

# Errata

## *Automatic Sequences: Theory, Applications, Generalizations*

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### Version of April 4 2023

Page xiii: Replace “introduced in Chapter 5” with “introduced in Chapter 4”. (Joel Noche, December 6 2011)

Page xv: The correct address for the errata page is

<https://cs.uwaterloo.ca/~shallit/asas.html> .

(P. Stockmeyer, January 27 2005; JOS, March 31 2020)

Page 5: In line 12, replace the first  $k$  with  $-k$ . (D. Pal, September 21 2006)

Page 6: The correct definition for  $D$  should be

$$D = \{D_{j,a} : j \geq 0, a \in \Sigma\}.$$

(P. Stockmeyer, January 27 2005)

Page 5: Insert  $\dots$  before  $a_{k-2}$  in the rhs of line 13. (Joel Noche, December 6 2011)

Page 6: In line  $-8$ , the domain of  $d$  should be  $A \times A$ , not  $A$ . (D. Pal, September 21 2006)

Page 22: In line 11, remove the  $P$ . (Kalle Saari, September 5 2006)

Page 24: In Line 1, the definition of critical exponent is correct, but poorly worded. It should read that “ $e$  is defined to be the sup, over all  $\alpha > 1$ , such that  $\mathbf{w}$  contains an  $\alpha$ -power”. (V. Linek, May 19 2004)

Page 24: In Exercise 2, add the requirement that the set  $S$  be closed. (Kalle Saari, November 5 2004)

Page 28: In exercise 48 (b), replace  $\varphi$  with  $\mu$ . (Joel Noche, December 6 2011)

Page 29: Exercise 57 is phrased poorly. It means that the maximum is to be taken over all words of length  $n$ . (Kalle Saari, September 5 2006)

Page 31: In Open Problem 12 (c), replace “infinitely many indices” with “infinitely many even indices”. (Kalle Saari, September 5 2006)

Page 33: In line  $-14$ , replace the second big- $O$  with  $\Omega$ . (Kalle Saari, September 5 2006)

Page 33: In line 13, change “weakly cubefree” to “weakly squarefree”. (Narad Rampersad, April 12 2004)

Page 42: In line  $-6$ , replace  $a_1p$  with  $a_1pq^{n-1}$  and  $a_0$  with  $a_0q^n$ . (Kalle Saari, September 5 2006)

Page 44: In Section 2.4, we should have proved that continued fraction expansions of real numbers are essentially unique (so that the statement on page 61, line  $-12$  would be true). (Kalle Saari, September 5 2006)

Page 46: The term “partial quotient” is used for the first time after Theorem 2.4.3, but it was not defined yet. (Dalia Krieger, January 24 2006)

Page 49: In line 4, append “in  $[0, 1)$ ” to the end of the sentence. (Kalle Saari, September 5 2006)

Page 54: In line 13, remove the extra occurrence of “of length”. (Kalle Saari, September 5 2006)

Page 55: In line 3, replace “II” with “I”. In line 8, replace “I” with “II”. (Kalle Saari, September 5 2006)

Page 58: In line 3, the leftmost “2” in the last row of the matrix should be “3”. (Kalle Saari, September 5 2006)

Page 62: In the third line from the bottom, replace “where  $0 \leq a_i < p$  for  $i \geq 0$ ” with “where  $0 \leq a_i < p$  for  $i \geq -m$ ”. (Joseph Meleshko, July 18 2020)

Page 63: In line  $-10$ , replace “reminder” with “remind”. (Eric Rowland, September 4 2008)

Page 63: In Example 2.11.1, last line, the second  $0 \cdot 3^5$  should be  $0 \cdot 3^6$ . (Joel Noche, December 6 2011)

Page 69: In line 3, replace “2.11” with “2.13.5”. (Kalle Saari, September 5 2006)

Page 84: In line  $-8$ , the notation for suffix “ $\triangleright$ ” was introduced but not defined. (Kalle Saari, September 5 2006)

Page 86: In line  $-10$ , replace  $k^{-j}[x_n]$  with  $k^{-n}[x_n]$ . (Kalle Saari, September 5 2006)

Page 99: In line  $-2$ , in the statement of Theorem 3.5.6, replace  $(-1)_{2;11}^e(n)$  with  $(-1)^{e_{2;11}(n)}$ . (Kalle Saari, September 5 2006)

Page 114: In exercise 23, the summation should be  $\sum_{0 \leq i \leq n} r_n$ . (Andrej Vukovic, March 31 2020)

Page 115: In exercise 37, the inequality on  $a_i$  should read  $0 \leq a_i \leq i$ . (Stepan Holub, October 23 2012)

