Angles and Lengths in Reconfigurations of Polygons and Polyhedra*

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Abstract. We explore the possibility of reconfiguring, or "morphing", one simple polygon to another, maintaining simplicity, and preserving some properties of angles and edge lengths. In linkage reconfiguration, edge lengths must be preserved. By contrast, a *monotone morph* preserves edge directions and changes edge lengths monotonically. A monotone morph is only possible for *parallel* polygons—ones with corresponding edges parallel. Our main results are that monotone morphs exist for parallel pairs of polygons that are: (1) convex; or (2) orthogonally convex. Our morphs either move vertices in straight lines, or change few edge lengths at once. On the negative side, we show that it is NP-hard to decide if two simple parallel orthogonal polygons have a monotone morph. We also establish which of these results extend to 3-dimensional polyhedra.

1 Introduction

In transforming one geometric object to another we often wish to preserve certain geometric properties. For example, it is possible to continuously reconfigure one planar triangulation to another combinatorially identical planar triangulation while maintaining simplicity [5, 13, 7]. More generally, one can "morph" between different straight line drawings of the same embedded graph while maintaining simplicity. There are unfortunately no known bounds on how edge lengths and angles might change during these morphs, and in fact the method of [13] explicitly shrinks edges to infinitesimals.

Studies have been done on transforming configurations while restricting the amount of change to angles and/or edge lengths. The most stringent requirement on edge lengths is that they not change at all, i.e., the source and target configuration have the same edge lengths. This is the problem of linkage reconfiguration, which can be done (while maintaining simplicity) only for a limited class of graphs: between any two simple chains in the plane with corresponding edges of the same length, there is a transformation that preserves simplicity and edge lengths [6, 12]. The result extends to cycles (or equivalently, polygons) but not to trees [2].

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This paper is about preserving simplicity and *angles*. We say that two combinatorially identical configurations of straight line segments and vertices in the plane are *parallel* if corresponding edges are parallel. Guibas, Hershberger, and Suri [10], and independently, Grenander et al. [9], prove that between any two parallel simple polygons in the plane there is a transformation that preserves simplicity and angles. Such a transformation—preserving simplicity and angles—will be called a *parallel morph*. Thomassen had proved earlier [13] that orthogonal polygons have parallel morphs. In papers related to this one we establish other classes of graphs that have parallel morphs: chains and trees drawn in any dimension [3], and orthogonal graphs drawn in the plane [4].

Preserving angles OR edge lengths should not mean ignoring the other measure. Linkage results preserve edge lengths and change angles nicely: for chains, the angles open to π, then close to their target values [6]; for convex polygons the angles change monotonically [1]. The situation for parallel morphs is not so good: the parallel morph between polygons given by Guibas et al. shrinks edges to infinitesimals. They morph via an intermediate "reduced form" polygon in which the longest and shortest edges have a length ratio that is a product of $\Theta(n)$ infinitesimals, n the size of the polygon. They note that even if each infinitesimal could be replaced by a constant, the length ratio would still be exponential in n. The algorithms in the other papers share this defect.

Our goal in this paper is to find parallel morphs that alter edge lengths as little as possible. The strongest possibility is a *monotone* morph—a parallel morph in which each edge changes length monotonically from its initial to its final value.

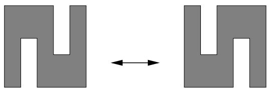

Fig. 1. Two polygons that do not admit a monotone morph.

Monotone morphs are often not possible—see Figure 1—and we show in Section 4 that it is NP-hard to decide if two parallel polygons have a monotone morph, even for the special case of orthogonal polygons, where there are only two possible edge directions.

As main results, we prove that monotone morphs do exist in two cases: (1) between any two parallel convex polygons (Section 3); and (2) between any two parallel orthogonally convex polygons (Section 5). Result (1) is easily obtained via the *linear* morph which moves all vertices at once at uniform speed in a straight line from source to target. The linear morph does not work in case (2), and, in any case, it is not always desirable to move everything at once. For both (1) and (2) we give "discrete" morphs that change few edge lengths at once—3 for the convex case, and 2 for the orthogonally convex case.

