# The Tower of Hanoi and Finite Automata

Jean-Paul Allouche and J. Shallit

**Abstract** Some of the algorithms for solving the Tower of Hanoi puzzle can be applied "with eyes closed" or "without memory". Here we survey the solution for the classical tower of Hanoi that uses finite automata, as well as some variations on the original puzzle. In passing, we obtain a new result on morphisms generating the classical and the lazy tower of Hanoi, and a new result on auomatic sequences.

## 1 Introduction

A huge literature in mathematics or theoretical computer science deals with the tower of Hanoi and generalizations. The reader can look at the references given in the bibliography of the present paper, but also at the papers cited in these references (in particular in [5, 13]). A very large bibliography was written by Stockmeyer [27]. Here we present a survey of the relations between the tower of Hanoi and monoid morphisms or finite automata. We also give a new result on morphisms generating the classical and the lazy tower of Hanoi (Theorem 4), and a new result on automatic sequences (Theorem 5).

Recall that the tower of Hanoi puzzle has three pegs, labeled I, II, III, and $N$ disks of radii $1, 2, \ldots, N$. At the beginning the disks are placed on peg I, in decreasing order of size (the smallest disk on top). A move consists of taking the topmost disk from one peg and moving it to another peg, with the condition that no disk should cover a smaller one. The purpose is to transfer all disks from the initial peg to another one (where they are thus in decreasing order as well).

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The usual (*recursive*) approach for solving the Hanoi puzzle consists in noting that, in order to move $(N + 1)$ disks from a peg to another, it is necessary and sufficient first to move the smallest $N$ disks to the third peg, then to move the largest disk to the now empty peg, and finally to transfer the smallest $N$ disks on that third peg. An easy induction shows that the number of moves for $N$ disks is thus $2^N - 1$ and that it is optimal.

Applying this recursive algorithm with a small number of disks (try it with 3 disks), shows that it transfers 1 disk (the smallest) from peg I to peg II; then continuing the process, the sub-tower consisting of the disks of radii 1 and 2 will be reconstructed on peg III; and the sub-tower consisting of the disks of radii 1, 2 and 3 will be reconstructed on peg II. More generally, let $\mathscr{S}_N$ be the sequence of moves that transfers the tower with the smallest $N$ disks from peg I to peg II if $N$ is odd, and from peg I to peg III if $N$ is even. Then, for any positive integer $k \leq N$, the sequence $\mathscr{S}_N$ begins with the sequence $\mathscr{S}_k$. In other words, there exists an infinite sequence of moves $\mathscr{S}_\infty$, such that, for any integer $N$, the first $(2^N - 1)$ moves of $\mathscr{S}_\infty$ solve the Hanoi puzzle by moving the tower of $N$ disks from peg I to peg II or III according to whether $N$ is odd or even.

Let us denote by $a$, $b$, $c$ the moves that take the topmost disk from peg I to peg II, resp. from peg II to peg III, resp. from peg III to peg I. Let $\overline{a}$, $\overline{b}$, $\overline{c}$ be the inverse moves (e.g., $\overline{c}$ moves the topmost disk from peg I to peg III). Then, as the reader can easily check

$$\mathscr{S}_\infty = a\,\overline{c}\,b\,a\,c\,\overline{b}\,a\,\overline{c}\,b\,\overline{a}\,c\,b\,a\,\overline{c}\,b\,a\,\cdots$$

Before studying this infinite sequence, we note that applying this algorithm leads to the discovery (and to the proof) of a "simpler" algorithm for the puzzle's solution, where

- the first, third, fifth, etc., moves only concern the smallest disk, which moves circularly from peg I to peg II, from peg II, to peg III, from peg III to peg I, and so forth;
- the second, fourth, sixth, etc., moves leave the smallest disk fixed on its peg. Hence, they consist in looking at the topmost disk of each of the two other pegs, and in moving the smaller to cover the larger.
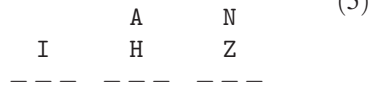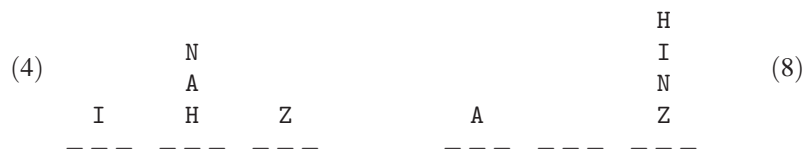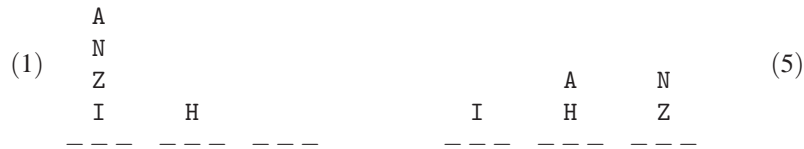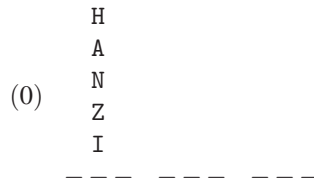
*Remark 1*. We note that this "simpler" algorithm cannot be performed "without memory" nor "with eyes closed" (i.e., without looking at the pegs): namely at the even steps, we need to know the sizes of the topmost disks and compare them. The next section addresses the question of finding an algorithm that can be applied "with bounded memory" and "with eyes closed".

The "simpler" algorithm where the smallest disk moves circularly every second move is attributed to Raoul Olive, the nephew of Édouard Lucas in [20]. Also, reconstructing the tower of Hanoi on peg II or peg III according to the parity of the number of disks can be seen as a "dual" of the strategy of R. Olive, where the smallest disk moves circularly either clockwise or counter-clockwise, according to the parity of the number of disks and the desired final peg where the tower of Hanoi is reconstructed.

