On Levels in Arrangements of Curves, III: Further Improvements

Timothy M. Chan*

February 18, 2008

Abstract

We revisit the problem of bounding the combinatorial complexity of the k-*level* in a two-dimensional arrangement of n curves. We give a number of small improvements over the results from the author's previous paper (FOCS'03). For example:

- For pseudo-parabolas, we obtain an upper bound of $O(n^{3/2}\log n)$, which improves the previous bound of $O(n^{3/2}\log^2 n)$.
- For 3-intersecting curves, we obtain an upper bound of $O(n^{2-1/(3+\sqrt{7})}) = O(n^{1.823})$, the first improvement over the previous bound of $O(n^{11/6}) = O(n^{1.834})$.
- For s-intersecting curves or curve segments with $s \geq 3$, we obtain an upper bound of $O(n^{2-\frac{1}{2s}-\delta_s})$ if s is odd, and $O(n^{2-\frac{1}{2(s-1)}-\delta_s})$ if s is even, for some constant $\delta_s > 0$.
- For pseudo-segments, we obtain an upper bound of $O(n^{4/3}\log^{1/3-\delta} n)$ for some constant $\delta > 0$; the previous bound was $O(n^{4/3}\log^{2/3} n)$.
- For s-intersecting curve segments such that all but B pairs intersect at most once, we obtain an upper bound of $O((n^{4/3} + n^{1+\delta}B^{1/3-\delta})\log^{1/3-\delta} n + B)$ for some constant $\delta > 0$.

We also observe that better concrete bounds for k-levels for constant values of n could in principle lead to better asymptotic bounds for arbitrary n.

1 Introduction

The topic of this paper, the combinatorial complexity of k-*levels* in arrangements, perhaps "needs no introduction" among discrete and computational geometers. We refer the readers to the previous papers [5, 6] of this trilogy for more on the problem's background and extensive history. Our focus here is on the 2-d setting.

The case of k-levels of lines, equivalent to the k-*set* problem, is the most well-known version studied. A number of proofs of an $O(n^{3/2})$ upper bound,[1] first shown by Lovász, have been (re)discovered [14, 12, 13, 11, 2]. The current best upper bound of $O(n^{4/3})$ is due to Dey [10]. Erdős *et al.* [12] conjectured that $O(n^{1+\varepsilon})$ is an upper bound for any constant $\varepsilon > 0$. The current best lower bound is $n2^{\Omega(\sqrt{\log n})}$ by Tóth [21].

*School of Computer Science, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada (tmchan@uwaterloo.ca).
Work supported by NSERC.

[1] We only state bounds in terms of n, since any upper bound in terms of n can be converted to an upper bound in terms of n and k in a systematic manner, as shown in [1].

In the previous paper [6], we proposed an interesting proof of the old $O(n^{3/2})$ upper bound that is fundamentally different from all the previous proofs. The approach involves a new simple inequality relating the sizes of different levels in the arrangement. In this paper, we refine this approach further. Our first result (see Section 2) is a new proof of an $O(n^{3/2-\delta})$ upper bound, where δ is a specific but very small positive constant.

Now, this "new result" should draw one immediate criticism—that we are actually not making progress, but regressing! Certainly, the $O(n^{3/2-\delta})$ bound would have been more stunning had it been discovered before 1997—before Dey's simple $O(n^{4/3})$ proof came along [10], the only proof that beat $O(n^{3/2})$ was a rather complicated effort by Pach, Steiger, and Szemerédi from 1989 [16], with an improvement that was "merely" $O(n^{3/2}/\log^* n)$, where $\log^*$ denotes the very slow-growing iterated logarithm function. Although we cannot rewrite history (alas), we believe that the new proof is still worthy of study for several reasons:

First, the proof, though longer than in [6], is based on completely elementary arguments. We do not even need Euler's formula on planar graphs or the probabilistic method (both of which are used to establish the Crossing Lemma required within Dey's proof [10]).

Second, the inequality approach in [6], as we have said, is intrinsically different from all previous approaches to the k-set problem, and intellectual curiosity compels us to find out how far this approach can go, even if Dey's approach turns out to win at the end.

Third, and most importantly, the inequality approach can be applied to the k-level problem in more general settings where Lovász's and Dey's approaches do not lead to the best results, or simply do not work at all! For instance, (i) our previous paper [6] has obtained the current best results for k-levels for most classes of curves in 2-d, all using the inequality approach; (ii) another paper of the author [7] has extended the approach to get the first nontrivial k-level results for surfaces in 3-d; (iii) yet another paper [8] illustrated the power of the approach on a bichromatic variant of the k-set problem. In this paper, our attention will be devoted to (i) in particular, i.e., on applying our refined inequality approach to obtain still better results for k-levels of curves.

For example:

- Consider an arrangement of n *almost pseudo-segments*, where each pair intersects at most once except for B "bad" pairs, and where a bad pair is allowed to intersect any constant number of times. Dey's approach is not applicable unless one *cuts* the arrangement first to eliminate the bad pairs, but in doing so, the bound worsens as B gets large. Specifically, a bound $O((n^{4/3} + n^{2/3}B^{2/3})\log^{2/3} n)$ can be shown, according to [5, Theorems 2.1 and 3.3]. In contrast, the inequality approach yields $O(n^{3/2} + B)$ [6], which has a much better dependency on B. Our new proof implies a further improved bound of $O(n^{3/2-\delta} + B)$ for this problem. Moreover, by combining the two approaches, we get a bound that subsumes previous results for all B: $O((n^{4/3} + n^{1+\delta'}B^{1/3-\delta'})\log^{1/3-\delta'} n + B)$ for some constant $\delta' > 0$.

- Consider an arrangement of n *pseudo-parabolas*, where each pair intersects at most twice. This case has received much attention in the past, and a long chain of intermediate results can be found in the literature [20, 5, 17, 15]: $O(n^{23/12})$, $O(n^{17/9})$, $O(n^{16/9}\log^{2/3} n)$, $O(n^{8/5})$, and lastly $O(n^{3/2}\log^2 n)$. In this paper, we use our refined inequality to derive the best upper bound to date: $O(n^{3/2}\log n)$. Although a single logarithmic factor reduction may not necessarily impress everyone, $O(n^{3/2}\log n)$ is a natural barrier, matching the best result known for a related combinatorial problem: cutting pseudo-parabolas into pseudo-segments [15]. Indeed, if one could get better results for the latter problem, our refined inequality would automatically

class of curves	best previous result	new result
pseudo-segments	$O(n^{4/3}\log^{2/3} n)$	$O(n^{4/3}\log^{1/3-\delta} n)$
almost pseudo-segments	$O(\min\{n^{3/2}+B,\ (n^{4/3}+ n^{2/3}B^{2/3})\log^{2/3} n\}$	$O((n^{4/3}+n^{1+\delta}B^{1/3-\delta}) \times\log^{1/3-\delta} n+B)$
pseudo-parabolas	$O(n^{3/2}\log^2 n)$	$O(n^{3/2}\log n)$
pairwise-intersect. pseudo-parabolas	$O(n^{13/9})$	$O(n^{13/9-\delta})$
3-intersecting curves	$O(n^{11/6})=O(n^{1.834})$	$O(n^{2-1/(3+\sqrt{7})})=O(n^{1.823})$
5-intersecting curves	$O(n^{19/10})=O(n^{1.9})$	$O(n^{2-2/(15+\sqrt{17})})=O(n^{1.896})$
s-intersect. curve segments (s odd)	$O(n^{2-\frac{1}{2s}})$	$O(n^{2-\frac{1}{2s}-\delta_s})$
s-intersect. curve segments ($s\geq 4$ even)	$O(n^{2-\frac{1}{2(s-1)}})$	$O(n^{2-\frac{1}{2(s-1)}-\delta_s})$
degree-5 polynomials	$O(n^{161/90}\log^{182/45} n)$	$O(n^{161/90-\delta})$

Table 1: Upper bounds on the complexity of the k-level. The best previous results are all from [6] except for the first entry, which is from [5]. Here, $\delta, \delta_s > 0$ are absolute constants (not necessarily the same in all entries).

lead to better results for k-levels for pseudo-parabolas.