Page 121: In line 10, we attribute to D. McIlroy the connection between  $S(n)$  and merging algorithms. However, this connection was noticed earlier by D. E. Knuth, *Mathematical Analysis of Algorithms, Proceedings of IFIP Congress 71*, North-Holland, 1971, pp. 19–27;

see pp. 23–24. This paper is also reprinted in *Selected Papers on Analysis of Algorithms*, Center for the Study of Language and Information, 2000, pp. 1–18. (Donald Knuth, August 19 2006)

Page 124: In line 18, replace “odd than even” with “even than odd”. (Kalle Saari, September 5 2006)

Page 135: In the last centered display, the  $B_{qq'}$  needs a complement bar over it. (Joel Seiferas, February 18 2014)

Page 140: In line 5, replace “is is” with “is”. (Eric Rowland, September 19 2008)

Page 147: In line –15, replace “all inputs” with “for all inputs”. (Eric Rowland, September 4 2008)

Page 155: In the table in the center of the page, the entry for 14 has an unneeded +. (Kalle Saari, September 5 2006)

Page 161: In the proof of Theorem 5.2.7, the implication arrows that introduce both parts of the proof are in the wrong direction. (Kalle Saari, September 5 2006)

Page 162: Delete the word “the” before “each of the fibers” in the proof of Theorem 5.3.2. (Olga Miltchman, December 1 2007)

Pages 164–165: The transducers presented in Figures 5.14 and 5.15 are completely wrong. In general you need  $k + \ell + 1$  states. (Dominique Perrin, March 23 2023)

Page 165: In the proof of Theorem 5.4.2, we should specify that  $q_0 = 0$  and in the last displayed line we should replace  $q$  with  $q_0$ . (Afshin Amini, January 24 2014)

Page 165: In the proof of Theorem 5.4.3, we should specify that  $\rho$  is a map from  $\Gamma$  to  $\Delta$ , and in the definition of the DFAO  $M'$ , replace  $\Gamma$  with  $\Delta$ . (Afshin Amini, January 13 2014)

Page 166: In line 5 of Example 5.5.1, replace “ $y = 2^{2m+1}$ ” with “ $y = 2^{2m+1} - 1$ ”. (Kalle Saari, September 5 2006)

Page 166: In line 11, the word “nonautomatic” is misspelled. (Eric Rowland, September 4 2008)

Page 167: In the proof of Corollary 5.5.3, the expression  $\overbrace{1 \cdots 1}^n$  should be  $\overbrace{k - 1 \cdots k - 1}^n$ . (Greg Dresden, November 11 2013)

Page 168: In the statement of Theorem 5.6.3, part (d), it should read “ $r \in R$ ”, not “ $r \in S$ ”. (Narad Rampersad via an anonymous referee, July 7 2005)

Page 169: In the second displayed equation, replace the  $q$  with  $a$ . Further down, replace “guesses  $s$ ” with “guessed  $s$ ”. (Eric Rowland, December 20 2011)

Page 171: Ali Aberkane points out that Open problem 1 of section 5.9 is solved, and the critical exponent is 4. This follows from the paper Allouche and Bousquet-Mélou [1994b]

cited on page 483. They prove that the only 4th powers are 0000 and 1111, and there are no 5th powers. It follows that there are no  $4^+$  powers. (James Currie, February 24 2004)

Page 175: In line  $-3$ , change  $\Sigma^*$  to  $\Sigma_k^*$ . (Zaid Shakwet, November 17 2009)

Page 181: In line 8, remove the extra occurrence of the word “that”. (Kalle Saari, September 5 2006)

Page 183: In the 2nd line of the proof of Theorem 6.5.4, replace “automatic” with “2-automatic”. (Narad Rampersad, July 10 2006)

Page 185: In the last set of displayed equations on this page, each item should be enclosed in  $\tau(\dots)$  instead. (JOS, January 10 2017)

Page 192: In line 9, replace “ $\tau'(a)$ ” with “ $\tau(a, q)$ ”. (Kalle Saari, September 5 2006)

Page 195: In line 1, replace “automatic, sequence” with “automatic sequence”. (Eric Rowland, December 20 2011)

Page 202: In Exercise 9, correct the first few terms of  $a_n$  to be 11264224288 $\dots$ . (Eric Rowland, April 17 2010)

Page 204: In Exercise 19, the formula in (c) and the two formulas in part (d) are off by a factor of two. Replace the right side of (c) by  $2/\pi$ ; the first in (d) by  $2\sqrt{2}/\pi$ , and the second by  $\sqrt{2}/\pi$ . (March 19 2011; August 5 2012)

Page 209: Our discussion of the period-doubling sequence should have mentioned (a) M. Morse and G. A. Hedlund, Symbolic dynamics, *Amer. J. Mathematics* **60** (1938), 815–866, where it appears on p. 844 after the words “Upon reducing...” and (b) M. Garcia and G. A. Hedlund, The structure of minimal sets, *Bull. Amer. Math. Soc.* **54** (1948), 954–964, where it appears under the name  $\beta$  on p. 962.