Result (1) is analogous—by exchanging the roles of angles and edge lengths—to the result [1] that there is a discrete morph between any two convex polygons with the same edge lengths that preserves edge lengths and changes angles monotonically. (Here "discrete" means changing few angles at once.) We note that these two results cannot be unified: it is not always possible to morph between a pair of convex polygons where both angles and edge-lengths change, but all angles and edge-lengths change monotonically.

Finally, we go up to 3 dimensions, and consider monotone morphs of parallel polyhedra. The linear morph still works for convex polyhedra, but discrete monotone morphs do not always exist. Result (2) does not extend either: we give an example of a pair of parallel orthogonally convex polyhedra with no monotone morph.

At the beginning of this section we mentioned the relationship of our work to the general topic of morphing. Further description of this background can be found in the full version of the paper. Our work is also related to "parallel redrawings" [14, 11] which are like parallel morphs except that simplicity need not be maintained. For more information see the full version of the paper.

2 Definitions

We define a *configuration* as a set of points in d-dimensional space and straight line segments between some pairs of those points. Two such configurations P, Q are *parallel* if there is an isomorphism between them with corresponding edges parallel, and endpoints in the same order with respect to the coordinate axes. If P and Q are parallel configurations then a connected set of edges that are co-planar in P will also be co-planar in Q. Thus, in particular, if P and Q are the (connected) edge sets of parallel 3-dimensional polyhedra then corresponding faces are parallel as well.

A *morph* is a transformation, or "metamorphosis" of one configuration to another. More technically, for a source configuration P and a target configuration Q, a *morph* from P to Q is a continuously changing family of configurations, $R(t), 0 \leq t \leq 1$ with $R(0) = P$ and $R(1) = Q$. A *parallel morph* between two simple parallel configurations is one where $R(t)$ is simple, and parallel to $R(0)$, for all t, $0 \leq t \leq 1$. A *monotone* morph is a parallel morph in which edge lengths change monotonically.

For an edge e, let $e(P)$ and $e(Q)$ denote the corresponding parallel edges in P and Q. We say that edge e is a $+$*-edge* of P if $||e(P)|| < ||e(Q)||$—i.e., if the length of e needs to increase during a monotone morph from P to Q. We define $-$*-edges* analogously. Any $+$-edge of P corresponds to a $-$-edge of Q and vice versa. We say that e is a 0-edge if it has the same length in P and Q—i.e., its length must stay unchanged during a monotone morph from P to Q.

If the parallel configurations form a simple cycle in the plane, we are thus morphing parallel polygons. An *orthogonal polygon* is a polygon such that each edge is horizontal or vertical. An orthogonal polygon is *orthogonally convex* if every horizontal and every vertical line intersects the interior of the polygon in a (possibly empty) connected line segment. An *orthogonal polyhedron* is a polyhedron such that every face is perpendicular to one of the three coordinate axes. An orthogonal polyhedron is *orthogonally convex* if every plane perpendicular to a coordinate axis intersects the polyhedron in a (possibly empty) connected, orthogonally convex polygon.

3 Morphing Convex Polyhedra

In this section we show that, in any dimension, any two parallel convex polyhedra admit a monotone morph. Our first morph changes the lengths of all non-0-edges simultaneously. In the second subsection we explore discrete-step morphing, where only a small number of edges change length in each step.

3.1 Linear morphing

Let P and Q be parallel configurations consisting of points and straight line segments between some pairs of those points. Suppose that P and Q are parallel configurations with corresponding points $p_1, \dots p_n$ and $q_1, \dots, q_n$ respectively. The *linear morph* between P and Q is given by the configurations $R(t), 0 \leq t \leq 1$ with vertices $r_1(t), \dots, r_n(t)$ defined by $r_i(t) = (1-t)p_i + tq_i$, and with edges corresponding to those of P (and Q).

From the definition, one easily verifies the following result:

Lemma 1. *For any two parallel configurations the linear morph is a monotone morph, except that it does not necessarily maintain simplicity.*

Though linear morphs do not always preserve simplicity, they do so for convex polyhedra.

