathcal{I}_n$, then $F_{n+1}(I_j^h) = M_j^h$.

Let A denote the incidence matrix of I. We will investigate the incidence matrix of I, which we will denote A_j^h, with the purpose of showing that $I \in \mathcal{I}_n$. A_j^h is constructed as follows: We start with the matrix A, and duplicate the row and column corresponding to x, placing the copy next to the original. These two duplicate rows and columns are x_0 and x_1. The resulting matrix implies that $\alpha H x_1$, so we add a "1" in the row corresponding to α so that we have $\alpha H x_0$. Now let r_i correspond to the z_ith row in the matrix for all i. We add enough 1's in rows z_i through $z_{i+1} - 1$ so that the nose $r_i N s_i$ has now become $r_i N s_{i-1}$ and $r_1 N x_1$. Note that this furthermore shifts each hollow such that we now have $p_i H q_{i-1}$. Finally, we add enough 1's in the row corresponding to x_0 so that we have $x_0 N s_k$. This procedure is exemplified in Figures 4, 5, and 6.

Let $a, b \in I_j^h$ where a immediately precedes b in the trace of I_j^h. Since $p_1 H x_1$, $x_0 H q_k$, and the pair x_0, x_1 have different predecessor sets, neither a nor b are the elements x_0 or x_1. Suppose $\text{pred}(a) = \text{pred}(b)$ and $\text{succ}(a) = \text{succ}(b)$. Then a does not nose any element, and no element hollows a in I^a. These statements are also true in I by our construction of I_j^h. This implies that $\text{pred}(a) = \text{pred}(b)$ and $\text{succ}(a) = \text{succ}(b)$ in I, a contradiction. Thus, I_j^h is irredundant. We must now verify the connectedness of I_j^h. By our construction of I_j^h, no element noses an element adjacent to it in the trace; the noses $r_1 N x_1$, $x_0 N s_l$, and $r_i N s_{i-1}$ are all between non-adjacent elements. Since all other nose relationships were inherited from I, this shows that I_j^h is connected. We conclude that $I_j^h \in \mathcal{I}_{n+1}$.

Case 2: P_j is a positive soft peak. Let I_j^h have the following properties: $p_1 H x_1$, $x_1 H \beta$, $\alpha H x_0$, $x_0 H q_k$, $r_1 N x_1$, $x_1 N \delta$, and $x_0 N s_k$. If $\{p_i\}_{i=1}^k$ is empty, then set $x_0 H x_1$. Analogously

	1	α/r_1	p_1	r_2	p_2/r_3	x	q_1/s_1	q_2/s_2	β/s_3	10
1	0	0	0	1	1	1	1	1	1	1
α/r_1	0	0	0	0	0	0	1	1	1	1
p_1	0	0	0	0	0	0	0	1	1	1
r_2	0	0	0	0	0	0	0	1	1	1
p_2/r_3	0	0	0	0	0	0	0	0	1	1
x	0	0	0	0	0	0	0	0	0	1
q_1/s_1	0	0	0	0	0	0	0	0	0	1
q_2/s_2	0	0	0	0	0	0	0	0	0	1
β/s_3	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0

The incidence matrix of a semiorder in $\mathcal{I}_{10}$. The associated Riordan path is shown in Figure 5.

	1	α/r_1	p_1	r_2	p_2/r_3	x_0	x_1	q_1/s_1	q_2/s_2	β/s_3	10
1	0	0	0	1	1	1	1	1	1	1	1
α/r_1	0	0	0	0	0	0	0	1	1	1	1
p_1	0	0	0	0	0	0	0	0	1	1	1
r_2	0	0	0	0	0	0	0	0	1	1	1
p_2/r_3	0	0	0	0	0	0	0	0	0	1	1
x_0	0	0	0	0	0	0	0	0	0	0	1
x_1	0	0	0	0	0	0	0	0	0	0	1
q_1/s_1	0	0	0	0	0	0	0	0	0	0	1
q_2/s_2	0	0	0	0	0	0	0	0	0	0	1
β/s_3	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0

The incidence matrix after duplication.