Page 218: In line  $-12$ , replace “ $h^t(x)$ ” with “ $h^t(y)$ ”. (Kalle Saari, September 5 2006)

Page 220: Delete line 11 (the one that begins “The following theorem is the first...”). (Kalle Saari, September 5 2006)

Page 230: In line 6, replace “ $w_1w_2\dots w_r$ ” with “ $b_1b_2\dots b_r$ ”. (Kalle Saari, September 5 2006)

Page 244: In exercise 23, replace “nonerasing morphism” with “nontrivial nonerasing morphism”. (Tim Smith, November 27 2012)

Page 248: In line 7, replace the period after  $0 \rightarrow 1$  with comma. (Kalle Saari, September 5 2006)

Page 254: The left side of the equation in line 2 is missing the term  $|\lambda^{k-1}|$ . (Kalle Saari, September 5 2006)

Page 262: The second part of the statement of Theorem 8.3.12 (b) is missing an hypothesis. In order to conclude that  $\lim_{n \rightarrow \infty} M^n/r^n$  has rational entries, add the extra hypothesis that  $r$  is rational. (D. Krieger, March 10 2006)

Page 262: In line  $-7$ , replace  $\lim_{n \rightarrow \infty}$  with  $\lim_{k \rightarrow \infty}$ . (Kalle Saari, September 5 2006)

Page 265: In line  $-7$ , add 1 to the exponents of the two rightmost entries on the second row of the matrix  $M_k^n$ . (Kalle Saari, September 5 2006)

Page 267: In line 13, replace  $2 \cdot 3^\ell$  in the subscript of the summation with  $2 \cdot 3^\ell$ . In line  $-2$ , replace “The first sum” with “As  $n \rightarrow \infty$ , the first sum”. (Kalle Saari, September 5 2006)

Page 268: In Proposition 8.4.4 (b), the displayed equation should be  $\lim_{s \rightarrow 1^+} (s-1) \sum_{n \geq 1; x_n = a} \frac{1}{n^s}$ . (Johannes Morgenbesser, August 2 2010)

Page 268: The proof of Theorem 8.4.5 should have been divided more clearly into parts (a) and (b) (to follow the statement). (Kalle Saari, September 5 2006)

Page 286: In line  $-11$ , replace  $f_\alpha$  with  $\mathbf{f}_\alpha$ . (Moshe Schwartz, June 12 2016)

Page 300: The definition for  $T_x$  should have an intersection, not a union, and the comment about the union being non-disjoint should be removed. (Michel Rigo, March 9 2005)

Page 303: At the end of line 5, delete  $\leq k$ . (JOS, February 25 2017)

Page 303: Lemma 10.2.7 is known in the literature as Fekete’s lemma; we should have cited M. Fekete, Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten, *Math. Zeitschrift* **17** (1923), 228–249.

Page 306: In line  $-18$ , the equation  $o_n + o_{n+1} = 4n + 4$  holds for  $n \geq 2$ , not  $n \geq 1$ . (Kalle Saari, September 5 2006)

Page 309: In line 6, replace “ $u_1 \geq 2$ ” with “ $|u_1| \geq 2$ ”. (Kalle Saari, September 5 2006)

Page 309: In line  $-12$ , replace “Exercise 35” by “Exercise 34” twice. (Narad Rampersad, February 22 2007)

Page 311: The proof of Theorem 10.4.12 as given is incorrect. (The problem is that  $j$  is not uniquely specified for all words of length  $n$ .) It should be replaced with the following simpler proof.