Theorem 1. *For any two parallel convex polyhedra the linear morph is a monotone morph.*

Proof. Since P and Q are both convex and the morph is parallel, all intermediate polyhedra are also convex and hence simple.

3.2 Discrete-step morphing

The linear morph described in Section 3.1 changes all (non-static) edge lengths simultaneously. Below, we consider monotone morphing where only a small number of edges change in length at one time.

Discrete-step morphing in 2D

Theorem 2. *For any two parallel convex polygons P and Q with n vertices, there exists a monotone morph from P to Q that consists of at most n steps, each of which changes the lengths of at most three edges.*

Proof. Let P and Q be the source and target polygons, where the vertices of P are $p_0, \ldots, p_{n-1}, p_n = p_0$ and the corresponding vertices of Q are $q_0, \ldots, q_{n-1}, q_n = q_0$. For $i = 1, \ldots, n$, define $v_i = (q_i - q_{i-1}) - (p_i - p_{i-1})$. We morph by picking a small set of non-zero vectors v_i that have the 0-vector as a convex combination, and altering the lengths of the corresponding edges until one becomes a 0-edge. After n such steps all edges must be 0-edges. We begin by describing what to do with such a small set of vectors, and later argue existence.

Claim. If three non-0 vectors v_i, v_j, v_k have the 0-vector as a convex combination, then there exists a monotone morph that changes the lengths of only those three edges and converts at least one of them into a 0-edge.

Proof. (Sketch) Assume that $i < j < k$ and that $\lambda_i v_i + \lambda_j v_j + \lambda_k v_k = 0$, for some $\lambda_i, \lambda_j, \lambda_k \geq 0$ and $\lambda_i + \lambda_j + \lambda_k = 1$. For $t > 0$, define $P(t) = \langle p_1(t), \ldots, p_n(t) \rangle$ as follows (see also Figure 2):

1. $p_\ell(t) = p_\ell$ for $1 \leq \ell < i$.
2. $p_\ell(t) = p_\ell + t\lambda_i v_i$ for $i \leq \ell < j$.
3. $p_\ell(t) = p_\ell + t\lambda_i v_i + t\lambda_j v_j$ for $j \leq \ell < k$.
4. $p_\ell(t) = p_\ell + t\lambda_i v_i + t\lambda_j v_j + t\lambda_k v_k = p_\ell$ for $k \leq \ell \leq n$.

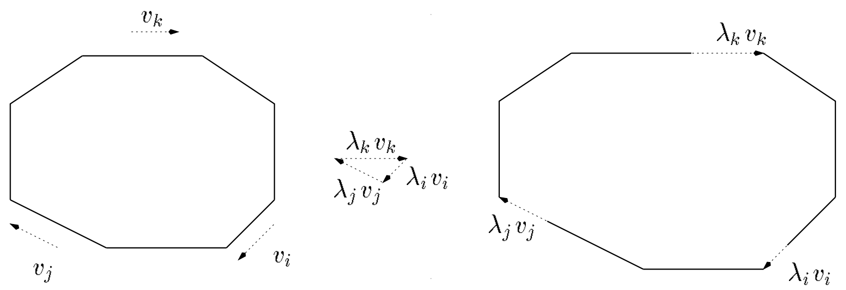

Fig. 2. Discrete-step monotone morphing for convex polygons. The target polygon Q is not shown.

One can verify that this is a monotone morph that keeps $P(t)$ parallel to P. Since the polygon was (and remains) convex, it remains simple, so we can increase t until one of the three changing edges becomes a 0-edge, which happens at $t = 1/\max\{\lambda_i, \lambda_j, \lambda_k\}$.

We note that a similar monotone morph exists if there are two non-0 vectors that have a convex combination to 0. It remains to show that one of these two situations occurs.

Claim. If the morph is not finished, i.e. not all edges are 0-edges, then there exist two or three non-0 vectors among $v_1, \ldots, v_n$ that have a convex combination to the 0-vector.

