APPROXIMATION OF BEST SEPARATORS BY EIGENFUNCTIONS OF $p$-LAPLACIANS

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Abstract. Partitioning a graph with $n$ vertices into two parts with minimal deviation from equal size and a minimal number of cut edges can be understood as looking at the geometric problem of dividing some domain $\Omega$ into two regions of equal measure by a surface $\Gamma$ with possibly small surface measure in some non-Euclidian metric. In [1] the geometric problem is related to minimizing the functional

$$F_1(u) = \frac{\int_{\Omega} |Du| \, dV}{||u||_1} \to \min, \ u \in BV, \int_{\Omega} \text{sign} \, u \, dx = 0.$$ 

$F_1$ is still unpleasant for algorithmic purposes, hence the problem is replaced by

$$F_p(u) = \frac{||\nabla u||_p^p}{||u||_p^p} \to \min, \ u \in H^{1,p}, \int_{\Omega} |u|^{p-2}u \, dV = 0, \ u \neq 0, \ p \in (1, 2].$$

Minimizers $u \in H^{1,p}(\Omega)$ satisfy necessarily the Euler Lagrange equation

$$-\nabla \cdot (||\nabla u||_p^{-2}\nabla u) = F_p(u)|u|^{p-2}u \text{ in } \Omega, \ \nu \cdot \nabla u = 0 \text{ on } \partial \Omega,$$

$\nu$ outer normal (see [1] for convergence results). This can be seen as a nonlinear eigenvalue problem, which can be solved by a steepest descent or Newton’s method for fixed $p > 1$ and embedding starting form $p = 2$. The case $p = 2$ is the usual spectral bisection, minimizing the $L^2$-norm of the gradient but not fulfilling $\int_{\Omega} \text{sign} \, u \, dx = 0$. The figure shows a grid and a separator approximation. The main practical interest in these computational expensive separators is seen in comparing heuristic methods in cases without obvious symmetry to best approximations. Another question is how much weight is given to minimal operation count and to load balance in case of non-convex ‘domains’. Remark: Phase separation models minimizing ‘surface energy’ in the limit (see [2]) are well suited to approximate the total minimal surface for devinding a domain (or graph) into $n$ partitions with specified measures $m_i, i = 1, \ldots, n$.


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