- Finally, and most generally, consider an arrangement of n *s-intersecting* curve segments, where each pair intersects at most s times. The previous paper [6] gave the first nontrivial upper bound, $O(n^{2-\frac{1}{2s}})$. The same paper offered an improvement for even values of s, to $O(n^{2-\frac{1}{2(s-1)}})$, using combinatorial results on cutting s-intersecting curve segments into $(s-1)$-intersecting ones. For odd values of s, unfortunately nontrivial cutting bounds are known to be *impossible*. Nonetheless, through the techniques in this paper, we still manage to get an improved bound $O(n^{2-\frac{1}{2s}-\delta_s})$ for some constant $\delta_s > 0$ for odd s. Thus, officially $O(n^{2-\frac{1}{2s}})$ has now been beaten for all values of s. (For even s, we also get an improvement over the $O(n^{2-\frac{1}{2(s-1)}})$ bound.)

In addition to the above, we can get small improvements to many of the sundry other results from the previous paper [6] as well. Table 1 summarizes all our new results (see Section 3).

We note that it is possible to work out some concrete bounds for these unspecified constants δ and δ_s. We do not do so here, because the values of these constants are too embarassingly small, and it is hard to determine their optimal values in our proofs. Without additional ideas, it is highly unlikely that optimizing the constants would allow us to surpass Dey's result for the k-set problem, for example. Although all our results are incremental in nature, no progress has been reported on any of the cases since 2003, and it is hoped that the ideas here might inspire further progress.

There is one instance where our improvement is more substantial, namely, the case of s-intersecting curves (not curve segments) for odd s. Here, we find a cute proof (see Section 4) where the constants can be worked out explicitly, with reasonable values for small s (see the above table).

In Section 5, we end with some additional observations about potential alternative approaches to the k-set problem.

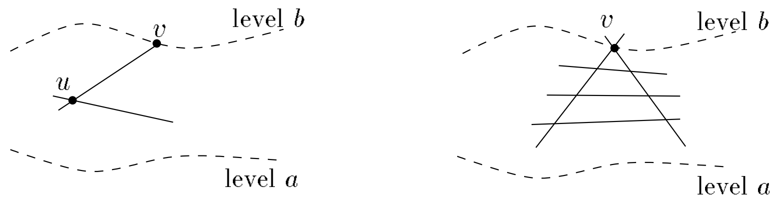

Figure 1: In the left diagram, a vertex $u \in V_{a,b}$ is charged to a vertex $v \in \partial V_{a,b}$. In the right diagram, at most $2(b-a-1)$ vertices are charged to v. (Levels are drawn abstractly in dashed lines in all diagrams.)

2 (Almost) Pseudo-Lines

2.1 The basic charging argument

We are given an arrangement of n curves in the plane in general position, where each curve is the graph of a total continuous function and two curves can intersect at most $O(1)$ times. The *level* of a point q refers to the number of curves strictly below q. The main goal of this paper is to find nontrivial upper bounds (as functions of n) on the number of vertices in the arrangement that have level exactly k.

Let $L_{a,b}$ be the locus of all points that have level in the interval $(a, b]$. Let $V_{a,b}$ be the set of vertices that have level in the interval (a, b). Let $\partial V_{a,b}$ be the set of all vertices that have level a or b.

If two curves γ_1 and γ_2 intersect more than once, a *lens* refers to the portion of $\gamma_1 \cup \gamma_2$ between two consecutive intersection points of $\gamma_1 \cap \gamma_2$. Let $\Lambda_{a,b}$ be the set of all lenses completely inside $L_{a,b}$. Let ℓ_v denote the vertical line through a point v.

We begin by reviewing the simple charging argument from the previous paper [6] where vertices in $V_{a,b}$ are charged to vertices in $\partial V_{a,b}$. The following definitions capture the essence of the argument and immediately lead to the main inequality of that paper.

Definition 2.1 Suppose $u \in V_{a,b}$ is defined by curves γ_1 and γ_2, and $v \in \partial V_{a,b}$ is incident on γ_1 or γ_2. If γ_1 and γ_2 are completely inside $L_{a,b}$ between ℓ_u and ℓ_v, then we say u is *(a,b)-charged* to v.

Definition 2.2 Suppose $u \in V_{a,b}$ is defined by curves γ_1 and γ_2. If γ_1 and γ_2 both have level in $(a, b]$ at $x = -\infty$ or $x = \infty$, or form a lens in $\Lambda_{a,b}$, then we say u is *(a,b)-exceptional.*

Lemma 2.3 $|V_{a,b}| \ \le\ (b-a-1)|\partial V_{a,b}| + O((b-a)^2 + |\Lambda_{a,b}|)$.

Proof: Observe that each non-exceptional vertex $u \in V_{a,b}$ is charged to exactly two vertices (one on each side of ℓ_u). Conversely, at most $2(b-a-1)$ non-exceptional vertices are charged to each vertex $v \in \partial V_{a,b}$ (at most two on each curve that has level in $[a, b]$ at ℓ_v, excluding the two curves defining v). See Figure 1. There are at most $O((b-a)^2 + |\Lambda_{a,b}|)$ exceptional vertices. The lemma follows. □

Letting $t_i = |V_{k-i,k+i}|$, $\Delta t_i = t_{i+1} - t_i$, and $\lambda(n, i) = |\Lambda_{k-i,k+i}|$, we get the inequality

$$t_i \ \le\ 2i\Delta t_i + O(i^2 + \lambda(n, i)), \tag{1}$$

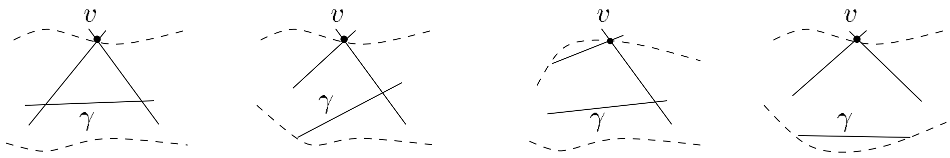

Figure 2: Examples of when (v, γ) forms a non-helper, helper, or strong helper.

with $t_n = O(n^2)$ as the trivial base case.

Solving this recurrence is straightforward (e.g., see Lemma A.1 in the appendix). For an almost pseudo-line arrangement with $\lambda(n, i) \leq B$, the recurrence implies an upper bound of $t_1 = O(n^{3/2}+B)$ on the number of k-level vertices.

2.2 Helpers

To improve upon $O(n^{3/2} + B)$, we try to identify scenarios—configurations called "helpers"—where there is some slack in the inequality. (The helper concept was also used in another paper of the author [7], but for a different version of the k-level problem, with different technical definitions.)