Proof. First observe that

$$\sum_{i=1}^{n} v_i = \sum_{i=1}^{n} (q_i - q_{i-1}) - (p_i - p_{i-1}) = \sum_{i=1}^{n} q_i - \sum_{i=0}^{n-1} q_i - \sum_{i=1}^{n} p_i + \sum_{i=0}^{n-1} p_i = 0,$$

since $q_0 = q_n$ and $p_0 = p_n$. Let $\{v'_1, \ldots, v'_m\} \subseteq \{v_1, \ldots, v_n\}$ be those vectors that are not the 0-vector; since P and Q are not identical there is at least one of them. Clearly $\sum_{i=1}^{m} v'_i = \sum_{i=1}^{n} v_i = 0$, since adding 0-vectors does not make a difference to the sum. Since the vectors v'_i have the 0-vector as a convex combination, thus, by Caratheodory's theorem (see e.g. [15]), there is a subset of at most three of these vectors that has the 0-vector as a convex combination.

Discrete-step morphing in 3D

Theorem 3. *A parallel morph of parallel convex polyhedra may need to change all edge lengths simultaneously.*

Proof. Let P be a polyhedron with all faces triangular, and let Q be a scaled-up version of P. Note that in any parallel morph of a triangle all three edge lengths must change simultaneously. Thus, by connectivity, any parallel morph from P to Q must change all edge lengths simultaneously.

4 The NP-Hardness of Monotone Morphing

Theorem 4. *Given parallel, orthogonal polygons P, Q (without holes) in the plane, it is NP-hard to decide whether P and Q admit a monotone morph.*

We only sketch the proof here, and give full details in the full version of the paper. The reduction is from 3-SAT, i.e., we are given n variables $x_1, \ldots, x_n$ and m clauses $c_1, \ldots, c_m$ that are conjunctions of three literals, and we want to know whether a satisfying boolean assignment exists.

We create two polygons P and Q that are almost identical except for a comb-like structure which has moved downward in Q. Also, some edges (marked with bold lines in Figure 3) are "elastic edges", which are capable of increasing and decreasing in length even in a monotone morph. (This is achieved by replacing each edge by two consecutive parallel edges.)

In order for the comb-like structure to move downward, the blocks that interleave with it have to move to the right temporarily. Each such block (or row of blocks) corresponds to a clause, and moving it to the right corresponds to satisfying the clause. A block can move to the right only if there is a gap in one of the blocks in its row (and elastic edges are set up such that any such gap can indeed be used to move the block to the right.) For each variable we have two columns of blocks. One column has flexibility to move up or down, and encodes whether the variable is set to true or false. (A surrounding frame prevents this column from moving too far.) The second column leaves gaps in each row depending on whether the variable or its negation is in the clause of this row. Thus, one can show that there is a monotone morph from P to Q if and only if the instance of 3-SAT is satisfiable.

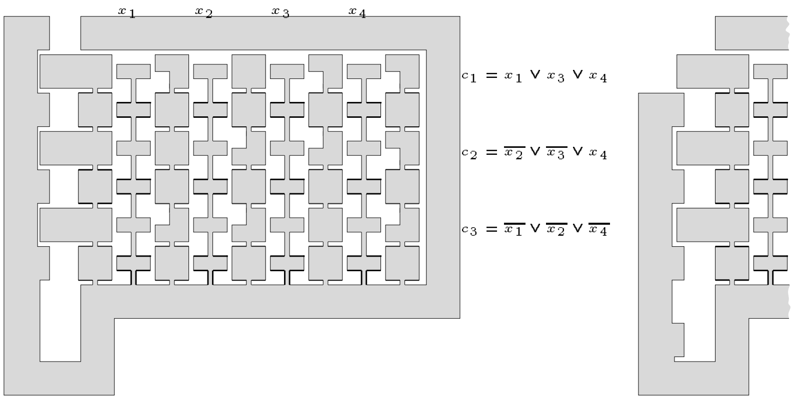

Fig. 3. The construction for the NP-hardness.