*Remark 2.* Several variations on this game have been introduced: the cyclic tower of Hanoi (wherein, using the notation above, only the moves $a$, $b$, and $c$ are allowed), the lazy tower of Hanoi (using only the moves $a$, $\overline{a}$, $b$, $\overline{b}$), the colored tower of Hanoi, Antwerpen towers, $d$ pegs instead of 3 pegs, etc. There are also variations studied in cognitive psychology: the tower of Hanoi itself [25], the towers of London [24], and the towers of Toronto [21]. We do not resist to propose a modest contribution to the world of variations on the tower of Hanoi, in honor of the organizers of the Symposium "La ≪ Tour d'Hanoï ≫, un casse-tête mathématique d'Édouard Lucas (1842-1891)". Start with three pegs, and disks indexed by a given word on the usual Latin alphabet. Move as usual the topmost disk from a peg to another, the rule being that no two consecutive vowels should appear. Start from HANOI. Well, O and I are already consecutive. Let us say that "O = oh = Zero = Z", and let us thus replace HANOI with HANZI. Here are a few permitted moves: starting with HANZI, we get successively

```
                            H
                            A
                            N
        (0)                 Z
                            I
                           ___  ___  ___


          A
          N
 (1)      Z                                           (5)
          I      H                      A        N
                               I        H        Z
         ___  ___  ___                 ___  ___  ___


          N                                     I
 (2)      Z        A                            N     (6)
          I        H                   A        Z
                                       H
         ___  ___  ___                 ___  ___  ___


                   N                            I
 (3)      Z        A                            N     (7)
          I        H                            Z
                              A        H
         ___  ___  ___                 ___  ___  ___


                                                H
                   N                            I
 (4)                                            N     (8)
                   A                            Z
          I        H        Z          A
         ___  ___  ___                 ___  ___  ___
```
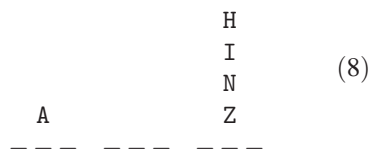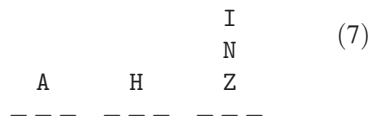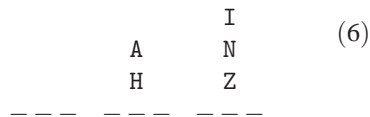
*Exercise: concoct two variations giving respectively*

```
H                D              H                G
A                A              A                A
N   ⟶ ... ⟶     N      and     N   ⟶ ... ⟶     U
O                E              O                Z
I                K              I                Y
```

## 2 The infinite sequence of moves $\mathscr{S}_\infty$ and an easy way of generating it

We will focus on the infinite sequence $\mathscr{S}_\infty$ defined above. It is a sequence on six symbols, i.e., a sequence over the 6-letter alphabet $\{a, b, c, \overline{a}, \overline{b}, \overline{c}\}$). This sequence can also be seen as a sequence of moves that "tries" to reconstruct a tower of Hanoi with infinitely many disks, by reconstructing the sub-towers with the smallest $N$ disks for $N = 1, 2, 3, \ldots$ on peg II or III acording to the parity of $N$.

Group the letters of $\mathscr{S}_\infty$ pairwise, and write this sequence of pairs of letters just under the sequence $\mathscr{S}_\infty$:

$$
\begin{array}{cccccccc}
a & \overline{c} & b & a & c & \overline{b} & a & \overline{c} \cdots \\
(a\,\overline{c}) & (b\,a) & (c\,\overline{b}) & (a\,\overline{c}) & (b\,\overline{a}) & (c\,b) & (a\,\overline{c}) & (ba) \cdots
\end{array}
$$

We observe that under any of the six letters we always find the same pair of letters, e.g., there always is an $(a\overline{c})$ under an $a$. More precisely, if we associate a 2-letter word with each letter in $\{a, b, c, \overline{a}, \overline{b}, \overline{c}\}$ as follows

$$
\begin{aligned}
a &\longrightarrow a\overline{c} & \overline{a} &\longrightarrow ac \\
b &\longrightarrow c\overline{b} & \overline{b} &\longrightarrow cb \\
c &\longrightarrow b\overline{a} & \overline{c} &\longrightarrow ba
\end{aligned}
$$

we can obtain the infinite sequence $\mathscr{S}_\infty$ by starting with $a$ and iterating the map above, where the image of a word is obtained by "gluing" (*concatenating*) together the images of letters of that word:

$$
a \longrightarrow a\,\overline{c} \longrightarrow a\,\overline{c}\,b\,a \longrightarrow a\,\overline{c}\,b\,a\,c\,\overline{b}\,a\,\overline{c} \longrightarrow \cdots
$$

This result was proven in [4]. We give more details below.

**Definition 1.** Let $\mathscr{A}$ be an *alphabet*, i.e., a finite set. A *word* on $\mathscr{A}$ is a finite sequence of symbols from $\mathscr{A}$ (possibly empty). The set of all words on $\mathscr{A}$ is denoted by $\mathscr{A}^*$. The *length* of a word is the number of symbols that it contains (the length of the empty word $\varepsilon$ is 0). The *concatenation* of two words $a_1 a_2 \cdots a_r$ and $b_1 b_2 \cdots b_s$ of lengths $r$ and $s$, respectively, is the word $a_1 a_2 \cdots a_r b_1 b_2 \cdots b_s$ of length $r + s$ obtained

by gluing them in order. The set $\mathscr{A}^*$ equipped with concatenation is called the *free monoid* generated by $\mathscr{A}$.

A sequence of words $u_\ell$ of $\mathscr{A}^*$ is said to *converge* to the infinite sequence $(a_n)_{n\geq 0}$ on $\mathscr{A}$ if the length of the largest prefix of $u_\ell$ that coincides with the prefix of $(a_n)_{n\geq 0}$ of the same length tends to infinity with $\ell$.

*Remark 3.* It is easy to see that $\mathscr{A}^*$ equipped with concatenation is indeed a monoid: concatenation is associative, and the empty word $\varepsilon$ is the identity element. This monoid is *free*: this means intuitively that there are no relations between elements other than the relations arising from the associative property and the fact that the empty word is the identity element. In particular this monoid is not *commutative* if $\mathscr{A}$ has at least two distinct elements.