Definition 2.4 Suppose $v \in \partial V_{a,b}$, and the curve γ has level in $[a, b]$ at ℓ_v (with γ not equal to the two curves defining v). If at most one (a, b)-non-exceptional vertex on γ is (a, b)-charged to v, then we say (v, γ) is an (a, b)-*helper*. If no vertex on γ is (a, b)-charged to v, then we say (v, γ) is a *strong* (a, b)-*helper*. See Figure 2.

Let $H_{a,b}$ be the set of all (a, b)-helpers, and $J_{a,b}$ be the set of all strong (a, b)-helpers.

Part (i) of the lemma below states that improving Lemma 2.3 indeed amounts to proving the existence and abundance of helpers. We also provide a companion inequality in part (ii), involving strong helpers in the reverse direction.

Lemma 2.5

(i) $|V_{a,b}| \leq (b - a - 1)|\partial V_{a,b}| - \frac{1}{2}|H_{a,b}| + O((b - a)^2 + |\Lambda_{a,b}|)$.

(ii) $|V_{a,b}| \geq \frac{b-a-1}{2}|\partial V_{a,b}| - \frac{1}{2}|J_{a,b}|$.

Proof: For (i), simply observe that the number of charges from non-exceptional vertices in $V_{a,b}$ to vertices in $\partial V_{a,b}$ is at least $2|V_{a,b}| - O((b - a)^2 + |\Lambda_{a,b}|)$ and at most $2(b - a - 1)|\partial V_{a,b}| - |H_{a,b}|$.

For (ii), observe that the number of charges from vertices in $V_{a,b}$ to vertices in $\partial V_{a,b}$ is at most $2|V_{a,b}|$ and at least $(b - a - 1)|\partial V_{a,b}| - |J_{a,b}|$. □

Unfortunately, we do not see any easy argument to show the abundance of helpers so that the inequality in part (i) above can be applied. While the plan of finding helpers might sound simple, the execution of the plan is far from obvious.

Before we proceed, a comment about the inequality in part (ii) is in order, as at first glance it seems totally irrelevant to our endeavor of proving *upper* bounds for k-levels. To understand its significance, consider the "critical" case, where $t_i \approx n^{3/2}i^{1/2}$ and $\Delta t_i \approx \frac{1}{2}n^{3/2}/i^{1/2}$ for all i, say, with equal number of $(k + i)$-level vertices as $(k - i)$-level vertices. Applying (ii) to the interval

$(a,b)=(k-i,k+i)$ yields no useful information whatsoever. However, consider instead an interval away from the "center", e.g., $(a,b)=(k+i,k+1.4i)$. Then $|V_{a,b}|\approx\frac{1}{2}(t_{1.4i}-t_i)\approx\frac{1}{2}(\sqrt{1.4}-1)n^{3/2}i^{1/2}\approx 0.0916n^{3/2}i^{1/2}$ and $\frac{b-a-1}{2}|\partial V_{a,b}|\approx 0.2i\cdot\frac{1}{2}(\Delta t_i+\Delta t_{1.4i})\approx 0.2i\cdot\frac{1}{4}(1+1/\sqrt{1.4})n^{3/2}/i^{1/2}\approx 0.0923n^{3/2}i^{1/2}$. So (ii) does provide new information after all—namely, that the number of strong helpers must be large for a "side" interval like $(k+i,k+1.4i)$, at least when things are supposedly tight.

Of course, what we want is different—that the number of helpers is large for a "center" interval of the form $(k-i,k+i)$. Note that we do not need the existence of many helpers for every center interval, just sufficiently many center intervals.

2.3 A new second charging argument

By a stroke of luck, we show that there is actually a relationship linking the number of strong helpers in side intervals to the number of helpers in center intervals. This relationship is described in the lemma below, which is where most of the hard work lie. The argument involves a second layer of charging, where this time helpers are charged to helpers instead of vertices to vertices. Specifically, we charge strong helpers in side intervals (a',a) and (b,b') to helpers in center intervals (a,b) and (a',b'). The argument is longer and requires a careful case analysis that nonetheless fits together neatly.

One minor technicality: for the proof to go through, we redefine $\Lambda_{a,b}$ as the set of all lenses completely inside $L_{a-0.2(b-a),b+0.2(b-a)}$. This redefinition would expand the set of (a,b)-exceptional vertices, but without changing asymptotic bounds on $\lambda(n,i)$.

Lemma 2.6 *Let $a'<a<b<b'$ with $a-a',b'-b\le 0.2(b-a)$. Then $|H_{a,b}|+|H_{a',b'}|\ge c(|J_{a',a}|+|J_{b,b'}|)-O(b-a)^2$ for an absolute constant $c>0$.*

Proof: Let (v,γ_0) be a strong (a',a)-helper. Say v is defined by two curves γ_1 and γ_2, with γ_1 (resp. γ_2) inside $L_{a',a}$ immediately to the left (resp. right) of v.

Let v_1 be the rightmost intersection of $\gamma_1\cup\gamma_0$ with $\partial V_{a',a}$ to the left of v. Similarly, let v_2 be the leftmost intersection of $\gamma_2\cup\gamma_0$ with $\partial V_{a',a}$ to the right of v. By the definition of strong helpers, we know that γ_j does not intersect γ_0 between ℓ_{v_j} and ℓ_v for each $j\in\{1,2\}$. If v_j does not exist, we simply ignore (v,γ_0). Note that the number of ignored helpers is at most $O(b-a)^2$, since if v_j does not exist, γ_0 and γ_j would both have levels in $(a,b]$ at $x=-\infty$ or $x=\infty$.

Take a curve γ that has level in $[a,b]$ at ℓ_v, and level not in $[a,a']\cup[b,b']$ at ℓ_{v_1} and not in $[a,a']\cup[b,b']$ at ℓ_{v_2} (with $\gamma\ne\gamma_1,\gamma_2$). There are at least $b-a-2(a-a')-2(b'-b)-O(1)=\Omega(b-a)$ choices for γ.

Let w_1 be the rightmost intersection of γ with $\partial V_{a',b'}$ between ℓ_{v_1} and ℓ_v, if it exists. Similarly, let w_2 be the leftmost intersection of γ with $\partial V_{a',b'}$ between ℓ_v and ℓ_{v_2}, if it exists. Note that if w_j does not exist, the level of γ at ℓ_{v_j} must be in $(a,b]$.

We claim that at least one of the following is an (a,b)- or (a',b')-helper: (v,γ), (v,γ_0), (v_1,γ), (v_2,γ), (w_1,γ_0), (w_2,γ_0). We send a charge from (v,γ_0) to this helper.

The proof of this claim is divided into cases, depending on whether each of $\{v,v_1,v_2\}$ has level a or a', and whether each of $\{w_1,w_2\}$ have level a' or b' or does not exist. This case analysis is best understood, and justified, through pictures rather than long prose. See Figures 3 and 4. (It might help the readers to focus on the pure pseudo-line setting without lenses first.)

- Case 1: v has level a.