5 Orthogonally Convex Polygons and Polyhedra

In this section we show that there is a monotone morph between any two parallel orthogonally convex polygons, but that the analogous result for polyhedra fails.

5.1 Morphing Orthogonally Convex Polygons in 2D

Theorem 5. *For any two parallel orthogonally convex polygons P and Q there exists a monotone morph from P to Q. Furthermore, if P and Q have n vertices, there exists a morph that consists of at most n steps, each of which changes the lengths of two edges.*

Several times in this proof we use the technique of monotonically morphing P to an intermediate polygon P' and Q to an intermediate polygon Q' so that the changes in edge lengths match the labels, i.e., if e is a $+$-edge of P then $||e(P)|| \leq ||e(P')|| \leq ||e(Q')|| \leq ||e(Q)||$. This reduces the problem to that of finding a monotone morph from P' to Q', because such a morph, preceded by the morph from P to P', and followed by the reverse of the morph from Q to Q', provides a monotone morph from P to Q.

Applying this idea for the first time, we will morph the horizontal edges of P and Q, without changing the lengths of the vertical edges. Those are morphed later.

The edges of an orthogonally convex polygon can be partitioned into four *chains* by breaking the cycle of edges at the leftmost and rightmost vertical edges, and the uppermost and lowermost horizontal edges, as shown in Figure 4. We will refer to these chains as upper-left, upper-right, lower-left, and lower-right, in the obvious way. Each of the four extreme edges is considered to be part of two chains.

We morph to realize certain properties of these chains. In particular, a chain is *positive* [*negative*] if all horizontal edges in it are $+$-edges or 0-edges [$-$-edges or

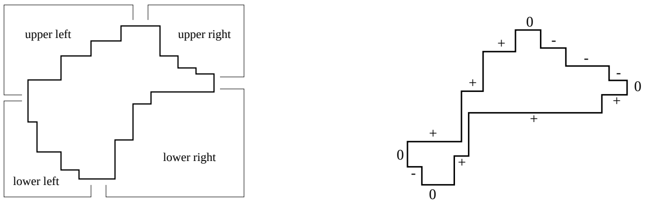

Fig. 4. The four chains, and a polygon in alternating form.

0-edges]; and a chain is *mixed* if it contains both +- and −-edges. A polygon is in *alternating form* if its chains are alternately positive and negative. See Figure 4. Our monotone morph has the following steps: (1) morph to eliminate mixed chains; (2) morph to put each polygon in alternating form; (3) morph so that all horizontal edges are 0-edges. The morphs in steps (1) and (2) (see Figure 5) strictly increase the polygon (by containment) and are thus simple; the morphs in step (3) (see Figure 6) need a more careful justification of simplicity.

Lemma 2. *Given a pair of parallel orthogonally convex polygons P and Q, we can morph P and Q monotonically to arrive at two intermediate polygons P' and Q' with no mixed chains.*

Proof. Without loss of generality consider the horizontal edges of the upper-left chain of P and Q. Suppose the chain has two edges e_1 and e_2, in that order, with opposite sign. In one of P or Q, e_1 must be a −-edge and e_2 a +-edge. Then the morph shown in Figure 5 is monotone and preserves simplicity, so it can be continued until either e_1 or e_2 becomes a 0-edge. Repeating this process yields the result.

Lemma 3. *Given a pair of parallel orthogonally convex polygons P and Q with no mixed chains, we can morph P and Q monotonically to arrive at two intermediate polygons P' and Q' in alternating form.*

Proof. Because we can morph in P or in Q (which has opposite signs) it suffices to show how to eliminate consecutive chains with +-edges. If both top chains of P contain +-edges then, because they are not mixed chains, the width of P must be less than the width of Q. In this case a lower chain of P must also have a +-edge. Thus we reduce to the case where, say, the two left chains of P contain +-edges. Then the morph shown in Figure 5 is monotone and preserves simplicity, so it can be continued until one edge becomes a 0-edge. Repeating this process yields the result.