**Definition 2.** Let $\mathscr{A}$ and $\mathscr{B}$ be two alphabets. A *morphism* from $\mathscr{A}^*$ to $\mathscr{B}^*$ is a map $\varphi$ from $\mathscr{A}^*$ to $\mathscr{B}^*$, such that, for any two words $u$ and $v$, one has $\varphi(uv) = \varphi(u)\varphi(v)$. A morphism of $\mathscr{A}^*$ is a morphism from $\mathscr{A}^*$ to itself.

If there exists a positive integer $k$ such that $\varphi(a)$ has length $k$, the morphism $\varphi$ is said to be *k-uniform*.

*Remark 4.* A morphism $\varphi$ from $\mathscr{A}^*$ to $\mathscr{B}^*$ is determined by the values of $\varphi(a)$ for $a \in \mathscr{A}$. Namely, if the word $u$ is equal to $a_1 a_2 \cdots a_n$ with $a_j \in \mathscr{A}$, then $\varphi(u) = \varphi(a_1)\varphi(a_2)\cdots\varphi(a_n)$.

**Definition 3.** An infinite sequence $(a_n)_{n\geq 0}$ taking values in the alphabet $\mathscr{A}$ is said to be *pure morphic* if there exist a morphism $\varphi$ of $\mathscr{A}^*$ and a word $x \in \mathscr{A}^*$ such that

- the word $\varphi(a_0)$ begins with $a_0$, (there exists a word $x$ such that $\varphi(a_0) = a_0 x$);
- iterating $\varphi$ starting from $x$ never gives the empty word (for each integer $\ell$, $\varphi^\ell(x) \neq \varepsilon$);
- the sequence of words $\varphi^\ell(a_0)$ converges to the sequence $(a_n)_{n\geq 0}$ when $\ell \to \infty$.

*Remark 5.* It is immediate that

$$
\begin{aligned}
\varphi(a_0) \ &= a_0 x \\
\varphi^2(a_0) &= \varphi(\varphi(a_0)) = \varphi(a_0 x) = \varphi(a_0)\varphi(x) = a_0 x \varphi(x) \\
\varphi^3(a_0) &= \varphi(\varphi^2(a_0)) = \varphi(a_0 x \varphi(x)) = \varphi(a_0)\varphi(x)\varphi^2(x) = a_0 x \varphi(x)\varphi^2(x) \\
&\phantom{=}\ \ \vdots \\
\varphi^\ell(a_0) &= a_0 x \varphi(x)\varphi^2(x)\cdots\varphi^{\ell-1}(x) \\
&\phantom{=}\ \ \vdots
\end{aligned}
$$

**Definition 4.** An infinite sequence $(a_n)_{n\geq 0}$ with values in the alphabet $\mathscr{A}$ is said to be *morphic* if there exist an alphabet $\mathscr{B}$ and an infinite sequence $(b_n)_{n\geq 0}$ on $\mathscr{B}$ such that

- the sequence $(b_n)_{n\geq 0}$ is pure morphic;
- there exists a 1-uniform morphism from $\mathscr{B}^*$ to $\mathscr{A}^*$ sending the sequence $(b_n)_{n\geq 0}$ to the sequence $(a_n)_{n\geq 0}$ (i.e., the sequence $(a_n)_{n\geq 0}$ is the pointwise image of $(b_n)_{n\geq 0}$).

If the morphism making $(b_n)_{n\geq 0}$ morphic is $k$-uniform, then the sequence $(a_n)_{n\geq 0}$ is said to be *k-automatic*. The word "automatic" comes from the fact that the sequence $(a_n)_{n\geq 0}$ can be generated by a finite automaton (see [8] for more details on this topic).

*Remark 6.* A morphism $\varphi$ of $\mathscr{A}^*$ can be extended to infinite sequences with values in $\mathscr{A}$ by defining $\varphi((a_n)_{n\geq 0}) = \varphi(a_0a_1a_2\cdots) := \varphi(a_0)\varphi(a_1)\varphi(a_2)\cdots$
It is easy to see that a pure morphic sequence is a *fixed point* of (the extension to infinite sequences of) some morphism: actually, with the notation above, it is *the* fixed point of $\varphi$ beginning with $a_0$. A pure morphic sequence is also called an *iterative fixed point* of some morphism (because of the construction of that fixed point), while a morphic sequence is the pointwise image of an iterative fixed point of some morphism, and a $k$-automatic sequence is the pointwise image of the iterative fixed point of a $k$-uniform morphism.

We can now state the following theorem [4, 5]:

**Theorem 1.** *The Hanoi sequence $\mathscr{S}_\infty$ is pure morphic. It is the iterative fixed point of the 2-morphism $\varphi$ on $\{a,b,c,\overline{a},\overline{b},\overline{c}\}^*$ defined by*

$$\varphi(a) := a\overline{c}, \;\; \varphi(b) := c\overline{b}, \;\; \varphi(c) := b\overline{a}, \;\; \varphi(\overline{a}) := ac, \;\; \varphi(\overline{b}) := cb, \;\; \varphi(\overline{c}) := ba.$$

*In particular the sequence $\mathscr{S}_\infty$ is 2-automatic.*

*Remark 7.* Using the automaton-based formulation of Theorem 1 above (see, e.g., [5]), it is possible to prove that the $j$th move in the algorithm for the optimal solution of the tower of Hanoi can be determined from the binary expansion of $j$, hence "with eyes closed" (i.e., without looking at the towers), and with bounded memory (the total needed memory is essentially remembering the morphism above, which does not depend on the number of disks).

It may also be worth noting that the tower of Hanoi sequence is *squarefree*; it contains no block of moves $w$ immediately followed by another occurrence of the same block [2]. Also note that this is not the case for all variations on this puzzle: for example the lazy tower of Hanoi sequence (see Remark 2 and Theorem 3) is not squarefree, since it begins with $a\, b\, a\, \overline{b}\, \overline{a}\, b\, a\, b\, a\cdots$.

