Certificate of impossibility of Hilbert-Artin representation of given degree for definite polynomials and functions

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Polynomial Optimization Problem

Unconstrained polynomial minimization problem

$$f^* \stackrel{\text{def}}{=} \inf\{f(\xi) \mid \xi \in \mathbb{R}^n\}$$

where $f \in \mathbb{R}[\bar{X}] = \mathbb{R}[X_1, \ldots, X_n]$.

The problem is equivalent to compute

$$f^* = \sup\{a \in \mathbb{R} \mid f - a \geq 0 \text{ on } \mathbb{R}^n\}$$

Sums of Squares (SOS) Relaxation

- If f attains a minimum on $\mathbb{R}^n$

$$f - f^* = \mathrm{SOS} \mod I$$

  Gradient Ideal Nie/Demmel/Sturmfels, Math. Program., 2005.

- Otherwise

$$f - f^* = \mathrm{SOS} \mod P$$

  - Gradient Tentacle Schweighofer, SIAM J. Optim. 2006.
  - Tangency Variety Hà/Pham, SIAM J. Optim. 2008.
  - Polar Variety Guo/Safey El Din/Zhi, ISSAC 2010.

Semidefinite Programming (SDP) $\Longrightarrow$ Numerical Lower Bound

Hilbert-Artin Representation of PSD polynomials

Emil Artin's 1927 Theorem (Hilbert's 17th Problem):

$$\forall \xi_1, \ldots, \xi_n \in \mathbb{R} \colon\ f(\xi_1, \ldots, \xi_n) \geq 0$$

$$\Updownarrow$$

$$\exists u_i, v_j \in \mathbb{R}[\bar{X}] \colon\ f(\bar{X}) = \frac{\sum_{i=1}^{l} u_i^2}{\sum_{j=1}^{l} v_j^2}$$

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$$\Updownarrow$$

$$\exists e \geq 0,\ W^{[1]} \succeq 0,\ W^{[2]} \succeq 0,\ W^{[2]} \neq \mathbf{0} :$$

$$f(\bar{X}) \cdot (m_e(\bar{X})^T W^{[2]} m_e(\bar{X})) = m_d(\bar{X})^T W^{[1]} m_d(\bar{X})$$

where $d = e + (\deg f)/2$, $m_e(\bar{X})$ and $m_d(\bar{X})$ are vectors of terms.

$W \succeq 0 \Leftrightarrow P^T L^T D L P$, D diagonal, $D_{i,i} \geq 0$ (Cholesky)

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where $d = e + (\deg f)/2$, $m_e(\bar{X})$ and $m_d(\bar{X})$ are vectors of terms.