Lemma 4. *Given a pair of parallel orthogonally convex polygons P and Q in alternating form, we can morph P and Q monotonically to arrive at two intermediate polygons P' and Q' whose horizontal edges are all 0-edges.*

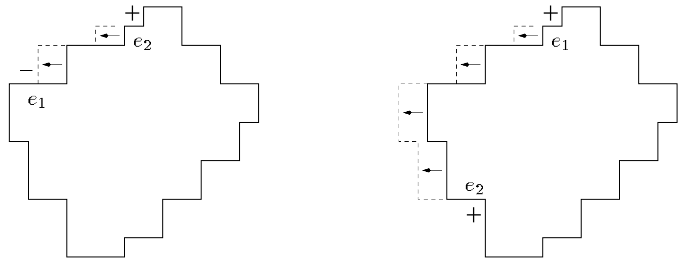

Fig. 5. Morphing to eliminate a mixed chain, and morphing to eliminate two positive left chains.

Proof. Let $l^+(P)$ be the horizontal length of P's left positive chain not counting the last (topmost) edge. Define $l^-(P)$, $l^+(Q)$, and $l^-(Q)$ for the other left chains analogously. Note that $l^+(P) \leq l^-(Q)$ and $l^+(Q) \leq l^-(P)$, since these are corresponding chains that must be morphed into each other.

Claim. If not all horizontal edges are 0-edges, then in at least one of P or Q we have $l^- > l^+$.

Proof. If $l^-(P) \leq l^+(P)$ and $l^-(Q) \leq l^+(Q)$ then

$$l^-(P) \leq l^+(P) \leq l^-(Q) \leq l^+(Q) \leq l^-(P)$$

so all are equal, and all left chains have only 0-edges. Since P and Q are in alternating form, this implies that the right chains have only 0-edges, too, and we are done.

Assume (by renaming if necessary) that $l^+(P) < l^-(P)$. Assume (by flipping upside down, if necessary) that the upper left chain of P is positive. The idea is to trade off a $+$-edge in P's upper left chain against either a $-$-edge in P's upper right chain, or a $+$-edge in P's lower right chain. See Figure 6. This maintains simplicity for as long as $l^+(P) \leq l^-(P)$. We morph in such a way until $l^+(P') = l^-(P')$. Similarly we morph Q if necessary until $l^+(Q') = l^-(Q')$; then by the claim all horizontal edges are 0-edges.

This completes the description of the monotone morph between two parallel orthogonally convex polygons. One verifies that each step changes the lengths of at most two edges. It remains to bound the number of steps, which we show to be at most $n/2$ for the horizontal edges (and hence n in total). With every step of our morph, we either convert a horizontal edge into a 0-edge (this happens at most once per edge), or we achieve that $l^+(P) = l^-(P)$ or $l^+(Q) = l^-(Q)$ (each of these also happens at most once). The last motion must convert two edges into 0-edges, and hence accounts for the last two types of steps. So there are $n/2$ steps for horizontal edges and $n/2$ steps for vertical edges.

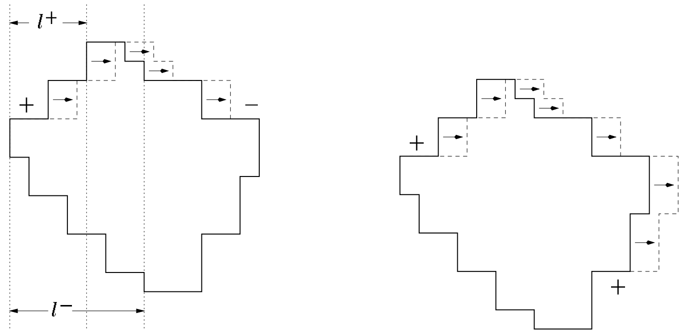

Fig. 6. Morphing to trade-off a +-edge in the upper left chain against an edge of the right chain.

5.2 Morphing Orthogonally Convex Polyhedra in 3D

Moving into three dimensions, we find that not all pairs of orthogonally convex polyhedra admit a monotone morph. In order to show this, we find it convenient to first consider the problem of monotonically morphing between parallel orthogonal graphs embedded in the plane. See [4] for more on morphing graphs. It is not true that all such graphs admit a monotone morph.

