The Complexity of a Single Face of a Minkowski Sum

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Abstract

This note considers the complexity of a free region in the configuration space of a polygonal robot translating amidst polygonal obstacles in the plane. Specifically, given polygonal sets P and Q with k and n vertices, respectively ($k < n$), the number of edges and vertices bounding a single face of the complement of the Minkowski sum $P \oplus Q$ is $\Theta(nk\alpha(k))$ in the worst case. The lower bound comes from a construction based on lower envelopes of line segments; the upper bound comes from a combinatorial bound on Davenport-Schinzel sequences that satisfy two alternation conditions.

1 Introduction and Background

Let A and B be two sets in $\mathbb{R}^2$. The *Minkowski sum* (or vector sum) of A and B, denoted $A \oplus B$, is the set $\{a + b \mid a \in A, b \in B\}$.

The Minkowski sum is a useful concept in robot motion planning and related areas [2, 11, 12, 13]. For example, consider an obstacle A and a robot B that moves by translation. We can choose a reference point r rigidly attached to B and suppose that B is placed such that the reference point coincides with the origin. If we let B' denote a copy of B rotated by $180°$, then $A \oplus B'$ is the locus of placements of the reference point where $A \cap B \neq \emptyset$. This sum is often called a *configuration-space obstacle* or *C-obstacle* because B collides with A under rigid motion along a path γ exactly when the reference point r, moved along γ, intersects $A \oplus B'$.

We confine ourselves to the Minkowski sum of polygonal sets, which is a polygonal set [4]. Let P and Q be two polygonal sets, not necessarily connected, with k and n vertices respectively. The boundary of $P \oplus Q$ comes from an arrangement of $O(nk)$ line segments, which has complexity bounded by $O(n^2k^2)$, and this bound is tight in the worst case [10, 14].

In applications such as motion planning [5] and assembly planning [18], however, we only need to know the *face complexity*—the number of segments that bound a single face of the complement of the Minkowski sum $P \oplus Q$ in the worst case. Figure 1 depicts the outer face of a sum $P \oplus Q$. Davenport-Schinzel sequence analysis, which is described in section 3.1, shows that the face complexity is $O(nk\,\alpha(nk))$ [14], where $\alpha(\cdot)$ is the functional inverse of Ackermann's function.

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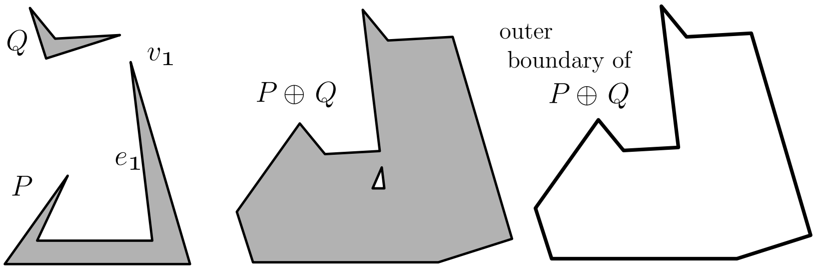

Figure 1: Two polygons, their Minkowski sum, and the outer face of the sum

There is a collection of kn segments that gives rise to a single face with complexity $\Omega(nk\,\alpha(nk))$ [17]. Speaking at the 5th CCCG [12], Milenkovic conjectured that the special structure of the segments in a Minkowski sum would imply $O(nk)$ complexity. We establish the true bound to be $\Theta(nk\,\alpha(k))$ with $k < n$. Our lower bound is based on the lower envelope construction of Wiernik and Sharir [17]. The upper bound comes from recent work of Har-Peled [7] on generalized combination lemmas; its combinatorial analysis of *double Davenport-Schinzel sequences* can be seen as a generalization of an analysis of Huttenlocher et al. [9].

These bounds are worth noting in the context of motion planning, where it is common to assume that P is a robot polygon with small fixed complexity and Q, the set of obstacles, has large complexity. In this setting, our bound states that the complexity of a single face in the complement of the Minkowski sum of the obstacles and the robot polygon is $\Theta(n)$.

2 The Lower Bound on the Face Complexity

We establish an $\Omega(nk\,\alpha(k))$ lower bound (with $k < n$) even for simple polygons P and Q. One can modify the construction to make P and Q be star-shaped polygons (which implies that $P \oplus Q$ is star-shaped).