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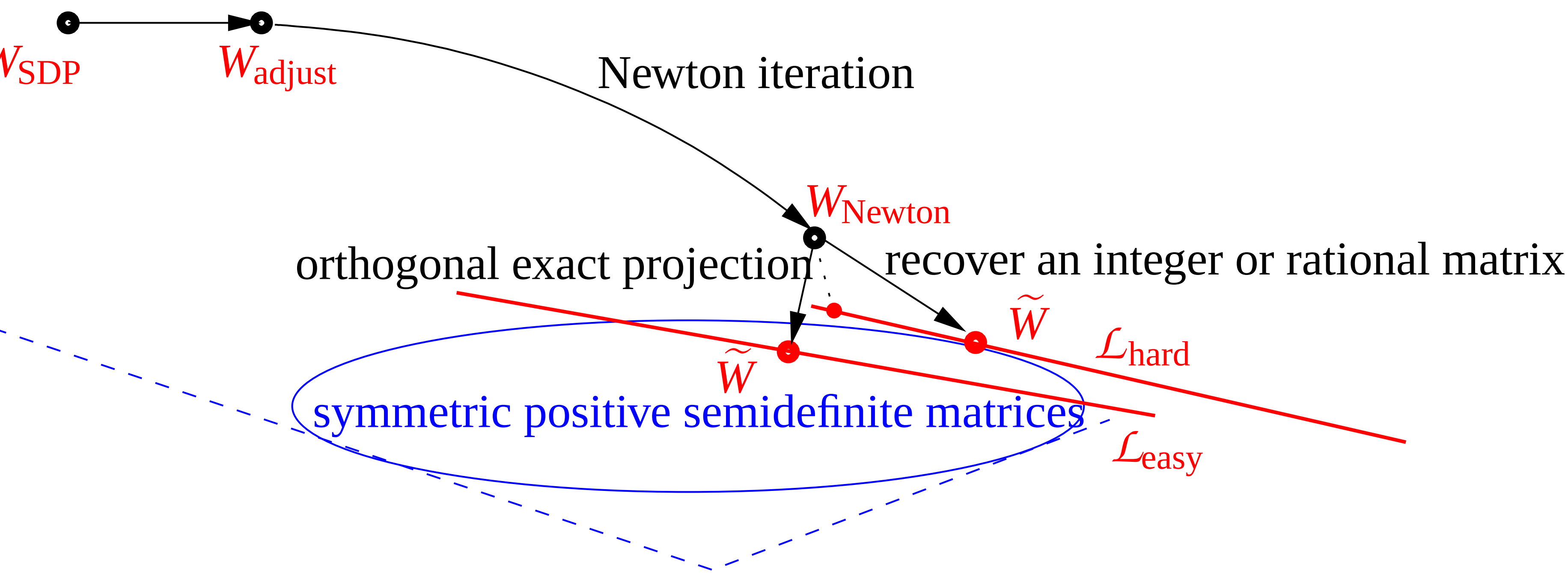
Exact Certification of PSD Polynomial

Denote $W = \begin{bmatrix} W^{[1]} & \\ & W^{[2]} \end{bmatrix}$, then the affine linear hyperplane is

$$\mathcal{L} = \left\{ \; W \;\middle|\; f(\bar{X}) \cdot (m_e(\bar{X})^T W^{[2]} m_e(\bar{X})) = m_d(\bar{X})^T W^{[1]} m_d(\bar{X}) \; \right\}$$



References: "Easy Case" Peyrl, Parrilo '07,'08;
"Hard Case" Kaltofen, Li, Yang, Zhi '08,'09

Our Result

f e is too small $\Longrightarrow \nexists\, W^{[1]} \succeq 0, W^{[1]} \succeq 0, W^{[2]} \neq \mathbf{0}$, s.t.

$$f(\bar{X}) \cdot (m_e(\bar{X})^T W^{[2]} m_e(\bar{X})) = m_d(\bar{X})^T W^{[1]} m_d(\bar{X})$$

SDP solver in Maltab can only **numerically** detect it!

Notation: $\mathrm{SOS}/\mathrm{SOS}_{2e} = \{\sum u_i^2 / \sum v_j^2 \mid u_i, v_j \in \mathbb{R}[\bar{X}],\ \deg v_j \le e\}$.

Our result: Given integer $e \geq 0$, we give **exact** certification if $f(\bar{X}) \notin \mathrm{SOS}/\mathrm{SOS}_{2e}$.

Remark: More general, we can certify $f(\bar{X}) \notin \mathrm{SOS}/\mathrm{SOS}_{2e}$ with

erms in v_j are **restricted** in a given subset of m_e.

Reduce to Semidefinite Programming

$f(\bar{X}) \notin \mathrm{SOS}/\mathrm{SOS}_{2e}$ if and only if $\nexists\, W^{[1]} \succeq 0, W^{[2]} \succeq 0$, s.t.

$$f(\bar{X}) \cdot (m_e^T W^{[2]} m_e) = m_d^T W^{[1]} m_d, \quad \mathrm{Tr} W^{[2]} = 1,$$

where $d = e + (\deg f)/2$.