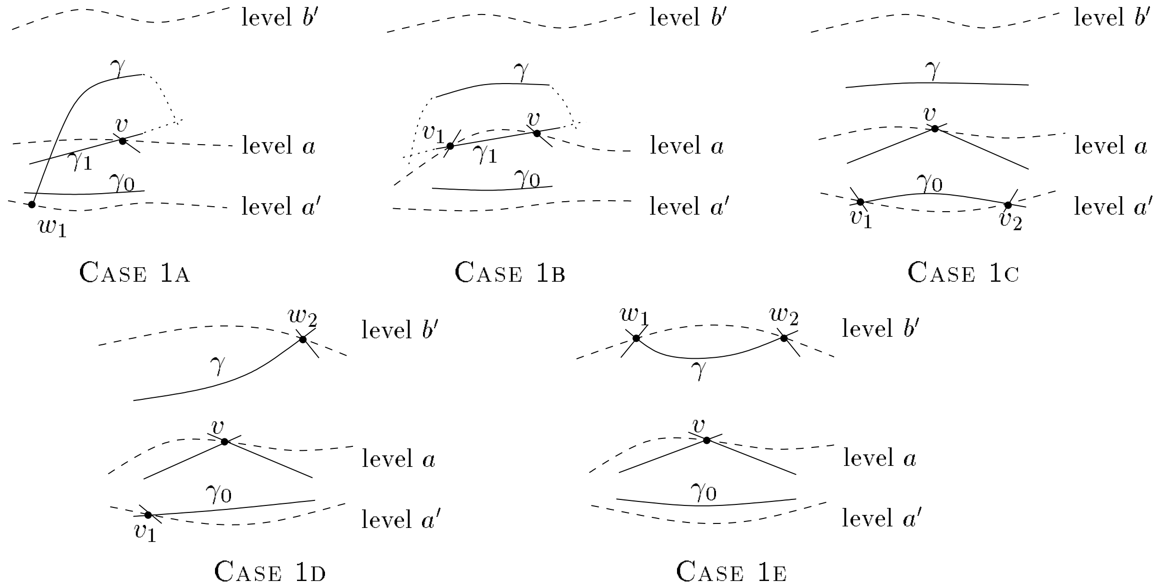

Figure 3: Proof of Lemma 2.6: Case 1.

- Case 1a: w_j exists and has level a' for some $j \in \{1, 2\}$.
  Then (v, γ) is an (a, b)-helper (otherwise γ would intersect γ_j at least twice, at exceptional vertices—a contradiction).
- Case 1b: w_j does not exist and v_j has level a for some j.
  Then (v, γ) or (v_j, γ) is an (a, b)-helper (otherwise γ would intersect γ_j at least twice, at exceptional vertices).
- Case 1c: w_1 does not exist and v_1 has level a', and w_2 does not exist and v_2 has level a'.
  Then (v_1, γ) and (v_2, γ) are (a', b')-helpers (otherwise γ would intersect γ_0 at least twice, at exceptional vertices).
- Case 1d: w_1 does not exist and v_1 has level a', and w_2 exists and has level b' (or vice versa).
  Then(v_1, γ) and (w_2, γ_0) are (a', b')-helpers (otherwise γ would intersect γ_0 at least twice, at exceptional vertices).
- Case 1e: w_1 exists and has level b', and w_2 exists and has level b'.
  Then (w_1, γ_0) and (w_2, γ_0) are (a', b')-helpers (otherwise γ would intersect γ_0 at least twice, at exceptional vertices).

- Case 2: v has level a'.
  - Case 2a: w_j exists and has level b' for some j.
    Then (v, γ) and (w_j, γ_j) are (a', b')-helpers (otherwise γ would intersect γ_j at least twice, at exceptional vertices).
  - Case 2b: w_j does not exist and v_j has level a' for some j.
    Then (v, γ_0) is an (a', b')-helper (and so are (v, γ), (v_1, γ_0), and (v_1, γ), actually).

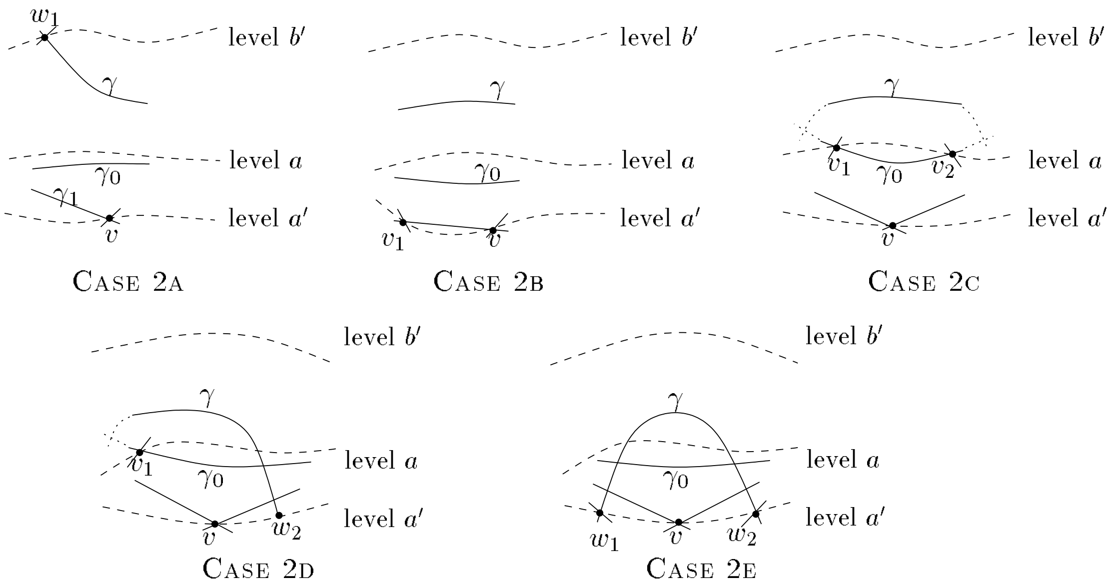

Figure 4: Proof of Lemma 2.6: Case 2.

– Case 2c: w_1 does not exist and v_1 has level a, and w_2 does not exist and v_2 has level a. Then (v_1, γ) or (v_2, γ) is an (a, b)-helper (otherwise γ would intersect γ_0 at least twice, at exceptional vertices).

– Case 2d: w_1 does not exist and v_1 has level a, and w_2 exists and has level a' (or vice versa).

Then (v_1, γ) is an (a, b)-helper (otherwise γ would intersect γ_0 at least twice, at exceptional vertices).

– Case 2e: w_1 exists and has level a', and w_2 exists and has level a'.

Then (w_1, γ_0) and (w_2, γ_0) are (a', b')-helpers (since γ intersects γ_0 at least twice, at exceptional vertices).

Each strong (a', a)-helper (and, in a similar manner, each strong (b, b')-helper) sends out $\Omega(b-a)$ charges. Each helper receives $O(b-a)$ charges: for example, given an (a, b)-helper (v_1, γ) from Case 1b, we can recover v uniquely, and there are at most $O(b-a)$ choices for γ_0; given an (a', b')-helper (w_2, γ_0) from Case 1d, we can recover γ uniquely, and there are at most $O(b-a)$ candidates for γ_2, and once γ_2 is known, we can recover v uniquely as well; the other cases are similar. This concludes the proof of the lemma. □

We now put the inequalities together. Let $i \le i'$. Applying Lemma 2.5(i) to $(a, b) = (k-i, k+i)$ and $(a, b) = (k-i', k+i')$ yields

$$t_i + t_{i'} \;\le\; 2i\Delta t_i + 2i'\Delta t_{i'} - \tfrac{1}{2}(|H_{k-i,k+i}| + |H_{k-i',k+i'}|) + O(i^2 + \lambda(n, i)).$$

Applying Lemma 2.5(ii) to $(a, b) = (k-i', k-i)$ and $(a, b) = (k+i, k+i')$ yields

$$t_{i'} - t_i \;\ge\; \tfrac{i'-i-1}{2}(\Delta t_i + \Delta t_{i'}) - \tfrac{1}{2}(|J_{k-i',k-i}| + |J_{k+i,k+i'}|).$$

Combining with Lemma 2.6 for $(a,b)=(k-i,k+i)$ and $(a',b')=(k-i',k+i')$ finally leads to a new inequality for all $i \le i' \le 1.4i$:

$$t_i + t_{i'} \;\le\; 2i\Delta t_i + 2i'\Delta t_{i'} - c\left[\tfrac{i'-i-1}{2}(\Delta t_i + \Delta t_{i'}) - (t_{i'} - t_i)\right] + O(i^2 + \lambda(n,i)), \qquad (2)$$

with the trivial base case $t_n = O(n^2)$.