Theorem 2.1 *Given $k < n$, there exists a simple polygon P with $\Theta(k)$ edges and a simple polygon Q with $\Theta(n)$ edges such that the outer face of $P \oplus Q$ has $\Omega(nk\,\alpha(k))$ edges.*

Proof: Let $s_1, s_2 \ldots, s_k$ be k segments such that their lower envelope $\mathcal{L}$ has $\Omega(k\,\alpha(k))$ edges [17]. We may assume that the segments lie inside the unit square $(0,1)\times(0,1)$. Define P by extending the k segments $s_1+(1,0), s_2+(2,0), \ldots, s_k+(k,0)$ vertically to the line $y = 1$, as in Figure 2. This gives us a polygon with $\Theta(k)$ edges. Define Q by extending the $n+k$ points $(1,0),(2,0),\ldots,(n+k,0)$ vertically to the line $y = 1$. By thickening the edges, this gives us a polygon with $\Theta(n+k) = \Theta(n)$ edges. In Figure 2, one can see that $P \oplus Q$ is a polygon whose outer face includes n translated copies of $\mathcal{L}$ and is thus of size $\Omega(nk\,\alpha(k))$. ∎

3 The Upper Bound on the Face Complexity

To prove the upper bound, we employ a useful combinatorial theorem of Har-Peled [7]. We include the details to make this note self-contained.

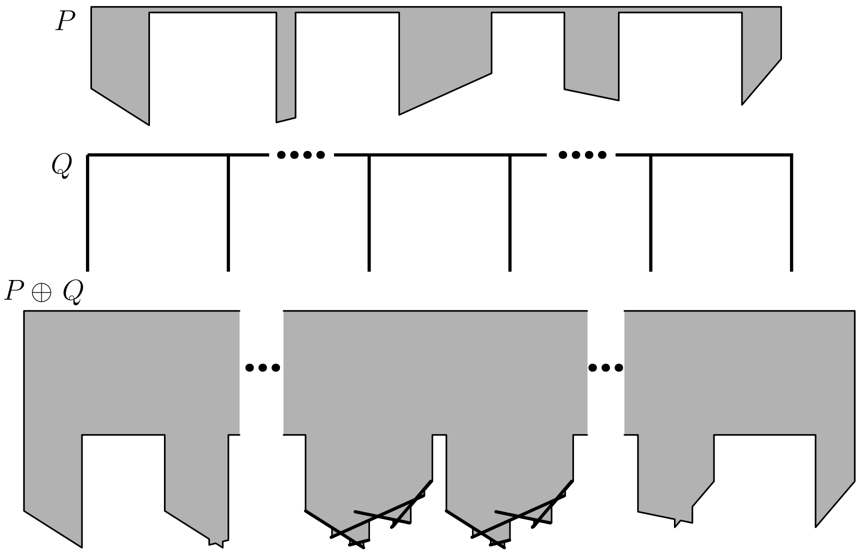

Figure 2: The polygons P and Q and sum $P \oplus Q$ with $\Omega(nk\,\alpha(k))$ complexity

3.1 Double Davenport-Schinzel Sequences

Davenport-Schinzel sequence analysis is a combinatorial tool with many applications in computational geometry. We remind the reader of the basic definitions; for more information, see Sharir's survey [15, 16]. Let Σ be an alphabet with m symbols and s be a positive integer. A string $U = u_1 u_2 \ldots u_r$ of symbols in Σ is an (m, s)–*Davenport-Schinzel* sequence if it satisfies two conditions:

1. No adjacent repeats: $u_i \neq u_{i+1}$ for all $i < r$.
2. No $s+1$ alternations: For no distinct $a, b \in \Sigma$ is the alternating sequence $abab\ldots$ of length $s+2$ a subsequence of U.

Let $\lambda_s(m)$ denote the maximum length of an (m, s)–DS sequence. It is not hard to see that $\lambda_1(m) = m$ and that $\lambda_2(m) = 2m - 1$. Hart and Sharir [8] have shown that $\lambda_3(m) = \Theta(m\alpha(m))$, where $\alpha(m)$ is the functional inverse of Ackermann's function. Agarwal et al. [1] have obtained the best bounds known for $\lambda_s(m)$ with $s > 3$; for any fixed s, the bounds are slightly superlinear in m.

Huttenlocher et al. [9] studied a variant of Davenport-Schinzel sequences in which there are a small number of "active" symbols at any given time. Har-Peled [7] has generalized and strengthened their result to what could be called "double" Davenport-Schinzel sequences: sequences satisfying two alternation restrictions. Let $\Sigma^i = \{a^{ij} \mid j = 1, \ldots, n\}$ for $i = 1, \ldots, k$. Let $\Sigma = \bigcup_{1 \leq i \leq k} \Sigma^i$. Thus, we have nk symbols in k families of n symbols.

Theorem 3.1 *Let $U = u_1 u_2 \ldots u_r$ be a string of symbols of Σ satisfying:*

1. *No adjacent repeats: $u_\ell \neq u_{\ell+1}$, for $1 \leq \ell < r$.*
2. *No global alternating 5-seq.: For distinct $a, b \in \Sigma$, ababa is a forbidden subsequence of U.*
3. *No family alternating 4-seq.: For all $1 \leq i \leq k$ and distinct $a, b \in \Sigma^i$, abab is forbidden in U.*

Then the length $|U| = r = O(nk\,\alpha(k))$.