## 3 Another "mechanical" way of generating the sequence of moves in $\mathscr{S}_\infty$

We begin with an informal definition (for more details the reader can look at [3] and the references therein). *Toeplitz sequences* or *Toeplitz transforms* of sequences are obtained by starting from a periodic sequence on an alphabet $\{a_1,a_2,\ldots,a_r,\diamond\}$, where $\diamond$ is a marked symbol called "hole". Then all the holes in the sequence are replaced in order by the terms of a periodic sequence with holes on the same alphabet (possibly the same sequence). The process is iterated. If none of the periodic

sequences used in the construction begins with a ⋄, the process converges and yields a *Toeplitz sequence*.

A classical example is the paperfolding sequence that results from folding a strip of paper on itself infinitely many times, and from looking at the pattern of up and down creases after unfolding (see, e.g., [8]). This sequence can also be constructed by a Toeplitz transform as follows. Start with the 3-letter alphabet $\{0,1,\diamond\}$. Take the sequence $(0 \diamond 1 \diamond)^\infty := 0 \diamond 1 \diamond 0 \diamond 1 \diamond 0 \cdots$. Replace the sub-sequence of diamonds by the sequence $(0 \diamond 1 \diamond)^\infty$ itself, and iterate the process. This yields successively

$$0 \diamond 1 \diamond 0 \diamond 1 \diamond 0 \diamond 1 \diamond 0 \diamond 1 \cdots = (0 \diamond 1 \diamond)^\infty$$
$$0\,0\,1 \diamond 0\,1\,1 \diamond 0\,0\,1 \diamond 0\,1\,1 \cdots = (0\,0\,1 \diamond 0\,1\,1 \diamond)^\infty$$
$$0\,0\,1\,0\,0\,1\,1 \diamond 0\,0\,1\,1\,0\,1\,1 \cdots = (0\,0\,1\,0\,0\,1\,1 \diamond 0\,0\,1\,1\,0\,1\,1 \diamond)^\infty$$
$$\vdots$$

After having applied this process an infinite number of times, there are no ⋄ left. The limit sequence is equal to the paperfolding sequence

$$0\,0\,1\,0\,0\,1\,1\,0\,0\,0\,1\,1\,0\,1\,1 \cdots.$$

The Hanoi sequence $\mathscr{S}_\infty$ can be constructed in a similar way. Take the 7-letter alphabet $\{a,b,c,\overline{a},\overline{b},\overline{c},\diamond\}$. Start with the sequence $(a\,\overline{c}\,b \diamond c\,\overline{b}\,a \diamond b\,\overline{a}\,c \diamond)^\infty$. Replace the sequence of holes by the sequence itself. Then iterate the process. This gives sequences with "fewer and fewer holes" and "more and more coinciding with" the Hanoi sequence $\mathscr{S}_\infty$, namely

$$a\,\overline{c}\,b \diamond c\,\overline{b}\,a \diamond b\,\overline{a}\,c \diamond a\,\overline{c}\,b \diamond c\,\overline{b}\,a \diamond b\,\overline{a}\,c \diamond a\,\overline{c}\,b \diamond c\,\overline{b}\,a \diamond b\,\overline{a}\,c \diamond \cdots$$
$$a\,\overline{c}\,b\,a\,c\,\overline{b}\,a\,\overline{c}\,b\,\overline{a}\,c\,b\,a\,\overline{c}\,b \diamond c\,\overline{b}\,a\,c\,b\,\overline{a}\,c\,\overline{b}\,a\,c\,b\,a\,\overline{c}\,b\,a \diamond b\,\overline{a}\,c\,b \cdots$$
$$a\,\overline{c}\,b\,a\,c\,\overline{b}\,a\,\overline{c}\,b\,\overline{a}\,c\,b\,a\,\overline{c}\,b\,a\,c\,b\,\overline{a}\,c\,\overline{b}\,a\,c\,b\,a\,\overline{c}\,b\,a\,c\,\overline{b}\,a\,\overline{c}\,b\,\overline{a}\,c\,b \cdots$$
$$\vdots$$

The following theorem was proved in [5].

**Theorem 2.** *The infinite Hanoi sequence $\mathscr{S}_\infty$ is equal to the Toeplitz transform obtained by starting from the sequence $(a\,\overline{c}\,b \diamond c\,\overline{b}\,a \diamond b\,\overline{a}\,c \diamond)^\infty$, replacing the ⋄ by the elements of the sequence itself, then iterating the process an infinite number of times.*

## 4 Classical sequences hidden behind the Hanoi sequence

Several classical sequences are linked to the Hanoi sequence. We will describe some of them in this section.

### 4.1 Period-doubling sequence

A binary sequence $\mathscr{T}$ can be deduced from $\mathscr{S}_\infty$ by replacing each of $a$, $b$, $c$ by 1 and each of $\overline{a}$, $\overline{b}$, $\overline{c}$ by 0 (i.e., the non-barred letters by 1 and the barred letters by 0), thus obtaining

$$\mathscr{S}_\infty = a \ \overline{c} \ b \ a \ c \ \overline{b} \ a \ \overline{c} \ b \ \overline{a} \ c \ b \ a \ \overline{c} \ b \ \cdots$$
$$\mathscr{T} \ = 1 \ 0 \ 1 \ 1 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 1 \ 1 \ 0 \ 1 \ \cdots$$

It is not difficult to prove [5] that $\mathscr{T}$ is the iterative fixed point of the morphism $\omega$ on $\{0,1\}^*$ defined by $\omega(1) := 10$, $\omega(0) := 11$. This iterative fixed point is known as the *period-doubling sequence*. It was introduced in the study of iterations of unimodal continuous functions in relation with Feigenbaum cascades (see, e.g., [8, pp. 176, 209]).

### 4.2 Double-free subsets

Define a sequence of integers $\mathscr{U}$ by counting for each term of $\mathscr{S}_\infty$ the cumulative number of the non-barred letters up to this term

$$\mathscr{S}_\infty = a \ \overline{c} \ b \ a \ c \ \overline{b} \ a \ \overline{c} \ b \ \overline{a} \ c \ b \ a \ \overline{c} \ b \ \ \cdots$$
$$\mathscr{U} \ = 1 \ 1 \ 2 \ 3 \ 4 \ 4 \ 5 \ 5 \ 6 \ 6 \ 7 \ 8 \ 9 \ 9 \ 10 \ \cdots$$

The sequence $\mathscr{U}$ is equal to the sequence of maximal sizes of a subset $S$ of $\{1,2,\ldots,n\}$ with the property that if $x$ is in $S$ then $2x$ is not. (The sequences $\mathscr{T}$ in Subsection 4.1 above and $\mathscr{U}$ are respectively called A035263 and A050292 in Sloane's *Encyclopedia* [26], where it is mentioned that A050292 is the summatory function of A035263).