"Solving" this "recurrence" (or, more accurately, proving asymptotically tight bounds on t_1 for this system of linear inequalities) does not appear easy, though. Intuitively, one can see that (2) should give a strictly better bound than before, by plugging in $t_i \approx n^{3/2} i^{1/2}$ and realizing the inequality fails, as we have done earlier. Formally, the appendix describes one (crude) way to prove that improved bounds indeed exist. For instance, for an almost pseudo-line arrangement with $\lambda(n,i) \le B$, Lemma A.2 allows us to conclude that $t_i = O\left(n^{3/2-\delta} i^{1/2+\delta} + i^{1/2+\delta}\sum_{j=i}^{n}(j^2+B)/j^{3/2+\delta}\right) = O(n^{3/2-\delta} i^{1+\delta} + B)$ for some absolute constant $\delta > 0$. In particular, $t_1 = O(n^{3/2-\delta} + B)$.

3 Other Consequences

Almost pseudo-segments. We have so far assumed that the given curves are graphs of total functions. If we are given an arrangement of pseudo-segments, we can turn it into an arrangement of curves by adding upward near-vertical rays to the endpoints. In doing so, we have created new lenses, but the number of lenses through each endpoint that are inside $L_{k-O(i),k+O(i)}$ is $O(i)$, so $\lambda_s(n,i)$ increases by only $O(ni)$. For instance, for an almost pseudo-segment arrangement, now with $\lambda_s(n,i) = O(ni + B)$, the inequality (2) still yields $t_1 = O(n^{3/2-\delta} + B)$.

Almost pseudo-lines/segments, again. We can combine the above result with Dey's to get a better bound for almost pseudo-line/segment arrangements when $B \ll n^{3/2-\delta}$. This hybrid approach is based on an idea from the previous paper [6]. We first turn the arrangement into an arrangement of $O(n+B)$ pseudo-segments, by introducing a cut point inside each of the $O(B)$ lenses. These pseudo-segments can be further cut into $N = O((n+B)\log n)$ *extendible* pseudo-segments by a technique of the author [5]. Recall that a level in the arrangement of n extendible pseudo-segments has complexity $\tau_1(n) = O(n^{4/3})$ by a generalization [19] of Dey's result [10]. Using [6, Lemma 5.1], we get

$$t_1 \;\le\; O\left(N + \frac{t_j}{j}\right)\frac{\tau_1(j)}{j} \;=\; O\left((n+B)\log n + \frac{n^{3/2-\delta}j^{1/2+\delta} + B}{j}\right) j^{1/3}.$$

Choose $j = n$ if $B \le n$; $j = 1$ if $B > n^{3/2-\delta}$; and $j = n^{3+\delta'}/(B\log n)^{2+\delta'}$ otherwise, for a sufficiently small constant $\delta' > 0$. Then $t_1 = O((n^{4/3} + n^{1+\delta''}B^{1/3-\delta''})\log^{1/3-\delta''} n + B)$ for some constant $\delta'' > 0$.

Incidentally, even for pseudo-segment arrangements with $B = 0$, the result improves previous ones in the logarithmic factors.

Pseudo-parabolas. For a pseudo-parabola arrangement, $\lambda(n,i) = O(n^{3/2} i^{1/2}\log(n/i))$ can be derived [6] from a result of Marcus and Tardos [15] related to cutting pseudo-parabolas (see also [20, 17, 4, 3] for earlier related work). The old inequality (1) then yielded $t_1 = O(n^{3/2}\log^2 n)$ only. With the improved inequality (2), we immediately obtain a better bound: by Lemma A.2, $t_1 = O\left(n^{3/2-\delta} + \sum_{j=1}^{n}(n^{3/2}j^{1/2}\log n)/j^{3/2+\delta}\right) = O(n^{3/2}\log n)$, since δ is a positive constant.

Pairwise-intersecting pseudo-parabolas. For the special case of pseudo-parabola arrangement where every pair intersects twice, $\lambda(n,i) = O(n^{4/3}i^{2/3})$ can be derived [6] from a result of Agarwal *et al.* [3]. In the previous paper [6], we have combined the old inequality with Dey's result [10], to prove a k-level upper bound of $O(n^{13/9}\log^{1/3} n)$. With the improved inequality (2), the same approach now gives $O(n^{13/9-\delta})$.

s-Intersecting curve segments for odd $s \geq 3$. In the previous paper [6], we have shown an extension of the inequality (1) for any constant $s > 1$ for any arrangement of curves:

$$t_i \ \leq \ si\Delta t_i + O(i^2 + \lambda_s(n,i)), \tag{3}$$

where $\lambda_s(n,i)$ denotes the number of *s-lenses* completely inside $L_{k-O(i),k+O(i)}$. Here, for two curves γ_1 and γ_2 that intersect more than s times, an s-lens refers to the portion of $\gamma_1 \cup \gamma_2$ between $s+1$ consecutive intersection points of $\gamma_1 \cap \gamma_2$.

We can adapt our proof to obtain the following improved inequality for any *odd* constant s:

$$t_i + t_{i'} \ \leq \ 2si\Delta t_i + 2si'\Delta t_{i'} - c[\tfrac{i'-i-1}{2}(\Delta t_i + \Delta t_{i'}) - (t_{i'} - t_i))] + O(i^2 + \lambda_s(n,i)). \tag{4}$$

Definition 2.1 on the basic charging argument remains the same—this time, at most $2s$ (instead of 2) points on a curve can be charged to a given vertex $v \in \partial V_{a,b}$. In Definition 2.4 on helpers, "at most one" is changed to "at most $2s-1$"; the definition of strong helpers is unchanged. In Lemma 2.5(i), the coefficient $b-a-1$ is now increased by a factor of s. The rest, including the main proof of Lemma 2.6, goes through without any extra complications, by the oddity of s.

For an arrangement of s-intersecting curves with $\lambda_s(n,i) = 0$, the improved inequality implies, by Lemma A.2, that $t_1 = O(n^{2-\frac{1}{2s}-\delta_s})$ for some absolute constant $\delta_s > 0$.

The same result holds for s-intersecting curve segments, by adding near vertical rays at endpoints as before.

s-Intersecting curve segments for $s \geq 4$ even. For an arrangement of s-intersecting curve segments for s even, $s-1$ is odd and $\lambda_{s-1}(n,i) = O(n^{2-\frac{1}{s+1}}i^{\frac{1}{s+1}})$ is known [5, 6]. The refined inequality (4) then implies $t_1 = O(n^{2-\frac{1}{2(s-1)}-\delta_s})$ for some absolute constant $\delta_s > 0$, assuming that $s \geq 4$ (so that $s+1 > 2(s-1)$).