Revised proof of Theorem 10.4.12:

**Proof.** Let  $i$  be the least index such that

$$\min_{a \in \Sigma} |h^{i-1}(a)| \leq n \leq \min_{a \in \Sigma} |h^i(a)|.$$

Since  $\mathbf{u} = h(\mathbf{u})$ , it follows that  $\mathbf{u} = h^i(\mathbf{u})$ . Write  $\mathbf{u} = u_0 u_1 u_2 \dots$ ; then  $\mathbf{u} = h^i(u_0) h^i(u_1) \dots$ . Let  $w$  be a word of length  $n$ . Since each block  $h^i(u_j)$  is of length at least  $n$ , it follows that either  $w$  is contained entirely in such a block, or it straddles exactly two such blocks. Thus every word of length  $n$  is contained in a word of the form  $h^i(bc)$  where  $b, c \in \Sigma$ . Hence any such word  $w$  of length  $n$  is uniquely specified by giving  $b, c$  and the position within  $h^i(bc)$  where  $w$  starts. Using Theorem 1.4.3, it follows that there exists an  $e$ , depending only on  $h$ , such that  $|h^i(bc)| \leq 2W^{2e} \min_{a \in \Sigma} |h^{i-1}(a)| \leq 2W^{2e}n$ , where  $W = \text{Width}(h)$ . Hence  $p_{\mathbf{u}}(n) \leq 2k^2 W^{2e}n$ .

Page 319: In line  $-6$ , replace  $\rho$  with  $\alpha$ . Same typo in line  $-2$ . (Kalle Saari, September 5 2006)

Page 327: In Example 10.8.10, the orbit closure should also include the word  $0^\omega$ . (JOS, November 16 2011)

Page 329: In the proof of Theorem 10.9.2, part (a), the very last inequality should read  $R_x(n+1) \geq t+1$ . (Michel Rigo, March 9 2005)

Also, we should have said that we would write  $R$  in place of  $R_t$  for brevity. (Kalle Saari, September 5 2006)

Page 331: In line 7, the statement beginning “It follows that” is not true. A fix will be forthcoming. (Kalle Saari, September 5 2006)

Page 333: In line 8 replace “such for” with “such that for”. (Kalle Saari, September 5 2006)

Page 334: In line  $-18$ , replace “for  $\geq 2$ ” with “for  $n \geq 2$ ”. (Kalle Saari, September 5 2006)

Page 335: In the 4th line of Exercise 10, replace  $2^a + 2^{a-1} - b$  with  $2^a + 2^{a-1} + b$ . (Narad Rampersad, July 10 2006)

Page 339: In Exercise 42, replace “ $i \geq 1$ ” with “ $i \geq 0$ ”. (Joel Noche, December 6 2011)

Page 346: Lemma 11.1.2 is incorrect. It needs to be replaced by a different result. For a corrected proof, see M. Rigo and L. Waxweiler, A note on syndeticity, recognizable sets, and Cobham’s theorem, *Bull. of the EATCS*, No. 88 (February 2006), 169–173. This is also available at <http://www.discmath.ulg.ac.be/papers/coblw.pdf>. Also see <http://www.tucs.fi/publications/attachment.php?fname=TR713.pdf>. (Michel Rigo, April 11 2005; JPA, July 15 2006)

Page 347: In the last line on this page, replace  $\mathbb{Z}$  with  $\mathbb{N}$ . (JOS, February 26 2017)

Page 348: Line 19, replace  $0 \leq y < l^j$  with  $0 \leq t < l^j$ . (JOS, February 26 2017)

Page 350: In Theorem 11.2.2, change the mention of  $\mathbf{u}$  in the last line to  $\mathbf{s}$ . (Brent Bostick, June 29 2011)

Pages 354–359, Section 12.2. As presented, the proof of Christol’s theorem is very slightly incorrect. The problem is that in the claim “ $G = A(X)/B_0(X)$ ” on page 358, the resulting  $G$  is only guaranteed to be a formal Laurent series and not a formal power series. This was pointed out to us by Joost Winter in April 2015.

One way to fix this is as follows:

In Definition 12.2.1, define  $\Lambda_r$  from  $GF(q)((X))$  to  $GF(q)((X))$ .

In Lemma 12.2.2, replace  $i \geq 0$  by  $i$  (i.e.,  $i \in \mathbb{Z}$ ) in the statement, as well as three times in the part of the proof on the same page. In the second part of the proof next page, replace every  $j \geq 0$  by  $j$  and  $k \geq 0$  by  $k$ ; also every  $i \geq 0$  by  $i$ .

On page 358, note that  $G$  belongs (a priori) to  $GF(q)((X))$ . In the definition of  $\mathcal{H}$ , replace  $GF(q)[[X]]$  by  $GF(q)((X))$ .

Now at the very end of the proof we still invoke Lemma 12.2.4, but only after having noted that  $A$  belongs to  $\mathcal{H}$  and is an element of  $GF(q)[[X]]$ , so are all the elements of its orbit under all the  $\Lambda$ 's and their compositions (the semigroup generated by the  $\Lambda$ 's), so that everything we are now interested in belongs both to  $\mathcal{H}$  and  $GF(q)[[X]]$ .