### 4.3 Prouhet-Thue-Morse sequence

Reducing the sequence $\mathscr{U}$ in Subsection 4.2 modulo 2 (or, equivalently, taking the summatory function modulo 2 of the sequence $\mathscr{T}$ in Subsection 4.1 above) yields a sequence $\mathscr{V}$

$$\mathscr{U} \ = \ 1 \ 1 \ 2 \ 3 \ 4 \ 4 \ 5 \ 5 \ 6 \ 6 \ 7 \ 8 \ 9 \ 9 \ 10 \ \cdots$$
$$\mathscr{V} \ = \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ \ \cdots$$

which is the celebrated (Prouhet-)Thue-Morse sequence (see, e.g., [7]), deprived of its first 0. Recall that the Prouhet-Thue-Morse sequence can be defined as the unique iterative fixed point, beginning with 0, of the morphism defined by $0 \to 01$, $1 \to 10$.

### 4.4 Other classical sequences related to Hanoi

Other classical sequences are related to the tower of Hanoi or to its variations. We mention here the Sierpiński gasket [17], the Pascal triangle [14] (see also [16] and the references therein), the Stern diatomic sequence [16], but also the Stirling numbers of the second kind [18], the second order Eulerian numbers, the Lah numbers, and the Catalan numbers [19].

## 5 Variations on the tower of Hanoi and morphisms

As indicated in Remark 2, several variations on the tower of Hanoi can be found in the literature. We will first indicate a generalization of Theorem 1. Then we will give a new result on the classical tower of Hanoi and one of its avatars, and a new result on automatic sequences.

### 5.1 Tower of Hanoi with restricted moves

There are exactly five variations deduced "up to isomorphism" from the classical Hanoi puzzle by restricting the permitted moves, i.e., by forbidding some fixed subset of the set of moves $\{a, b, c, \overline{a}, \overline{b}, \overline{c}\}$; see [23]. The following result was proven in [6] (also see [4, 5] for the classical case and [5] for the cyclic case).

**Theorem 3.** *The five restricted tower of Hanoi problems give rise to infinite morphic sequences of moves, whose appropriate truncations describe the transfer of any given number of disks. Furthermore two of these infinite sequences are actually automatic sequences, namely the classical Hanoi sequence and the lazy Hanoi sequence, which are, respectively,* 2-*automatic and* 3-*automatic.*

We give below the examples of the cyclic tower of Hanoi and the lazy tower of Hanoi (defined above in Remark 2).

- There exists an infinite sequence over the alphabet $\{a, b, c\}$ that is the common limit of finite minimal sequences of moves for the *cyclic* tower of Hanoi that allow us to transfer $N$ disks from peg $I$ to peg $II$ or from peg $I$ to peg $III$. Furthermore this sequence is morphic: it is the image under the map (1-uniform morphism) $F : \{f, g, h, u, v, w\} \rightarrow \{a, b, c\}$ where $F(f) = F(w) := a$, $F(g) = F(u) := c$, $F(h) = F(v) := b$ of the iterative fixed point of the morphism $\psi$ defined on $\{f, g, h, u, v, w\}$ by

$$\psi(f) := fvf, \quad \psi(g) := gwg, \quad \psi(h) := huh,$$
$$\psi(u) := fg, \quad \psi(v) := gh, \quad \psi(w) := hf$$

- the *lazy* Hanoi sequence is the iterative fixed point beginning with $a$ of the morphism $\lambda$ defined on $\{a,b,\overline{a},\overline{b}\}^*$ by

$$\lambda(a) := a\,b\,a, \ \ \lambda(\overline{a}) := a\,b\,\overline{a}, \ \ \lambda(b) := \overline{b}\,\overline{a}\,b, \ \ \lambda(\overline{b}) := \overline{b}\,\overline{a}\,\overline{b}.$$

In particular it is 3-automatic.

## 5.2 New results

We say a sequence is *non-uniformly pure morphic* if it the iterative fixed point of a non-uniform morphism. We say that a sequence is *non-uniformly morphic* if it is the image (under a 1-uniform morphism) of a non-uniformly pure morphic sequence. For example, the famous Fibonacci sequence $abaababa\cdots$ is non-uniformly morphic, being generated by the morphism $a \to ab$, $b \to a$.

A sequence can be simultaneously automatic and non-uniformly morphic. An example is the sequence $\mathscr{L}$ formed by the lengths of the strings of 1's between two consecutive zeros in the Thue-Morse sequence $0\mathscr{V}$ (whose definition is recalled in Subsection 4.3).

$$\begin{aligned} 0\mathscr{V} &= 0\,1\,1\,0\,1\,0\,0\,1\,1\,0\,0\,1\,0\,1\,1\,0\,\cdots \\ &= 0\,(11)\,0\,(1)\,0\,(\ )\,0\,(11)\,0\,(\ )0\,(1)\,0\,(11)\,0\,\cdots \\ \mathscr{L} &= 2\,1\,0\,2\,0\,1\,2\,\cdots \end{aligned}$$

The sequence $\mathscr{L}$ is both [9] the iterative fixed point of the morphism

$$2 \to 210, \ \ 1 \to 20, \ \ 0 \to 1$$

and the image under the map $x \to x \bmod 3$ of the iterative fixed point of the 2-morphism

$$2 \to 21, \ \ 1 \to 02, \ \ 0 \to 04, \ \ 4 \to 20.$$

We have just seen that the five variations of the tower of Hanoi (with restricted moves) are morphic; two of them are actually automatic (namely the classical and the lazy tower of Hanoi), and one is not (the cyclic tower of Hanoi, see [1]). It is asked in [6] whether it is true that the other two variations are not $k$-automatic for any $k \geq 2$. Reversing that question in some sense, we could instead ask whether the classical Hanoi sequence and the lazy Hanoi sequence (which are, respectively, 2-automatic and 3-automatic) are *also* non-uniformly morphic. The following result seems to be new.