Graphs of degree-s polynomials. For yet another example, the previous paper has obtained the current best result for graphs of degree-5 polynomials, using inequality (3) with $s = 3$ as an ingredient. With inequality (4), we can now reduce the previous bound of $O(n^{161/90}\log^{182/45} n)$ to $O(n^{161/90-\delta'})$ for some absolute constant $\delta' > 0$.

Similar improved bounds for polynomials of other higher degrees can also be obtained probably, had the previous paper worked out the details for these cases.

4 A Simple Proof for s-Intersecting Curves for Odd $s \geq 3$

In this section, we describe a much simpler way to beat the $O(n^{2-\frac{1}{2s}})$ bound for s-intersecting curves for odd $s \geq 3$. As an added advantage, we can even state concrete values for the exponent. The caveat is that the proof cannot be adapted to arbitrary s-intersecting curve segments or to any of other

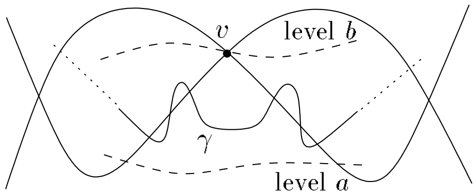

Figure 5: An example of a good vertex $v \in \partial V_{a,b}$ for $s = 3$. If $2s$ vertices on γ are charged to v, then we would get a contradiction.

cases considered in the previous sections (though the proof still works for *extendible* s-intersecting curve segments).

We only need to use a variant of the original charging argument from Section 2.1, without lenses (i.e., redefining $\Lambda_{a,b} = \emptyset$). The key new idea can be illustrated through the example in Figure 5: if $s = 3$ and $v \in \partial V_{a,b}$ is a "middle" vertex of a pair of curves intersecting 3 times, then we cannot have $2s$ vertices on a curve charged to v.

Inspired by this example, we make the following general definition:

Definition 4.1 Given two curves γ_1 and γ_2, an intersection point v is *good* if the number of intersections of γ_1 and γ_2 strictly to the right of v is odd, and *bad* otherwise.

Let $G_{a,b}$ (resp. $\partial G_{a,b}$) be the set of good vertices in $V_{a,b}$ (resp. $\partial V_{a,b}$) and $B_{a,b}$ (resp. $\partial B_{a,b}$) be the set of bad vertices in $V_{a,b}$ (resp. $\partial V_{a,b}$).

Lemma 4.2 *For any $0 \le \alpha \le 1$ and any arrangement of s-intersecting curves, we have*

$$|B_{a,b}| + (1-\alpha)|G_{a,b}| \;\le\; (b-a-1)\left[(s - \tfrac{s-1}{2}\alpha)|\partial B_{a,b}| + (s - \tfrac{1}{2} - \tfrac{s-1}{2}\alpha)|\partial G_{a,b}|\right] + O((b-a)^2).$$

Proof: Recall that each non-exceptional vertex $u \in V_{a,b}$ is charged to exactly two vertices in $\partial V_{a,b}$. Assign weight 1 to each bad vertex and weight $1 - \alpha$ to each good vertex.

Take a vertex $v \in \partial V_{a,b}$ defined by two curves γ_1 and γ_2, say with γ_1 below γ_2 immediately to the left of ℓ_v. Take a curve γ that has level in $[a, b]$ at ℓ_v (excluding the two curves defining v). If v is bad, then at most $2s$ vertices on γ can be charged to v; among these vertices, at most $2 \cdot \frac{s+1}{2}$ are bad and at most $2 \cdot \frac{s-1}{2}$ are good, for a total weight of at most $2s - (s-1)\alpha$. On the other hand, if v is good, then at most $2s - 1$ vertices on γ can be charged to v (otherwise, at $x = \infty$, we would have γ_1 above γ and γ above γ_2, making v bad). Among these vertices, at most $\frac{s+1}{2} + \frac{s-1}{2}$ are bad and at most $\frac{s-1}{2} + \frac{s-1}{2}$ are good, for a total weight of at most $2s - 1 - (s-1)\alpha$.

There are at most $O(b-a)^2$ exceptional vertices. The lemma follows. □

We now choose the parameter $\alpha = \alpha_s > 0$ so that

$$(1-\alpha_s)(s - \tfrac{s-1}{2}\alpha_s) = s - \tfrac{1}{2} - \tfrac{s-1}{2}\alpha_s.$$

Redefining $t_i = |B_{k-i,k+i}| + (1-\alpha_s)|G_{k-i,k+i}|$ and $\Delta t_i = t_{i+1} - t_i$, we then get the new inequality:

$$t_i \le (2s - (s-1)\alpha_s)\Delta t_i + O(i^2).$$

This yields $t_1 = O(n^{2 - \frac{1}{2s-(s-1)\alpha_s}})$ (e.g., by Lemma A.1).

The value of α_s can be found by solving the above quadratic equation, giving $\alpha_s = \frac{2}{(s-1)^2}\left[s-\sqrt{s^2-2\left(\frac{s-1}{2}\right)^2}\right]$. For example, for $s=3$, we have $\alpha_3 = \frac{1}{2}(3-\sqrt{7}) \approx 0.177$, yielding a k-level bound of $O(n^{2-1/(3+\sqrt{7})}) = O(n^{1.823})$. For $s=5$, we have $\alpha_5 = \frac{1}{8}(5-\sqrt{17}) \approx 0.109$, yielding a k-level bound of $O(n^{2-2/(15+\sqrt{17})}) = O(n^{1.896})$. Note that $(s-1)\alpha_s$ tends to $2-\sqrt{2} \approx 0.585$ as $s \to \infty$.

5 Additional Observations

We close with some further observations about the k-level problem that are independent of the preceding parts of this paper. While the observations are not deep, they do have interesting implications not noted before.

The first has to do with the probabilistic method. This method (specifically, Clarkson–Shor style of random sampling [9]) has been successful in obtaining asymptotically tight bounds for the ($\leq k$)-level problem, as well as a special case of the k-set problem for points in convex position. However, we are not aware of any explicit invocation of the method to the general k-level problem (except for small k [1]).

Fix a class of curves. Let $\tau(n,i)$ denote the maximum number of vertices of level in $[k-i, k+i]$ over all k and arrangements of a (multi)set of at most n curves in the class. We may assume $\tau(n,i) = \Omega(ni)$.

Lemma 5.1 $\tau(n, n/\sqrt{r}) \leq c_1(n/r)^2\,\tau(r,\sqrt{r})$ *for some constant* c_1.

Proof: Let S be the given set of n curves. Let V_S be the set of vertices v in the arrangement of S with $|\mathrm{lev}_S(v) - k| \leq n/\sqrt{r}$, where $\mathrm{lev}_S(v)$ denotes the level of v.

Form a multiset $R = \{\gamma_1, \ldots, \gamma_r\}$ where each γ_i is chosen independently and uniformly at random from S. (We could also use sampling without replacement, but calculations would be messier.) Let V_R be the set of vertices v in the arrangement of R with $|\mathrm{lev}_R(v) - rk/n| \leq c_0\sqrt{r}$ for some sufficiently large constant c_0.

Fix a vertex $v \in V_S$, defined by curves γ_1 and γ_2. Now, $\mathrm{lev}_R(v)$ is the sum of r independent 0-1 random variables each with mean $\mathrm{lev}_S(v)/n$. Thus, $\mathrm{lev}_R(v)$ has mean $\mu := r\mathrm{lev}_S(v)/n$ and variance $\sigma^2 := r(\mathrm{lev}_S(v)/n)(1-\mathrm{lev}_S(v)/n)$. By Chebyshev's inequality, with probability at least 3/4, we have $|\mathrm{lev}_R(v) - \mu| \leq 2\sigma$, implying $|\mathrm{lev}_R(v) - r\mathrm{lev}_S(v)/n| = O(\sqrt{r})$, further implying $|\mathrm{lev}_R(v) - rk/n| = O(\sqrt{r})$.