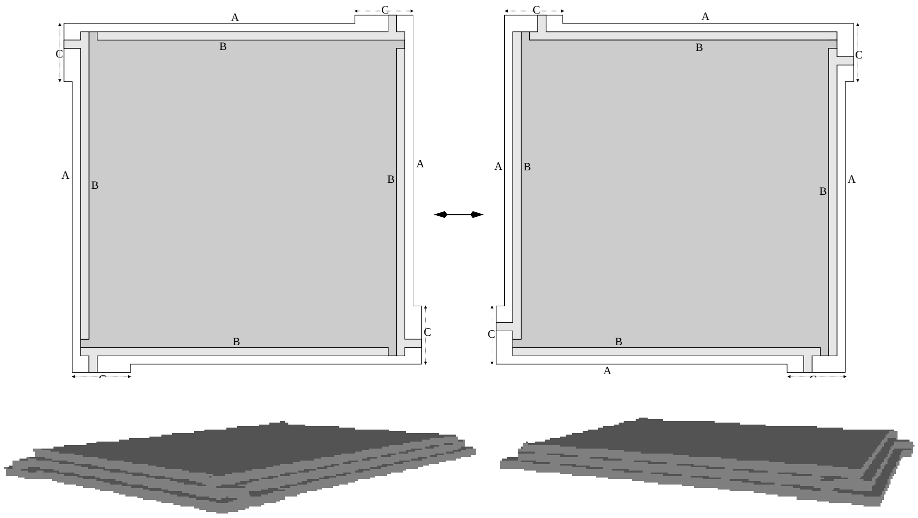

Fig. 7. Parallel, orthogonally convex polyhedra that do not admit a parallel morph.

Claim. The graphs shown in Figure 7 do not admit a monotone morph.

Proof. Let the leftmost graph in Figure 7 be the source graph, and let the graph on the right be the target. In these figures, letters denote edges of a particular length: $A = 35$, $B = 37$, $C = 7$. Each of the edges labeled C is actually composed of three incident, parallel edges of the graph.

Let us suppose that the source and target graphs admit a monotone morph. The central region of these graphs is shaded, serving to highlight that in the monotone morph, all edges bounding this region are static. The area contained in the shaded region is at all times greater than $|B|^2 = 37^2 = 1369$.

Now consider the edges that bound the outer face. The edges of length C do not change in length throughout the monotone morph. In a morph from source to target, there must be some time at which the two horizontal edges labeled C are aligned with respect to the horizontal axis. All of the other outer edges must have length $\leq A$. It is not difficult to show that the outer edges contains a region of area $\leq 4|A||C| + |C|^2 = 1029$. This is less than the area contained by the inner, shaded region. Therefore, by contradiction these graphs do not admit a monotone morph.

We use the graphs from Figure 7 to construct a pair of parallel, orthogonally convex polyhedra that do not admit a simple, monotone morph. Let x, y, z denote the axes of three-dimensional space. Generate two polyhedra such that the highest face (perpendicular the z axis) in both polyhedra is the same as the shaded region in the graphs. The other regions are transformed into faces of the polyhedra that are perpendicular with the z axis, at various heights. It is not difficult to see that such a pair of polyhedra can be constructed in which all faces are orthogonally convex polygons. See also Figure 7.

Theorem 6. *Not all pairs of orthogonally convex, parallel polyhedra admit a monotone morph, even if all faces are restricted to be orthogonally convex polygons.*

6 Conclusions

We have begun an exploration of parallel morphs that change edge lengths as little as possible, showing that monotone morphs are possible for convex and orthogonally convex polygons, but that, in general, deciding the existence of monotone morphs is NP-hard. We mention three open questions: (1) How badly must edge lengths change during parallel morphs between parallel polygons? (2) What other classes of configurations admit monotone morphs? (3) For what classes of configurations are there efficient algorithms to test for monotone morphs? We conjecture that this is possible for parallel orthogonal polygons that are monotone with respect to one axis (see Figure 1).

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