Another way to do this is to use the alternative solution of Joost Winter, available at [christol.pdf](#).

Page 355: In Lemma 12.2.3, the list of polynomials on line 3 should include  $B_0(X)$ . (JOS, January 24 2017)

Page 361: the last line on the page should read “it suffices to prove that the sequence  $(c(m) \bmod p)_{m \geq 1}$  is not  $q$ -automatic” (JOS, January 23 2017).

Page 371: Insert “(recall that  $Q(0, 0) \neq 0$ )” on the first line of page 371 just before “and so is zero”. (Jean Berstel, January 31 2008)

Page 381: In line 6, replace  $e_t = b - 1$  with  $e_t \geq b - 1$ . In line 14, replace  $e_n \neq b - 1$  with  $e_n < b - 1$ . In line -4, replace  $\tau(q) \neq b - 1$  with  $\tau(q) < b - 1$ . (Dan Roche, November 2006)

Page 382: In line 2, replace  $e_n = b - 1$  with  $e_n \geq b - 1$ . (Dan Roche, November 2006)

Page 395: In line -1, replace  $1/q_{|U_k V_k|^2}$  with  $1/q_{|U_k V_k|}^2$ . (Amy Glen, February 19 2004)

Page 396: In the statement of Proposition 13.7.4, part (b) the limit should read  $\lim_{k \rightarrow \infty}$ . (M. Mendès France, November 21 2003)

Page 398: In lines 4 and 7, replace the limit  $n \rightarrow \infty$  with  $k \rightarrow \infty$ . (Joel Noche, June 9 2009)

Page 399: Replace  $3 \cdot 2^k$  with  $3 \cdot 2^k$  twice in line -4 and one time in line -8. (Hanna Uscka-Wehlou, November 22 2008)

Page 404: In line -17, replace  $k$  in the subscript of the product with  $n$ . (Kalle Saari, September 5 2006)

Page 409: In lines 4,6, and 8 replace  $u_{n,m}$  with  $u_{m,n}$ . (Eric Rowland, December 20 2011)

Page 415: In line 7, remove the extra “can” in the sentence. (Kalle Saari, September 5 2006)

Page 425: In line 4, replace “an squarefree” with “a squarefree”. (Joel Noche, June 8 2009)

Page 425: Open Problem 14.8.2 was already solved by A. Carpi, Multidimensional unrepentive configurations, *Theoret. Comput. Sci.* **56** (1988), 233–241. (Also, the problem should impose the restriction  $\gcd(c, e) = 1$ .) (N. Rampersad, May 29 2006)

Page 432: In line 11, replace “a DFA” with “an NFA”. (Kalle Saari, September 5 2006)

Page 441: Replace “theorem” with “section” in the first line of section 16.2. (JOS, March 16 2006)

Page 442: In the statement of Theorem 16.2.3 (b), change  $\Sigma^k$  to  $\Sigma_k$ . (Eric Rowland, August 29 2015)

Page 443: In the second line of Theorem 16.2.5, change  $\Sigma^k$  to  $\Sigma_k$ . (Eric Rowland, August 29 2015)

Page 444: In line 11, replace  $e = \max_{0 \leq i < t}$  with  $e = \max_{0 \leq i < k}$ . (Eric Rowland, April 17 2010)

Pages 452–453: Exercise 35 should specify that you are counting partitions into *distinct* Fibonacci number parts. (Eric Rowland, May 23 2012)

Page 453: Exercise 36 duplicates Exercise 27(a) on Page 451. (Eric Rowland, December 20 2011)

Pages 476–477: The exercise numbers given for the solutions in A.10 Ch. 10 are shifted: 6, 7, 10, 12 are ok, 22 should be 21, and from 28 to 52 on,  $n$  should be replaced by  $n - 1$ . (JPA, April 26 2008)

Page 478: In the hint for Exercise 32 of Chapter 16, we should have cited the solution to Problem 10906 in *Amer. Math. Monthly* **110** (2003), 642–643. (K. Stolarsky, July 15 2005)

Page 495: In the title of Büchi [1960], replace “secord” with “second”. (Eric Rowland, May 16 2014).

Page 503: In line 2, remove the word “checked”. (Joel Noche, June 8 2009)

Page 528: The reference [de Luca 1990] should read A. de Luca, On the Burnside problem for semigroups, in *Mots: Mélanges Offerts à M.-P. Schützenberger*, Hermès, 1990, ed. M. Lothaire, pp. 185-200. (M.-w. Wang, March 18 2004)