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For symmetric matrices $C,\ W$, define inner product as

$$C \bullet W = \langle C, W \rangle = \sum_i \sum_j c_{i,j} w_{i,j} = \mathrm{Trace}(CW).$$

Let

$$m_d^T W^{[1]} m_d = \sum_\alpha (G^{[\alpha]} \bullet W^{[1]}) \bar{X}^\alpha,$$

and

$$-f(\bar{X}) \cdot (m_e^T W^{[2]} m_e) = \sum_\beta (H^{[\beta]} \bullet W^{[2]}) \bar{X}^\beta.$$

Infeasibility of Semidefinite Programming

Let

$$W = \begin{bmatrix} W^{[1]} & * \\ * & W^{[2]} \end{bmatrix}, \; A^{[\alpha]} = \begin{bmatrix} G^{[\alpha]} & \\ & H^{[\alpha]} \end{bmatrix}, \; A = \begin{bmatrix} \mathbf{0} & \\ & I \end{bmatrix}.$$

$$f(\bar{X}) \cdot (m_e^T W^{[2]} m_e) = m_d^T W^{[1]} m_d, \quad \mathrm{Tr} W^{[2]} = 1,$$

$$\Downarrow$$

$$\left.\begin{array}{ll} \sup\limits_{W \in \mathcal{S}^{k \times k}} & -C \bullet W \\ s.t. & \begin{bmatrix} \vdots \\ A^{[\alpha]} \bullet W \\ \vdots \\ A \bullet W \end{bmatrix} = \begin{bmatrix} \vdots \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \quad W \succeq 0 \end{array}\right\} (P)$$

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$f(\bar{X}) \notin \mathrm{SOS}/\mathrm{SOS}_{2e}$ if and only if SDP (P) is infeasible.

Dual Problem

The dual problem of SDP (P):

$$\left.\begin{array}{r} s^* = \inf\limits_{(y,s)\in\mathbb{R}^{m+1}} s \\ s.t.\ M(y,s) \succeq 0 \end{array}\right\} (D)$$

where

$$M(y,s) = \begin{bmatrix} M_d(y) & \\ & M_e((-f)y) + sI \end{bmatrix},$$

$y = (y_\alpha)_{\alpha\in\mathbb{N}^n_{2d}} \in \mathbb{R}^{\mathbb{N}^n_{2d}}$ and $m = \#\mathbb{N}^n_{2d}$, *moment matrix* $M_d(y)$ and *localizing moment matrix* $M_e((-f)y)$ are indexed by $\mathbb{N}^n_{2d}$. For $\alpha, \beta \in \mathbb{N}^n_{2d}$,

$$M_d(y)[\alpha,\beta] = y_{\alpha+\beta}$$

$$M_e((-f)y)[\alpha,\beta] = -\sum_\gamma f_\gamma y_{\alpha+\beta+\gamma}$$

Semidefinite Farkas' Lemma

Standard SDP:

$$\sup_{W \in \mathcal{S}^{n\times n}} \; -C \bullet W \qquad\qquad \inf_{y\in\mathbb{R}^l} \; b^T y$$

$$s.t.\; A_i \bullet W = b_i,\; i = 1\cdots l, \qquad\qquad s.t.\; C + \sum_{i=1}^{l} y_i A_i \succeq 0,$$

$$W \succeq 0.$$

Lemma. *[Jon Dattorro2005] If the set of vectors*

$$\begin{bmatrix} A_1 \bullet W \\ \vdots \\ A_l \bullet W \end{bmatrix}, \quad \forall W \succeq 0$$

is closed, then primal problem is feasible if and only if $\forall y \in \mathbb{R}^l$ *s.t.* $\sum_{i=1}^{l} y_i A_i \succeq 0$, *we have* $b^T y \geq 0$.

Closedness of the Convex Cone

Lemma. *[Robinson73] The cone* SOS_{2e} *is convex and closed.*

Lemma. *In our case, the subset of vector space*

$$\begin{bmatrix} \vdots \\ A^{[\alpha]} \bullet W \\ \vdots \\ A \bullet W \end{bmatrix}, \ \forall\, W \succeq 0,$$

is closed.

$\Longrightarrow$ Assumption in Farkas' Lemma is satisfied!