	1	α/r_1	p_1	r_2	p_2/r_3	x_0	x_1	q_1/s_1	q_2/s_2	β/s_3	10
1	0	0	0	1	1	1	1	1	1	1	1
α/r_1	0	0	0	0	0	0	1	1	1	1	1
p_1	0	0	0	0	0	0	0	1	1	1	1
r_2	0	0	0	0	0	0	0	1	1	1	1
p_2/r_3	0	0	0	0	0	0	0	0	1	1	1
x_0	0	0	0	0	0	0	0	0	0	1	1
x_1	0	0	0	0	0	0	0	0	0	0	1
q_1/s_1	0	0	0	0	0	0	0	0	0	0	1
q_2/s_2	0	0	0	0	0	0	0	0	0	0	1
β/s_3	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0

The incidence matrix after shifting rows α through x_0. The associated Riordan path is shown in in Figure 6.

Figure 4: Constructing the Incidence Matrix A_j^h from A with $j = 6$

to our previous case, this construction implies that if $I_j^h \in \mathcal{I}_n$, then $F_{n+1}(I_j^h) = M_j^h$.

Furthermore, the construction of the incidence matrix of I_j^h is identical to that of the case of a hard peak. In duplicating the row corresponding to P_j, the element x_1 is given the property $x_1 N \delta$, as required by our construction of I_j^h. Since the procedure for constructing the incidence matrix of I_j^h is the same for both the case of the hard peak and the soft peak, we conclude that I_j^h is an interesting semiorder in the case of a soft peak as well.

Case 3: P_j is a positive soft dip. Let I_j^h have the following properties: $p_1 H x_0$, $x_0 H q_k$, $x_1 H \beta$, $r_1 N x_1$, $x_1 N_\delta$, $\gamma N x_0$, and $x_0 N s_k$. Similar to the previous cases, we have that f $I_j^h \in \mathcal{I}_n$, then $F_{n+1}(I_j^h) = M_j^h$.

In constructing our matrix, we proceed identically to the case of the hard peak with one exception: rather than begin adding 1's with the row corresponding to α (this element does not exist in this case), we add a "1" to the row corresponding to γ after duplication so that $\gamma N x_0$. We then proceed exactly as in the case of a hard peak. Analogously, our resulting matrix represents an interesting semiorder, and thus we have $F_{n+1}(I_j^h) = M_j^h$.

Case 4: P_j is a positive slope. Let I_j^h have the following properties: $p_1 H x_0$, $x_0 H q_k$, $x_0 N s_k$, $r_1 N x_1$, $x_1 N \delta$, and $x_1 H \beta$. Similar to the previous cases, we have that f $I_j^h \in \mathcal{I}_n$, then $F_{n+1}(I_j^h) = M_j^h$.

Again, we construct our matrix in a manner identical to that of the case of the hard peak, with the following exception: Since α does not exist in this case, we simply ignore this element and begin adding 1's wherever necessary. This results in a matrix representing an interesting semiorder, proving that $F_{n+1}(I_j^h) = M_j^h$ for this case as well.

Case 5: P_j is a hard dip.

Lemma. *Let P be a hard dip of a Riordan path M with preimage I under F_n, let n_i be the number of noses initiated prior to P, and let n_t be the number of noses terminated prior to P. The height of P is equal to $n_i - n_t - 1$.*

Proof. Let P_i denote the ith point in $M \in \mathcal{R}_n$, let $1 < k < n$, let u denote the number of up-steps prior to P_k, let d denote the number of down-steps prior to P_k, let h denote the number of hard peaks prior to P_k, and finally let n_i and n_t denote the number of noses initiated and terminated prior to P_k, respecitvely.

