An Augmented Matrix Approach for Updating Finite Element Meshes in Surgical Simulations

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We present an algorithm to support interactive visualization of solid finite element models during deformation and cutting by quickly updating stiffness matrix factorizations. Topological mesh modifications and boundary condition changes are an essential part of many simulation scenarios, particularly surgical simulations. Integrating support for cutting with real-time finite element solution methods is a computational challenge due to two reasons: First, because of the high update rates required for graphical ad haptic rendering; Second, because the connectivity changes due to the cutting (removal and insertion of nodes and elements, and re-meshing around the cut) necessitate corresponding changes to the underlying system of linear equations.

The algorithm presented in this work provides a fast visualization functionality through an augmented matrix approach. The stiffness matrix of the initial mesh (from the discretization of PDEs corresponding to linear elasticity) is factored using the Cholesky factorization for symmetric positive definite matrices. This matrix forms the static $(1, 1)$ block of a block 2×2 matrix; the other three blocks are updated during the cutting. Updates to the mesh are reflected by adding columns to the $(1, 2)$ block, and rows of the identity matrix to the (2.1) block. The (2.2) block is zero. The augmented matrix is unsymmetric, inspite of the fact that the $(1, 1)$ block is symmetric.

As the mesh is cut, the solution to the stiffness matrix equations is updated by solving a reduced system of equations involving the Schur complement in an implicit manner. We have found that forming the Schur complement matrix is too time-consuming. Instead, we apply the Schur complement operator to vectors by solving systems of equations involving the L and D factors from the LDL^T factorization of the initial stiffness matrix. The Schur complement system is solved using a Krylov space iterative solver (GMRES) coupled with a preconditioner. The Schur complement solution involves several triangular solves involving the L factors of the initial stiffness matrix. Here we make use of the sparsity in the right-hand-side vectors (RHS) to reduce the time requirements for the large number of triangular solves by computing only the elements that occur in the closure of the nonzero elements of the RHS vector in the graph of L. We also make use of a projection involving the rows added in the augmented matrix, since these are extremely sparse with only one nonzero in each row.

We are able to provide ten to hundreds of updates per second for finite element meshes with hundreds of thousands of elements. To the best of our knowledge, this is the first such set of results for meshes of this size. We compare our solution to a conjugate gradients-based method; the latter works for well-conditioned stiffness matrices, but not for poorly conditioned meshes that arise when meshes are cut.