Main Contribution

Theorem. *The following are equivalent:*

1. $f \notin SOS/SOS_{2e}$,
2. $\exists$ *feasible point* $(y, s) \in \mathbb{R}^{m+1}$ *of* (D), *s.t.* $s < 0$,
3. $\exists$ ***rational vector*** $y' = (y'_\alpha) \in \mathbb{Q}^m$, *s.t.*, $M_d(y') \succeq 0$, $M_e(fy') \prec 0$.

Proof:

Semidefinite Farkas' Lemma

$+$

Dual problem (D) is strictly feasible

$\Downarrow$

$\exists$ **rational vector** $y' = (y'_\alpha) \in \mathbb{Q}^m$

Interpretation by Linear Forms on $\mathbb{R}[\overline{X}]$

Given $y = (y_\alpha) \in \mathbb{R}^{\mathbb{N}^n}$, define the linear form $L_y \in (\mathbb{R}[\overline{X}])^*$ by

$$L_y(f) = y^T \mathrm{CoeffVec}(f) = \sum_\alpha y_\alpha f_\alpha, \quad \text{for } f = \sum_\alpha f_\alpha \overline{X}^\alpha \in \mathbb{R}[\overline{X}].$$

For $u(\overline{X}), v(\overline{X}) \in \mathbb{R}[\overline{X}]$, we have

$$L_y(u^2) = \mathrm{CoeffVec}(u)^T M(y) \mathrm{CoeffVec}(u)$$

$$L_y(fv^2) = \mathrm{CoeffVec}(v)^T M(fy) \mathrm{CoeffVec}(v)$$

Interpretation by Linear Forms on $\mathbb{R}[\bar{X}]$

Theorem. *The following are equivalent:*

1. $f \notin SOS/SOS_{2e}$,
2. $\exists y' \in \mathbb{Q}^m$, *s.t.* $\forall v, u \in \mathbb{R}[\bar{X}]$ *with* $\deg v \le e$, $\deg u \le e + (\deg f)/2$, *we have* $L_{y'}(u^2) \ge 0$ *and* $L_{y'}(fv^2) < 0$.

If $f = \sum u_i^2 / \sum v_j^2$ with $\deg v_j \le e$, then

$$0 \le L_{y'}\left(\sum u_i^2\right) = \sum L_{y'}(fv_j^2) < 0$$

which is a contradiction.

Special Case: $e = 0$

$$e = 0 \implies \text{Certify } f(\bar{X}) \neq \sum_i u_i(\bar{X})^2, u_i(\bar{X}) \in \mathbb{R}[\bar{X}]$$

Theorem. *The following are equivalent:*

1. $f(\bar{X}) \neq \sum_i u_i(\bar{X})^2$, $u_i(\bar{X}) \in \mathbb{R}[\bar{X}]$,
2. $\exists y' \in \mathbb{Q}^m$, *s.t.* $\forall u \in \mathbb{R}[\bar{X}]$ *with* $\deg u \leq (\deg f)/2$. *we have* $L_{y'}(u^2) \geq 0$ *and* $L_{y'}(f) < 0$.

Ahmadi and Parrilo 2009: $y' \rightarrow$ *separating hyperplane*.

Exploit Sparsity in SOS

Theorem. *[Reznick78] For a polynomial* $p(\bar{X}) = \sum_\alpha p_\alpha \bar{X}^\alpha$, *let* $C(p)$ *be the convex hull of* $\{\alpha | \, p_\alpha \neq 0\}$. *If* $f = \sum_i g_i^2$ *then* $C(g_i) \subseteq \frac{1}{2}C(f)$.

$f \in \mathrm{SOS}/\mathrm{SOS}_{2e}$ if and only if

$$f(\bar{X}) \cdot (m_e^T W^{[2]} m_e) = m_d^T W^{[1]} m_d.$$

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$f \in \mathrm{SOS}/\mathrm{SOS}_{2e}$ if and only if

$$f(\bar{X}) \cdot (m_e^T W^{[2]} m_e) = m_d^T W^{[1]} m_d.$$

$$\Updownarrow$$

$$f(\bar{X}) \cdot (m_e^T W^{[2]} m_e) = m_{\mathcal{G}}^T W^{[1]} m_{\mathcal{G}}.$$

$m_{\mathcal{G}} \subseteq m_d \Rightarrow$ Sizes of the SDPs (P) and (D) decrease.