For $i > 1$, each point P_i that initiates a nose is preceded by an up-step, and each point that terminates a nose precedes a downstep, which follows directly from our definition of F_n. Thus we can associate exactly one up-step to each nose initiated after P_1, and exactly one down-step to each nose terminated.

Each up-step that is not part of a hard peak immediately precedes a point that initiates a nose, and each down-step that is not part of a hard peak is immediately preceded by a point that terminates a nose. A hard peak neither initiates nor terminates a nose, and contains both an up-step and a down-step. We now note that P_1 initiates a nose but is not preceded by an up-step.

Thus we arrive at the following two equations: $n_i = u - h + 1$ (our "+1" comes from the fact that we do not associate P_1 with an up-step) and $n_t = d - h$. Combining these, we get that $n_i - n_t = 1 + u - d$, or equivalently, $n_i - n_t - 1 = u - d$. We conclude that the height of P_k is equal to $n_i - n_t - 1$. □

In all previous cases, the fact that $M_j^h \in \mathcal{R}_{n+1}$ has been trivial. This case, however, provides the possibility of a horizontal step being added on the horizontal axis, which would result in M_j^h not being a Riordan path. We will thus require that P_j is of height greater than 0. Equivalently, this means that there are at least two unterminated noses initiated prior to the point P_j, by the previous lemma. One of these noses is initiated by the point corresponding to γ, and the rest ensure that our set $\{r_i\}_{i=1}^l$ is non-empty.

For this case, we let I_j^h have the following properties: $\gamma N x_0$, $x_0 N s_k$, $r_1 N x_1$, and $x_1 N \delta$. We add 1's to our incidence matrix just as in previous cases, starting with the row corresponding to γ and ending with that corresponding to x_0. Since there are no noses between adjacent points, connectedness is guaranteed, and irredundancy is guaranteed by an argument analogous to that of each previous case.

Finally, we consider the cases of a level point, a negative soft peak, and negative soft dip, and a negative slope. If P_j is a negative soft peak, negative soft dip, or negative slope, we have that M_j^h has a preimage under F_{n+1} by Theorem 5. If P_j is a level point, then adding a horizontal step to the point P_j is equivalent to adding a horizontal step to the point P_{j-1}. Continuing as necessary, we find that adding a horizontal step to P_j is equivalent to adding a horizontal step to P_{j-i} for some $1 < i < j - 2$, where P_{j-i} is not a level point. Since we have proven that $F_{n+1}(I_j^h) = M_j^h$ for every other case, we have therefore proven it for this case as well. □

Theorem 7. *Let $M \in \mathcal{R}_n$, and let $P_1, P_2, \ldots, P_n$ denote the ordered set of points in M, where P_1 is the left-most point. Now let M^a denote the path consisting of the ordered points $x, P_1, P_2, \ldots, P_n, y$, where P_i in M and $(x, P_1), (P_n, y)$ represent an up-step and a down-step, respectively. If M has a preimage in F_n, then M^a has a preimage under F_{n+2}.*

Proof. Let I be a semiorder such that $F_n(I) = M$, and let i_j denote the jth element in the trace of I for $1 \leq j \leq n$. Furthermore, let $\{p_i\}_{i=1}^k, \{q_i\}_{i=1}^k$ denote the set of elements of I such that $p_i H q_i$, and let $\{r_i\}_{i=1}^l$,$\{s_i\}_{i=1}^l$ denote the set of elements of I such that $r_i N s_i$. We will now define a semiorder I^a as the set $i_x, i_1, \ldots, i_n, i_y$ with noses $r_i N s_{i+1}$, $i_x N s_1$, $r_l N i_y$, and hollows $p_i H q_{i+1}$, $i_x H q_1$, and $p_k H i_y$. Note that each element $i_1, \ldots, i_n$ has the same number and type of nose/hollow relationships in I^a as in I.

We will first show that I^a is an interesting semiorder. The connectedness of I^a follows trivially from the connectedness of I, so we will focus on proving irredundancy.