**Theorem 4.** *The classical Hanoi sequence is non-uniformly pure morphic: it is the iterative fixed point of the (non-uniform) morphism $\xi$ defined on $\{a,b,c,\overline{a},\overline{b},\overline{c}\}^*$ by*

$$\begin{aligned} \xi(a) &:= a\overline{c}b, \ \ \xi(b) := \overline{b}, \ \ \xi(c) := \overline{a}c, \\ \xi(\overline{a}) &:= acb, \ \ \xi(\overline{b}) := b, \ \ \xi(\overline{c}) := ac. \end{aligned}$$

*The lazy Hanoi sequence is non-uniformly pure morphic: it is the iterative fixed point of the (non-uniform) morphism $\eta$ defined on $\{a,b,\overline{a},\overline{b}\}^*$ by*

$$\eta(a) := a\, b\, a\, \overline{b}, \quad \eta(b) := \overline{a}\, b,$$
$$\eta(\overline{a}) := a\, b\, \overline{a}\, \overline{b}, \quad \eta(\overline{b}) := \overline{a}\, \overline{b}.$$

*Proof.* We begin with the classical Hanoi sequence $\mathscr{S}_\infty$. An easy computation shows the following five equalities (where $\varphi$ is the morphism defined in Theorem 1).

$$\xi(a\overline{c}b) = \varphi(a\overline{c}b),$$
$$\xi(ac\overline{b}) = \varphi(ac\overline{b}),$$
$$\xi(\overline{a}cb) = \varphi(\overline{a}cb),$$
$$\xi(acb) = \varphi(acb),$$
$$\xi(\overline{a}c\overline{b}) = \varphi(\overline{a}c\overline{b}).$$

Now, grouping the elements by triples, i.e., writing the sequence $\mathscr{S}_\infty$ as

$$\mathscr{S}_\infty = (a\,\overline{c}\,b)\,(a\,c\,\overline{b})\,(a\,\overline{c}\,b)\,(\overline{a}\,c\,b)\,(a\,\overline{c}\,b)\,\cdots,$$

we know that only the five triples $a\overline{c}b$, $ac\overline{b}$, $\overline{a}cb$, $\overline{a}cb$, $acb$, and $\overline{a}c\overline{b}$ occur (see [15, Theorem 2], where this result is used to construct a square-free sequence on a 5-letter alphabet, starting with the classical Hanoi sequence). Thus

$$\begin{aligned}
\xi(\mathscr{S}_\infty) &= \xi((a\,\overline{c}\,b)\,(a\,c\,\overline{b})\,(a\,\overline{c}\,b)\,(\overline{a}\,c\,b)\,(a\,\overline{c}\,b)\,\cdots) \\
&= \xi(a\,\overline{c}\,b)\,\xi(a\,c\,\overline{b})\,\xi(a\,\overline{c}\,b)\,\xi(\overline{a}\,c\,b)\,\xi(a\,\overline{c}\,b)\,\cdots \\
&= \varphi(a\,\overline{c}\,b)\,\varphi(a\,c\,\overline{b})\,\varphi(a\,\overline{c}\,b)\,\varphi(\overline{a}\,c\,b)\,\varphi(a\,\overline{c}\,b)\,\cdots \\
&= \varphi((a\,\overline{c}\,b)\,(a\,c\,\overline{b})\,(a\,\overline{c}\,b)\,(\overline{a}\,c\,b)\,(a\,\overline{c}\,b)\,\cdots) \\
&= \varphi(\mathscr{S}_\infty) = \mathscr{S}_\infty.
\end{aligned}$$

Now we look at the lazy Hanoi sequence. Using the morphism $\lambda$ defined in the second example following Theorem 3, we note that

$$\begin{aligned}
\eta(aba\overline{b}) &= \lambda(aba\overline{b}), \\
\eta(ab\overline{a}\overline{b}) &= \lambda(ab\overline{a}\overline{b}), \\
\eta(a\overline{b}\overline{a}b) &= \lambda(a\overline{b}\overline{a}b), \\
\eta(a\overline{b}\overline{a}\overline{b}) &= \lambda(a\overline{b}\overline{a}\overline{b}), \\
\eta(\overline{a}bab) &= \lambda(\overline{a}bab), \\
\eta(\overline{a}\overline{b}ab) &= \lambda(\overline{a}\overline{b}ab), \\
\eta(\overline{a}\overline{b}\overline{a}b) &= \lambda(\overline{a}\overline{b}\overline{a}b), \\
\eta(\overline{a}\overline{b}\overline{a}b) &= \lambda(\overline{a}\overline{b}\overline{a}b).
\end{aligned}$$

Grouping as above the elements of the lazy Hanoi sequence by quadruples, we can write that sequence, say $\mathscr{H}_\infty$ as

$$\begin{aligned}
\mathscr{H}_\infty &= a\,b\,a\,\overline{b}\,\overline{a}\,b\,a\,b\,a\,\overline{b}\,\overline{a}\,\overline{b}\,a\,b\,\cdots \\
&= (a\,b\,a\,\overline{b})\,(\overline{a}\,b\,a\,b)\,(a\,\overline{b}\,\overline{a}\,\overline{b})\,(a\,b\cdots
\end{aligned}$$