For any fixed $i,j \in \{1,\ldots,r\}$, the probability that $r_i = \gamma_1$, $r_j = \gamma_2$, and $|\mathrm{lev}_{R-\{r_i,r_j\}}(v) - rk/n| \leq c_0\sqrt{r}$ is $\Omega(1/n^2)$. Therefore, the probability that $v \in V_R$ is $\Omega(r^2/n^2)$. We conclude that $E[|V_R|] = \Omega(r^2/n^2)|V_S|$, i.e., $|V_S| = O(n/r)^2 E[|V_R|] = O(n/r)^2\,\tau(r, O(\sqrt{r}))$. The lemma holds after readjusting r by a constant factor. $\Box$

Another relationship is noted in [6, Lemma 5.1] and proved by divide-and-conquer:

Lemma 5.2 *For any* $j = \Omega(i)$, *we have* $\tau(n,i) \leq c_2\tau(n,j)\tau(j,i)/j^2$ *for some constant* c_2.

Combining these two relationships, we get for some absolute constant c:

$$\tau(n,i) \;\leq\; \frac{c(n/r)^2\tau(r,\sqrt{r})\tau(n/\sqrt{r},i)}{(n/\sqrt{r})^2} \;=\; \frac{c\tau(r,\sqrt{r})}{r}\tau(n/\sqrt{r},i). \tag{5}$$

for all $n = \Omega(\sqrt{r}i)$.

If we plug in the known bound $\tau(n,i) = O(n^{4/3}i^{2/3})$ for lines [10], the above gives no improvements, disappointingly.

Still, if we fix the parameter r and expand (5) as a recurrence, using the trivial base case $\tau(O(i),i) = O(i^2)$, we can get the following:

$$\tau(n,i) \;\le\; O(n/i)^{\log_{\sqrt{r}}(c\tau(r,\sqrt{r})/r)}i^2. \tag{6}$$

We mention two implications of this bound for the standard case of lines, i.e., the original k-set problem:

- If Erdős *et al.*'s original conjecture [12] is true, i.e., the upper bound $O(n^{1+\varepsilon})$ holds for any fixed $\varepsilon > 0$, then $\tau(r,\sqrt{r}) = O(r^{3/2+\varepsilon})$. Fix a constant $0 < \delta < 1/3$. Then a sufficiently large constant r would guarantee that $\log_{\sqrt{r}}(c\tau(r,\sqrt{r})/r) < 1+\delta$, and this fact would be demonstrable by a finite proof since there are "only" a constant number of combinatorially different arrangement of r lines. Thus, (6) would imply a proof of $\tau(n,1) = O(n^{1+\delta})$.

  Unfortunately, the constants involved are too big for this proof approach to be feasible in practice—back-of-the-envelope calculations seem to indicate that r cannot be less than a million! Still, the observation suggests the possibility that better k-set results may be provable by some sufficiently lengthy form of case analysis. (The situation is somewhat analogous akin to the matrix multiplication problem over rings, where better bounds for a large constant input size would imply better asymptotic bounds.)

- A weaker conjecture, that $\tau(r,\sqrt{r}) = o(r^{5/3})$, would suffice to beat Dey's result, since a sufficiently large constant r would guarantee that $\log_{\sqrt{r}}(c\tau(r,\sqrt{r})/r) < 4/3$. Then (6) would imply $\tau(n,i) = O(n^{4/3-\delta}i^{2/3+\delta})$ for some constant $\delta > 0$.

  Viewed another way: if someone could hypothetically prove $\tau(n,\sqrt{n}) = O(n^{5/3}/\log^* n)$, say, then we would automatically have $\tau(n,\sqrt{n}) = O(n^{5/3-\delta})$ for some constant $\delta > 0$.

We end with one final thought. It has been pointed out that most of the known proofs for the k-set problem (including Lovász's and Dey's, but not ours) use only limited properties about k-sets (*antipodality* [18] in primal space, or *concave-chain decomposability* [10] in the dual), and that $n^{4/3}$ is the best possible under these approaches in 2-d. The moral of this whole paper is that there are potentially many other properties that can be exploited, so an eventual proof of Erdős *et al.*'s conjecture might not be as hopeless as it would seem.

References

[1] P. K. Agarwal, B. Aronov, T. M. Chan, and M. Sharir. On levels in arrangements of lines, segments, planes, and triangles. *Discrete Comput. Geom.*, 19:315–331, 1998.

[2] P. K. Agarwal, B. Aronov, and M. Sharir. On levels in arrangements of lines, segments, planes, and triangles. In *Proc. 13th ACM Sympos. Comput. Geom.*, pages 30–38, 1997.

[3] P. K. Agarwal, E. Nevo, J. Pach, R. Pinchasi, M. Sharir, and S. Smorodinsky. Lenses in arrangements of pseudo-circles and their applications. *J. ACM*, 51:139–186, 2004.

[4] B. Aronov and M. Sharir. Cutting circles into pseudo-segments and improved bounds for incidences and complexity of many faces. *Discrete Comput. Geom.*, 28:475–490, 2002.

[5] T. M. Chan. On levels in arrangements of curves. *Discrete Comput. Geom.*, 29:375–393, 2003.

[6] T. M. Chan. On levels in arrangements of curves, II: a simple inequality and its consequences. *Discrete Comput. Geom.*, 34:11–24, 2005.

[7] T. M. Chan. On levels in arrangements of surfaces in three dimensions. In *Proc. 16th ACM-SIAM Sympos. Discrete Algorithms*, pages 232–240, 2005. Full version submitted to *Discrete Comput. Geom.*

[8] T. M. Chan. On the bichromatic k-set problem. To appear in *Proc. 19th ACM-SIAM Sympos. Discrete Algorithms*, 2008.

[9] K. L. Clarkson and P. W. Shor. Applications of random sampling in computational geometry, II. *Discrete Comput. Geom.*, 4:387–421, 1989.

[10] T. K. Dey. Improved bounds on planar k-sets and related problems. *Discrete Comput. Geom.*, 19:373–382, 1998.

[11] H. Edelsbrunner and E. Welzl. On the number of line separations of a finite set in the plane. *J. Combin. Theory Ser. A*, 38:15–29, 1985.

[12] P. Erdős, L. Lovász, A. Simmons, and E. Straus. Dissection graphs of planar point sets. In *A Survey of Combinatorial Theory* (J. N. Srivastava, ed.), North-Holland, Amsterdam, Netherlands, pages 139–154, 1973.

[13] D. Gusfield. Bounds for the parametric minimum spanning tree problem. In *Proc. Humboldt Conf. on Graph Theory, Combinatorics, and Computing*, Utilitas Mathematica, pages 173–183, 1979.

[14] L. Lovász. On the number of halving lines. *Ann. Univ. Sci. Budapest, Eötvös, Sec. Math.*, 14:107–108, 1971.

[15] A. Marcus and G. Tardos. Intersection reverse sequences and geometric applications. *J. Combin. Theory Ser. A*, 113:675–691, 2006.

[16] J. Pach, W. Steiger, and E. Szemerédi. An upper bound on the number of planar k-sets. *Discrete Comput. Geom.*, 7:109–123, 1992.