Let $a, b \in I^a$ where a immediately precedes b in the trace of I^a. Suppose $\text{pred}(a) = \text{pred}(b)$ and $\text{succ}(a) = \text{succ}(b)$. Then a does not nose any element, and no element hollows a in I^a. These statements are also true in I by our construction of I^a. This implies that $\text{pred}(a) = \text{pred}(b)$ and $\text{succ}(a) = \text{succ}(b)$ in I, a contradiction. We conclude that $I^a \in \mathcal{I}_{n+2}$.

We will now show that $F_{n+2}(I^a) = M^a$. By our construction of I^a, the elements $i_2, \ldots, i_{n-1}$ map to points $P_2, \ldots, P_{n-2}$ under F_{n+2}, respectively. The points i_1 and i_n now map to positive and negative slopes, respectively, since i_1 necessarily hollows and noses two elements in I^a, and i_n is necessarily nosed and hollowed by two elements in I^a. This furthermore shows that (x, P_1) is an up-step and (P_n, y) is a downstep. Finally, due to the noses

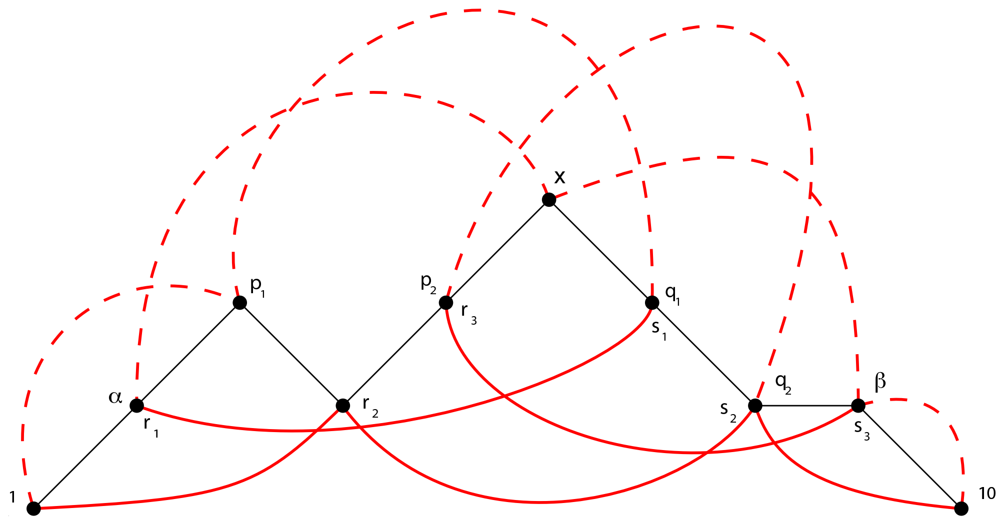

Figure 5: An example of a Motzkin path M.