Proof: We first fix a family $i \in \{1, \ldots, k\}$ and consider the sequence obtained from U by removing all elements of U that do not belong to Σ^i. This sequence might contain pairs of consecutive symbols that are identical. Contract strings of such identical symbols to just one symbol. Let the resulting string of symbols of Σ^i be U^i. Condition 3 implies that U^i is an $(n, 2)$–DS sequence, so its length is at most $\lambda_2(n) = 2n - 1$. Summed over all families $\sum_{1 \le i \le k} |U^i| = (2n - 1)k$.

Now consider the sequence U and subdivide it into blocks as follows. Start at the beginning of U and continue until $2k$ distinct symbols have been seen (or U has been exhausted). This forms the first block of U; remove it and repeat the process until U is exhausted. We prove that (1) U is hereby cut into at most $2n$ blocks and (2) each block has $O(k\,\alpha(k))$ symbols. The theorem follows immediately.

The second is easy: Any block U' of U contains at most $2k$ distinct symbols. By conditions 1 and 2, block U' is a $(2k, 3)$–DS sequence, so its length is at most $\lambda_3(2k) = O(k\,\alpha(k))$.

As for the first claim, any block U' except the last uses exactly $2k$ distinct symbols. For each $1 \le i \le k$, mark in U' the first occurrence, if any, of a symbol from Σ^i. This marks at most k symbols. Now traverse U' from left to right, considering, for each symbol a^{ij}, its first *unmarked* appearance in U', if any. There are at least k such appearances, and each corresponds to a new element of the sequence U^i described above. Because there are a total of $(2n - 1)k$ elements in all U^is, there are at most $2n$ blocks. ∎

3.2 The Face Complexity of the Minkowski Sum

Theorem 3.2 *Let P and Q be polygonal sets with k and n vertices respectively. The complexity of a face of the complement of the Minkowski sum $P \oplus Q$ is $O(nk\,\alpha(k))$.*

Proof: The segments that bound the Minkowski sum $P \oplus Q$ are the sums of a vertex of one polygonal set with an edge of the other [4]. We treat these asymmetrically and define a *vertex set* to be a sum of a fixed vertex of P with all the edges of Q and an *edge set* to be the sum of a fixed edge of P and all the vertices of Q. Figure 3(a) depicts the vertex set induced by v_1 of Figure 1; Figure 3(b) depicts the edge set induced by e_1.

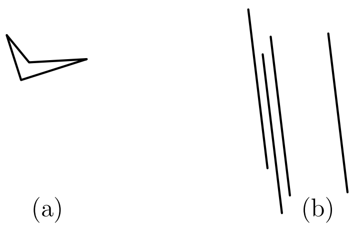

Figure 3: vertex & edge sets

Consider a face of the complement of $P \oplus Q$. We derive a double Davenport-Schinzel sequence on an alphabet consisting of $2k$ families (the vertex and edge sets) of $2n$ segments each. Starting at the rightmost point on the boundary of the face, walk around the boundary and list the segments encountered in order. When a segment s is encountered for the first time, split it into two, s and s', to ensure that each segment is traversed in a consistent order. If the face has more than one boundary component, then repeat for each component, and concatenate the resulting lists arbitrarily to form sequence U. Because no segment can appear in two components, concatenation cannot create adjacent repeated symbols or forbidden alternation patterns in U. (This is no longer true if we look at the boundaries of all faces.)

Because each segment of a vertex or edge set bounds a polygon, each has only one side exposed to the complement. Thus, an *ababa* subsequence in U would indicate that the two segments a and b had two points of intersection, which is impossible. In a similar manner, an $abab$ subsequence with a and b from the same vertex set would indicate that two edges of Q intersected; an $abab$ subsequence with a and b from the same edge set would indicate that two parallel segments intersected. Therefore, U is a double Davenport-Schinzel sequence and Theorem 3.1 bounds its length by $O(nk\,\alpha(k))$. ■

Remark: Sharir, in personal communication, has pointed out that this theorem can also be proved by decomposing P into $O(k)$ triangles, computing the Minkowski sum of each triangle with Q to form $O(k)$ arrangements with $O(n)$ complexity, and then applying Har-Peled's generalized combination theorem [7, Thm 3.1]. In fact, we can also apply the combination theorm directly to the arrangements of vertex and edge sets.

4 Open Problems

Can one further exploit the structure of vertex and edge sets to devise a "simple" deterministic algorithm for computing a face of the Minkowski sum (i.e., simpler than the general algorithm of Edelsbrunner *et al.* [3])? Can one exploit similar structure for constraint surfaces in other motion planning problems to improve bounds on the complexity of a connected component of free configuration space? One example is the three-dimensional configuration space of a polygon translating *and rotating* among polygons [6].

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