[17] R. Pinchasi and R. Radoičić. Topological graphs with no self-intersecting cycle of length 4. In *Proc. 19th ACM Sympos. Comput. Geom.*, pages 98–103, 2003.

[18] M. Sharir, S. Smorodinsky, and G. Tardos. An improved bound for k-sets in three dimensions. *Discrete Comput. Geom.*, 26:195–204, 2001.

[19] H. Tamaki and T. Tokuyama. A characterization of planar graphs by pseudo-line arrangements. In *Proc. 8th Int. Sympos. Algorithms and Computation*, Lect. Notes Comput. Sci., vol. 1350, Springer-Verlag, pages 123–132, 1997.

[20] H. Tamaki and T. Tokuyama. How to cut pseudoparabolas into segments. *Discrete Comput. Geom.*, 19:265–290, 1998.

[21] G. Tóth. Point sets with many k-sets. *Discrete Comput. Geom.*, 26:187–194, 2001.

A Appendix: Solving the "Recurrences"

In this appendix, we include a quick analysis of the inequalities that arise in the paper. We will not try to determine the best constants possible.

Below, the notation $\preceq$ hides factors of the form $1 + O(1/i)$. Floors and ceilings are ignored for the sake of simplicity. The given function $f(\cdot)$ are assumed to be well-behaved, in the sense that $i = \Theta(i')$ should imply $f(i) = \Theta(f(i'))$.

Lemma A.1 *Fix a constant s. Suppose $t_i \le si\Delta t_i + f(i)$ for all $i \le m$. Then*

$$t_i \preceq \left(\frac{i}{m}\right)^{1/s} t_m + O\left(\sum_{j=i}^{m} \left(\frac{i}{j}\right)^{1/s} \frac{f(j)}{j}\right).$$

Proof: Rewrite $t_j \le sj(t_{j+1} - t_j) + f(j)$ as $t_j \le \left(1 - \frac{1}{sj+1}\right) t_{j+1} + O\left(\frac{f(j)}{j}\right)$. The lemma follows by expansion, since $\prod_{\ell=i}^{j}\left(1 - \frac{1}{s\ell+1}\right) \le \exp\left(-\sum_{\ell=i}^{j} \frac{1}{s\ell+1}\right) \le \exp\left(-\frac{1}{s}\ln\frac{j}{i} + O(1/i)\right) \preceq \left(\frac{i}{j}\right)^{1/s}$. □

Lemma A.2 *Fix constants s and c. Suppose $t_i \le si\Delta t_i + f(i)$ and*

$$t_i + t_{i'} \le si\Delta t_i + si'\Delta t_{i'} - c\left[\tfrac{i'-i-1}{2}(\Delta t_i + \Delta t_{i'}) - (t_{i'} - t_i)\right] + f(i)$$

for all $i \le i' \le 1.4i$. Then there exists a constant $\delta > 0$ (depending only s and c) such that for all $i \le m$,

$$t_i = O\left(\left(\frac{i}{m}\right)^{1/s+\delta} t_m + \sum_{j=i}^{m} \left(\frac{i}{j}\right)^{1/s+\delta} \frac{f(j)}{j}\right).$$

Proof: We will show that for some constant $\delta > 0$,

$$t_i \preceq \left(\frac{1}{1.4}\right)^{1/s+\delta} t_{1.4i} + O(f(i)). \tag{7}$$

The lemma would then follow by expansion, after rewriting $f(i)$ as $\Theta\left(\sum_{j=i}^{1.4i}\left(\frac{i}{j}\right)^{1/s+\delta}\frac{f(j)}{j}\right)$.

To prove (7), it suffices to show that for some constant $\varepsilon_0 > 0$,

$$t_j \preceq \left(\left(\frac{j}{j'}\right)^{1/s} - \varepsilon_0\right) t_{j'} + O(f(i)) \qquad \text{for some } j, j' \in [i, 1.4i] \text{ with } j < j'. \tag{8}$$

Indeed this would imply, by Lemma A.1, that

$$t_i \preceq \left(\frac{i}{j}\right)^{1/s} \left(\left(\frac{j}{j'}\right)^{1/s} - \varepsilon_0\right) \left(\frac{j'}{1.4i}\right)^{1/s} t_{1.4i} + O(f(i)).$$

The coefficient is bounded by a constant strictly less than $\left(\frac{1}{1.4}\right)^{1/s}$, so (7) holds for a sufficiently small δ.

To prove (8), we first dispose of some easy cases. If $t_j \le (s-\varepsilon)j\Delta t_j$ for all $j \in [i, 1.1i]$ for some constant $\varepsilon > 0$, then by Lemma A.1, $t_i \le \left(\frac{1}{1.1}\right)^{1/(s-\varepsilon)} t_{1.1m}$, implying (8) for a sufficiently small ε_0. Similarly, if $t_j \le (s-\varepsilon)j\Delta t_j$ for all $j \in [1.3i, 1.4i]$, then (8) holds.

Thus, we may assume the existence of $j \in [i, 1.1i]$ and $j' \in [1.3i, 1.4i]$ such that $t_j > (s-\varepsilon)j\Delta t_j$ and $t'_j > (s-\varepsilon)j'\Delta t_{j'}$.

Write $j' = (1+\beta)j$ where $0.18 \le \beta \le 0.4$. Then

$$\begin{aligned} t_j + t_{j'} &\preceq sj\Delta t_j + s(1+\beta)j\Delta t_{j'} - c[(\beta/2)j(\Delta t_j + \Delta t_{j'}) - (t_{j'} - t_j)] + f(i) \\ &\le \frac{s - c\beta/2}{s-\varepsilon} t_j + \frac{s(1+\beta) - c\beta/2}{(s-\varepsilon)(1+\beta)} t_{j'} + c(t_{j'} - t_j) + f(i), \end{aligned}$$

implying

$$t_j \;\leq\; \frac{(c-1)(s-\varepsilon)+s-\frac{c\beta/2}{1+\beta}}{(1+c)(s-\varepsilon)-s+c\beta/2}\, t_{j'} \;+\; O(f(i)) \;\leq\; \left(\frac{s-\frac{\beta/2}{1+\beta}}{s+\beta/2}+O(\varepsilon)\right) t_{j'} \;+\; O(f(i)).$$

To complete the proof of (8), it suffices to verify that the above coefficient is bounded by $\left(\frac{1}{1+\beta}\right)^{1/s}-\varepsilon_0$ for some ε_0.

Indeed, let $g(\beta) := \frac{2s-\frac{\beta}{1+\beta}}{2s+\beta}$ and observe that the following strict inequality always holds for any $\beta > 0$:

$$g(\beta) \;<\; \left(\frac{1}{1+\beta}\right)^{1/s},$$

or equivalently, $h(\beta) := 2s(1+\beta)^{1/s} - \beta(1+\beta)^{1/s-1} - 2s - \beta < 0$. Confirming this inequality is a straightforward calculus exercise: $h(0) = 0$ and

$$\begin{aligned} h'(\beta) &= (1+\beta)^{1/s-1} + (1-1/s)\beta(1+\beta)^{1/s-2} - 1 \\ &= (1+\beta)^{1/s-2} \cdot [1 + (2-1/s)\beta - (1+\beta)^{2-1/s}] \;<\; 0. \end{aligned}$$

Consequently, $\min_{\beta\in[0.18,0.4]}\left[\left(\frac{1}{1+\beta}\right)^{1/s} - g(\beta)\right]$ is a strictly positive constant. Choosing ε sufficiently small, we are done. □