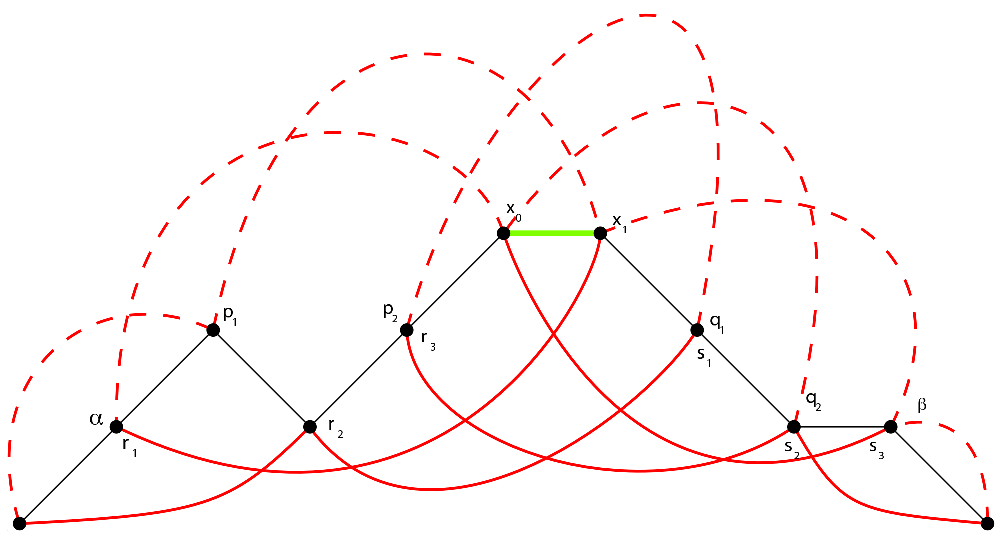

Figure 6: The Motzkin path M_5^h

and hollows constructed for elements i_x, i_y, we have that they properly correspond to the first and last elements of a Riordan path, namely M^a. □

Theorem 8. *The function F_n is a bijection for all $n > 2$.*

Proof. We will prove by induction on n that F_n is onto, with the case $n = 3$ shown in Figure 7, and case $n = 4$ given by Theorem 6 and the case $n = 3$. By Theorem 2, this will show that F_n is a bijection for all $n > 2$.

Let $n \geq 5$ and assume that F_k is a bijection for all $2 < k < n$. Suppose M has a horizontal step and let M' denote the Riordan path of length $n-1$ with this horizontal step deleted. By our inductive hypothesis, this shows that M' has a preimage under F_{n-1}. By Theorem 6, M has a preimage under F_n.

Now suppose that M has no horizontal steps, and suppose that there exists a point P_i in M such that P_i is on the horizontal axis and $1 < i < n$. Let M_1, M_2 denote the Riordan paths consisting of points $P_1, \ldots, P_i$ and $P_i, \ldots, P_n$, respectively. We have that $M_1 + M_2 = M$. By our inductive hypothesis and Theorem 4, we have that M has a preimage under F_n.

Finally, suppose that M has no horizontal steps and that there are no points other than P_1 and P_n on the horizontal axis. Now let M'' denote the path with the first and last steps deleted. Since there are no horizontal steps and no points on the horizontal axis in M, this implies that M'' has no points below the axis and no horizontal steps on the axis, i.e., it is a Riordan path. By our inductive hypothesis and Theorem 7, M has a preimage under F_n.

Since these cover every possible case for M, we have shown that F_n is onto. By Theorem 2, F_n is a bijection for all $n > 2$. □

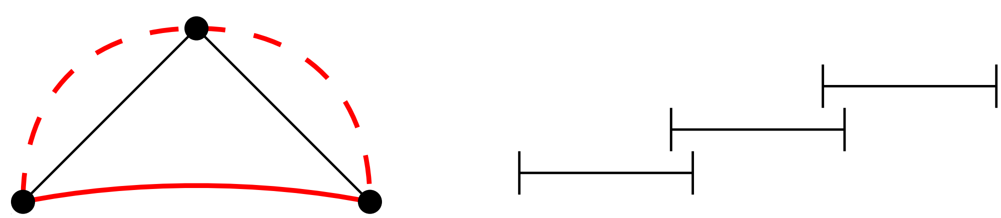

Figure 7: The Riordan path of base case $n = 3$, as well as its interval representation.

3 Acknowledgement

We are thankful to Jean-Paul Doignon for many helpful comments in the preparation of this manuscript. We would also like to thank the referee for their comments regarding presentation and clarity.

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2000 *Mathematics Subject Classification*: Primary 05A15; Secondary 06A07.
Keywords: Riordan numbers, interval orders, semiorders, lattice paths.

(Concerned with sequences A000108 and A005043.)

Received February 14 2007; revised version received July 18 2007. Published in *Journal of Integer Sequences*, July 23 2007.

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