Finding y' by Big-M Method

Employ the Big-M method [Vandenberghe96] to (P) and (D):

$$\left.\begin{array}{l} r^* = \displaystyle\sup_{W\in\mathcal{S}^{k\times k}, w\in\mathbb{R}} \; -C\bullet(W-w) - \mathcal{M}w \\ s.t. \begin{bmatrix} \vdots \\ A^{[\alpha]}\bullet(W-w) \\ \vdots \\ A\bullet(W-w) \end{bmatrix} = \begin{bmatrix} \vdots \\ 0 \\ \vdots \\ 1 \end{bmatrix}, W\succeq 0, w\geq 0, \end{array}\right\} (P^*)$$

$$\left.\begin{array}{l} s^* = \displaystyle\inf_{(y,s)\in\mathbb{R}^{m+1}} \; s \\ s.t.\; M(y,s)\succeq 0, \\ \mathrm{Tr} M(y,s) \leq \mathcal{M}, \end{array}\right\} (D^*)$$

(P^*) and (D^*) are strictly feasible $\Longrightarrow r^* = s^* \to -\infty$ as $\mathcal{M}\to\infty$.

Algorithm Degree Lower Bound Verification

Input: $f \in \mathbb{Q}[\bar{X}]$, $e \in \mathbb{Z}_{\geq 0}$.

Output: If $f \notin \mathrm{SOS}/\mathrm{SOS}_{2e}$, return certificate $y' \in \mathbb{Q}^m$.

1. Reduce the problem to SDPs (P) and (D).
2. Fix a big $\mathcal{M} \in \mathbb{Z}$ and modify (P), (D) to $(P^*), (D^*)$.
3. Solve (P^*), (D^*) by iteration until solution $p_k = (y^k, s^k)$ with $s^k < 0$ is obtained.
4. Find a strictly feasible point of (D) $\tilde{p} = (\tilde{y}, \tilde{s})$.
5. Fix $0 < t \leq 1$ and $\bar{p} = (1-t)p_k + t\tilde{p} = (\bar{y}, \bar{s})$ such that $\bar{s} < 0$.
6. Choose a rational point $p' = (y', s') \in B_\varepsilon(\bar{p})$ where $\varepsilon < \frac{1}{2}|\bar{s}|$.

SDPTools[Guo09]: **High precision** SDP solver in Maple based on the potential reduction method in [Vandenberghe96].

Generalization to Rational Functions

Problem: Given rational function $f/g \in \mathbb{Q}(\bar{X})$ with $g(\bar{X}) \geq 0$, integer $e \geq 0$, certify $f/g \notin \mathrm{SOS}/\mathrm{SOS}_{2e}$.

Let $d = e + \lceil(\deg f - \deg g)/2\rceil$ and

$$\Gamma_1 = \left\{ \bar{X}^{\alpha+\beta+\gamma} \mid \bar{X}^{\gamma} \in \mathrm{Terms}(g), \deg \bar{X}^{\alpha}, \deg \bar{X}^{\beta} \leq d \right\},$$

$$\Gamma_2 = \left\{ \bar{X}^{\alpha+\beta+\gamma} \mid \bar{X}^{\gamma} \in \mathrm{Terms}(f), \deg \bar{X}^{\alpha}, \deg \bar{X}^{\beta} \leq e \right\}.$$

Theorem. *The following are equivalent:*

1. $f/g \notin SOS/SOS_{2e}$,
2. $\Gamma_1 \not\supseteq \Gamma_2$ *or* $\exists y' \in \mathbb{Q}^m$ *s.t.* $\forall u, v \in \mathbb{R}[\bar{X}]$ *with* $\deg u \leq d$, $\deg v \leq e$, *we have* $L_{y'}(gu^2) \geq 0$ *and* $L_{y'}(fv^2) < 0$.