It is not hard to see that the 4-letter blocks (also called 4-*letter factors*) that can occur in this parenthesed version of $\mathscr{H}_\infty$ are among all 4-letter blocks of $\mathscr{H}_\infty$ beginning with $a$ or $\overline{a}$. These blocks are necessarily subblocks of images by $\lambda$ of 2-letter blocks of $\mathscr{H}_\infty$, i.e., of blocks $\{ab, a\overline{b}, \overline{a}b, \overline{a}\,\overline{b}, ba, b\overline{a}, \overline{b}a, \overline{b}\,\overline{a}\}$. Hence these 4-letter blocks are among $abab, ab\overline{a}b, \overline{a}bab, \overline{a}b\overline{a}b, \overline{a}bab, \overline{a}b\overline{a}b, \overline{a}\overline{b}\overline{a}b, \overline{a}b\overline{a}b$. We then have

$$
\begin{aligned}
\eta(\mathscr{H}_\infty) &= \eta(a\,b\,a\,\overline{b}\,\overline{a}\,b\,a\,b\,a\,\overline{b}\,\overline{a}\,\overline{b}\,a\,b\,a\,\overline{b}\cdots)\\
&= \eta((a\,b\,a\,\overline{b})\,(\overline{a}\,b\,a\,b)\,(a\,\overline{b}\,\overline{a}\,\overline{b})\,(a\,b\,a\,\overline{b})\cdots)\\
&= \eta(a\,b\,a\,\overline{b})\,\eta(\overline{a}\,b\,a\,b)\,\eta(a\,\overline{b}\,\overline{a}\,\overline{b})\,\eta(a\,b\,a\,\overline{b})\cdots\\
&= \lambda(a\,b\,a\,\overline{b})\,\lambda(\overline{a}\,b\,a\,b)\,\lambda(a\,\overline{b}\,\overline{a}\,\overline{b})\,\lambda(a\,b\,a\,\overline{b})\cdots\\
&= \lambda((a\,b\,a\,\overline{b})\,(\overline{a}\,b\,a\,b)\,(a\,\overline{b}\,\overline{a}\,\overline{b})\,(a\,b\,a\,\overline{b})\cdots)\\
&= \lambda(\mathscr{H}_\infty) = \mathscr{H}_\infty.
\end{aligned}
$$

$\square$

### 5.3 A general result on automatic sequences

In view of the previous subsection, it is natural to ask which non-uniformly morphic sequences are also $k$-automatic for some integer $k \geq 2$. Or, which automatic sequences are also non-uniformly morphic. We have just seen that the classical Hanoi sequence (hence the period-doubling sequence) and the lazy Hanoi sequence have this property. We prove here that *all* automatic sequences are also non-uniformly morphic.

**Theorem 5.** *Let* $(a_n)_{n\geq 0}$ *be an automatic sequence taking values in the alphabet* $\mathscr{A}$. *Then* $(a_n)_{n\geq 0}$ *is also non-uniformly morphic. Furthermore, if* $(a_n)_{n\geq 0}$ *is the iterative fixed point of a uniform morphism, then there exist an alphabet* $\mathscr{B}$ *of cardinality* $2 + \sharp\mathscr{A}$ *and a sequence* $(a'_n)_{n\geq 0}$ *with values in* $\mathscr{B}$, *such that* $(a'_n)_{n\geq 0}$ *is the iterative fixed point of some non-uniform morphism on* $\mathscr{B}^*$ *and* $(a_n)_{n\geq 0}$ *is the image of* $(a'_n)_{n\geq 0}$ *under a* 1-*uniform morphism.*

*Proof.* We prove the first assertion. Since the sequence $(a_n)_{n\geq 0}$ is the pointwise image of the iterative fixed point $(x_n)_{n\geq 0}$ of some uniform morphism, we may suppose, up to replacing $(a_n)_{n\geq 0}$ by $(x_n)_{n\geq 0}$ that $(a_n)_{n\geq 0}$ itself is the iterative fixed point beginning with $a_0$ of a uniform morphism $\gamma$ on $\mathscr{A}^*$. We may also suppose that the sequence $(a_n)_{n\geq 0}$ is not constant (otherwise the result is trivial). We claim that there exists a 2-letter word $bc$ such that $\gamma(bc)$ contains $bc$ as a factor. Namely, since $\gamma$ is uniform, it has exponential growth (i.e., iterating $\gamma$ on any letter gives words of length growing exponentially). Hence there exists a letter $b$ which is *expanding*, i.e., such that some power of $\gamma$ maps $b$ to a word that contains at least two occurrences of $b$ (see, e.g., [22]). Up to replacing $\gamma$ by this power of $\gamma$, we can write $\gamma(b) = ubvbw$ for some words $u, v, w$. Up to replacing this new $\gamma$ by $\gamma^2$, we can also suppose that $u$ and $w$ are not empty. Let $c$ be the letter following the prefix $ub$ of $ubvbw$. If $c \neq b$, then $v = cy$ for some word $y$, $\gamma(b) = ubcyw$ and $\gamma(bc) = \gamma(b)\gamma(c)$ contains $bc$ as a

factor; if $c = b$, then $\gamma(b) = ubbz$ for some word $z$, and $\gamma(bb) = ubbzubbz$ contains $bb$ as a factor. In any case, there exist two not necessarily distinct letters $b$ and $c$ such that $\gamma(b) = w_1 bc w_2$, $\gamma(bc) = w_1 bc w_3$, where $w_1, w_2$ are non-empty words. Note in particular that $b$ can be chosen distinct from $a_0$ ($w_1$ is non-empty).