Motzkin Polynomial

We prove that the well-known Motzkin polynomial

$$f(X,Y) = X^4Y^2 + X^2Y^4 + 1 - 3X^2Y^2$$

s not SOS. Otherwise, by exploiting sparsity, f can be written as $f(X) = \sum u_i(X,Y)^2$ where

$$u(X,Y) = u_{0,0} + u_{1,1}XY + u_{2,1}X^2Y + u_{1,2}XY^2.$$

The certificate we obtained is

$$y' = (y'_{0,0} = \frac{22011}{55402}, y'_{1,1} = 0, y'_{2,1} = 0, y'_{1,2} = 0, y'_{2,2} = \frac{358944}{9403},$$
$$y'_{3,2} = 0, y'_{2,3} = 0, y'_{4,2} = \frac{96310}{4693}, y'_{3,3} = 0, y'_{2,4} = \frac{96310}{4693}).$$

Continue

$$
\begin{aligned}
u(X,Y)^2 =& u_{0,0}^2 + u_{0,0}u_{1,1}XY + u_{0,0}u_{2,1}X^2Y + u_{0,0}u_{1,2}XY^2 \\
&+ u_{1,1}^2X^2Y^2 + u_{1,1}u_{2,1}X^3Y^2 + u_{1,1}u_{1,2}X^2Y^3 \\
&+ u_{2,1}^2X^4Y^2 + u_{2,1}u_{1,2}X^3Y^3 + u_{1,2}^2X^2Y^4.
\end{aligned}
$$

$$L_{y'}(u^2) = \frac{22011}{55402}u_{0,0}^2 + \frac{358944}{9403}u_{1,1}^2 + \frac{96310}{4693}u_{2,1}^2 + \frac{96310}{4693}u_{1,2}^2 \geq 0$$

and

$$L_{y'}(f) = \frac{96310}{4693} + \frac{96310}{4693} + \frac{22011}{55402} - 3 \times \frac{358944}{9403} = -\frac{178662293250763}{244479491 3158} < 0,$$

which implies f can not be written as SOS.

Ill-Posed Polynomial

Consider polynomial $f(X,Y) = X^2 + Y^2 - 2XY \in \mathbb{R}[X,Y]$. Since $f = (X-Y)^2$, we have $f^* = \inf f = 0$.

However, $\forall \varepsilon > 0$, $f_\varepsilon(X,Y) = (1-\varepsilon^2)X^2 + Y^2 - 2XY$ is not SOS. Take $x = y = C$, $f_\varepsilon(x,y) = -\varepsilon^2 C^2 \Rightarrow \inf f_\varepsilon = -\infty$. **Ill-posed!**

For $\varepsilon = 10^{-1},\ 10^{-2},\ 10^{-3},\ 10^{-4}$, SDP solver SeDuMi in Matlab can **numerically** detect f_ε is not SOS. But for $\varepsilon = 10^{-5}$ or smaller, it **fails!**

Our method in Maple can give **exact** certificate of f_ε being not SOS for $\varepsilon = 10^{-8}$ or smaller!

Reference: Hutton, Kaltofen and Zhi, ISSAC 2010

Examples with $e > 0$

Consider the even symmetric sextics in [Choi et al.1987]. Let

$$M_r(\bar{X}) = \sum_{i=1}^{n} X_i^r,$$

Then for integer $0 \le k \le n-1$, we define forms f_k by

$$f_0 = -nM_6 + (n+1)M_2M_4 - M_2^3,$$

and

$$f_k = (k^2+k)M_6 - (2k+1)M_2M_4 + M_2^3,\ 1 \le k \le n-1.$$

For $n = 4$, taking $e = 1$, we can certify $f_2 \notin \mathrm{SOS}/\mathrm{SOS}_2$.
For $n = 5$, taking $e = 1$, we can certify $f_2, f_3 \notin \mathrm{SOS}/\mathrm{SOS}_2$.

To our knowledge, they are the **first** PSD polynomials which can not be written as $\sum_i u_i^2 / \sum_j v_j^2$ with $\deg \sum_j v_j^2 = 2$!

THANK YOU!