Now, define a new alphabet $\mathscr{A}' := \mathscr{A} \cup \{b', c'\}$, where $b', c'$ are two new letters not in $\mathscr{A}$. Define the morphism $\gamma'$ on $\mathscr{A}'$ by: if the letter $y$ belongs to $\mathscr{A} \setminus \{b\}$, then $\gamma'(y) := \gamma(y)$. If $y = b$, define $\gamma'(b) := w_1 b' c' w_2$. Finally define $\gamma'(b')$ and $\gamma'(c')$ as follows: first recall that $\gamma(bc) = w_1 bc w_3$; cut the word $w_1 bc w_3$ into (any) two non-empty words of unequal length, say $w_1 bc w_3 := zt$, and define $\gamma'(b') := z$, $\gamma'(c') := t$. By construction, $\gamma'$ is not uniform. Its iterative fixed point beginning with $a_0$ clearly exists: we denote it by $(a'_n)_{n \geq 0}$. That sequence has the property that each $b'$ in it is followed by a $c'$. Let $\ell$ be *twice* the common length of the $\gamma(a_i)$. Write the initial sequence as $u_1 u_2 \cdots u_n \cdots$ where the $u_j$ are words of length $\ell$. Write the iterative fixed point of $\gamma'$ that begins with $a_0$ as $u'_1 u'_2 \cdots u'_n \cdots$, where the $u'_j$ have length $\ell$. Denote by $D$ the 1-uniform morphism that sends each letter of $\mathscr{A}$ on itself, and sends $b'$ on $b$ and $c'$ on $c$. For any letter $x$ belonging to $\mathscr{A}' \setminus \{b, b', c'\}$ we have $\gamma(x) = \gamma'(x)$, hence $D \circ \gamma'(x) = D \circ \gamma(x) = \gamma(x) = \gamma \circ D(x)$. For $x = b$, we have $D \circ \gamma'(b) = D(w_1 b' c' w_2) = w_1 bc w_2 = \gamma(b) = \gamma \circ D(b)$. Since $b'$ and $c'$ can occur in the sequence $(a'_n)_{n \geq 0}$ only as $b'c'$ "inside" the $u'_j$, and since $D \circ \gamma'(b'c') = D(w_1 bc w_3) = w_1 bc w_3 = \gamma(bc) = \gamma \circ D(b'c')$, we finally have that $D \circ \gamma'(u'_j) = \gamma \circ D(u'_j)$, for each word $u'_j$ (note that this is not true for *any* word on $\mathscr{A}'$, since $\gamma$ is uniform, while $\gamma'$ is not). Thus

$$D((a'_n)_{n \geq 0})) = D(\gamma'((a'_n)_{n \geq 0})) = D(\gamma'(u'_1 u'_2 \cdots)) = D(\gamma'((u'_1)(u'_2) \cdots))$$
$$= D(\gamma'(u'_1))D(\gamma'(u'_2)) \cdots = \gamma(D(u'_1))\gamma(D(u'_2)) \cdots = \gamma(u_1)\gamma(u_2) \cdots$$
$$= \gamma((u_1)(u_2) \cdots) = \gamma(u_1 u_2 \cdots) = \gamma((a_n)_{n \geq 0}) = (a_n)_{n \geq 0}$$

and we are done.
The second assertion is a consequence of the fact that we introduced only two new letters $b', c'$ in the proof above. $\square$

*Remark 8.* Our Theorem 4 is more precise than Theorem 5 for the classical and for the lazy tower of Hanoi, since the non-uniform morphisms involved are defined on the *same alphabet* as the corresponding uniform morphisms.

## 6 A little more on automatic sequences

Automatic sequences, such as the classical and the lazy Hanoi sequences, have numerous properties, in particular number-theoretical properties. We refer, e.g., to [8]. We give here a characteristic property of formal power series on a finite field in terms of automatic sequences Before doing this let us consider again the period-doubling sequence $\mathscr{T}$, i.e., the iterative fixed point of the morphism $\omega$ on $\{0, 1\}^*$ defined (see Section 4.1) by $\omega(1) := 10$, $\omega(0) := 11$. Let us identify $\{0, 1\}^*$ with the field of 2 elements, $\mathbb{F}_2$. The definitions of $\mathscr{T} = (t_n)_{n \geq 0}$ and of $\omega$ show that, for every $n \geq 0$, $t_{2n} = 1$, $t_{2n+1} = 1 + t_n$. Hence, denoting by $F$ the formal power series

$\sum t_n X^n$ in $\mathbb{F}_2[[X]]$, we have (remember we are in characteristic 2)

$$F := \sum_{n \geq 0} t_n X^n = \sum_{n \geq 0} t_{2n} X^{2n} + \sum_{n \geq 0} t_{2n+1} X^{2n+1} = \sum_{n \geq 0} X^{2n} + \sum_{n \geq 0} (1 + t_n) X^{2n+1}$$

$$= \frac{1+X}{1-X^2} + X \sum_{n \geq 0} t_n X^{2n} = \frac{1}{1-X} + X \left( \sum_{n \geq 0} t_n X^n \right)^2 = \frac{1}{1-X} + X F^2.$$

Remembering once more that we are in characteristic two, this can be written

$$X(1+X)F^2 + (1+X)F + 1 = 0$$

which means that $F$ is algebraic (at most quadratic – actually exactly quadratic since the sequence $\mathscr{T}$ is not ultimately periodic) on the field of rational functions $\mathbb{F}_2(X)$. This result is a particular case of a more general result that we give hereafter (see [10, 11]).

**Theorem 6.** *Let $\mathbb{F}_q$ be the finite field of cardinality $q$. Let $(a_n)_{n \geq 0}$ be a sequence on $\mathbb{F}_q$. Then, the formal power series $\sum a_n X^n$ is algebraic over the field of rational functions $\mathbb{F}_q(X)$ if and only if the sequence $(a_n)_{n \geq 0}$ is q-automatic.*

## 7 Conclusion

The tower of Hanoi and its variations have many mathematical properties. They also are used in cognitive psychology. It is interesting to note that psychologists, as well as mathematicians, are looking at the shortest path to reconstruct the tower on another peg. But, is really the shortest path the most interesting? If the answer is yes in terms of strategy for a puzzle, it is not clear that the answer is also yes for detecting all kinds of skills for a human being. During the Symposium "La ≪ Tour d'Hanoï ≫, un casse-tête mathématique d'Édouard Lucas (1842-1891)", one of the speakers said that finding the shortest path might not be the most interesting question. It reminded the first author (JPA) of a discussion he once had with the French composer M. Frémiot. After JPA presented the tower of Hanoi and an algorithmic solution to him, Frémiot composed a *Messe pour orgue à l'usage des paroisses* [12]. Interestingly enough, what he emphasized is the rule "no disk on a smaller one". His "Messe" used only that rule, without any attempt to reconstruct the tower with the smallest number of moves. In contrast to algorithms (robots?) that (try to) optimize a quantitative criterion, or to mathematicians who (try to) prove the optimality of a solution or study the set of all solutions, is it not the case that composers, and more generally, artists are interested in qualitatively (rather than quantitatively) exceptional elements of a given set, in "jewels" rather than in "generic" elements, in non-necessarily rational choices rather than in exhaustive studies or rigorous proofs...? René Char wrote "Le poète doit laisser des traces de son passage non des preuves. Seules les traces font rêver".

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