Chapter II: Continued Fractions

1. The Real Continued Fraction Algorithm.

Definition II.1.1.

A real (simple) continued fraction (CF) is an expression of the form

$$a_0 + 1/(a_1 + 1/(a_2 + \dots))$$

which may or may not terminate.

CF's are the subject of a vast literature because they have many interesting and useful properties. The traditional development of CF's is given, for example, in Hardy and Wright [8]. We will use some non-standard notation in our development to make our meaning more precise. The distinctions we will make below will be especially useful in section II.3.

Definition II.1.2.

$$\mathrm{val}(a_0, a_1, \dots, a_n) = a_0 + 1/(a_1 + \dots 1/a_n)$$

$$\mathrm{val}(a_0, a_1, \dots) = \lim_{n \to \infty} \mathrm{val}(a_0, \dots, a_n)$$

if the limit exists.

Thus val can be considered to be the "value" of the continued fraction, and is a function which maps (finite or infinite) sequences to real numbers.

Definition II.1.3. (CF algorithm)

$$\mathrm{cf}(x) = (a_0, a_1, \dots a_n, \dots)$$

where the a's are defined as follows:

$$x_0 = x$$

$$a_n = \mathrm{fl}(x_n),\ x_{n+1} = 1/(x_n - a_n) \quad (n \geq 0)$$

This expansion terminates with a_n if and only if there exists some N such that $a_N = x_N$. Otherwise, the result is an infinite sequence.

Thus cf is a function which maps real numbers to (finite or infinite) sequences.

Note: cf may be thought of as a "computer program" which takes a real number as input and outputs a sequence of integers, which may or may not terminate.

Traditionally, statements of theorems on continued fractions have not distinguished between the roles of cf and val. In a statement like

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots]$$

(traditional notation), it is not clear whether the statement is that the evaluation of the right side equals the left side, <u>or</u> whether the continued fraction algorithm applied to the left side produces the right side. The distinction between cf and val we have made clarifies this.

<u>Theorem</u> <u>II</u>.<u>1</u>.<u>1</u>.

$$\mathrm{val}(a_0, a_1, \ldots a_{n-1}, a_n + 1/x) = \mathrm{val}(a_0, a_1, \ldots a_n, x)$$

This is just the definition of val.

Another easily verified consequence of the definition is the following

Theorem II.1.2.

$$\mathrm{val}(a_0, a_1, \ldots a_n, 0, a_{n+1}, \ldots)$$
$$= \mathrm{val}(a_0, a_1, \ldots, a_n + a_{n+1}, \ldots)$$

We will write $|cf(x)| = N$ if, as in Definition II.1.3, the algorithm terminates with a_N. If the algorithm does not terminate, we write $|cf(x)| = \infty$.

Theorem II.1.3.

If x is rational, then $|cf(x)| < \infty$.

Proof.

Suppose x is rational. Then $x = r/s$, where $r,s \in \mathbb{Z}$, $s \geq 1$. Then application of the cf algorithm gives

$$x_0 = r/s,\ a_n = fl(x_n),\ x_{n+1} = 1/(x_n - a_n).$$

Now define

$$r_0 = r;\quad s_0 = s;\quad b_n = fl(r_n/s_n)$$
$$r_{n+1} = s_n;\quad s_{n+1} = r_n - b_n s_n = res(s_n, r_n).$$

where by res we mean the residue function with respect to floor.

Then I claim

$$a_n = b_n \text{ and } x_n = r_n/s_n.$$

Use induction. We find

$$x_0 = r_0/s_0 = r/s = x.$$

$$b_0 = fl(r_0/s_0) = fl(x_0) = a_0.$$

Now assume true for n. We find

$$\begin{aligned} x_{n+1} &= 1/(x_n - a_n) \\ &= 1/(r_n/s_n - fl(r_n/s_n)) \\ &= s_n/(r_n - s_n fl(r_n/s_n)) \\ &= r_{n+1}/s_{n+1}. \end{aligned}$$

Also

$$a_{n+1} = fl(x_{n+1}) = fl(r_{n+1}/s_{n+1}).$$

Now cf terminates iff $x_N = a_N$ for some N. Since the two methods of expansion above are the same, this condition is $b_N = r_N/s_N$, i. e., $s_{N+1} = 0$. From Theorem I.2.12 we see

$$0 \le s_{n+1} = res(s_n, r_n) < s_n.$$

Assume s_n is never 0. Then s_0, s_1, s_2, ... is an infinite strictly decreasing sequence of positive integers, which is clearly impossible. Hence cf terminates at some $s_{N+1} = 0$.

<u>Theorem</u> <u>II</u>.<u>1</u>.<u>4</u>.

Suppose $|cf(x)| < \infty$. Then $val(cf(x)) = x$.

<u>Proof</u>.

Since $|cf(x)| < \infty$, $|cf(x)| = N$. Write

$$cf(x) = (a_0, a_1, \ldots, a_N).$$

From Definition II.1.3, $x_n = a_n + 1/x_{n+1}$ for $n < N$.

Also, $x_N = a_N$.

Hence we find

$$\begin{aligned} x &= x_0 \\ &= a_0 + 1/x_1 \\ &= a_0 + 1/(a_1 + 1/x_2) \\ &\cdot \\ &\cdot \\ &\cdot \\ &= a_0 + 1/(a_1 + \ldots 1/(a_{N-1} + 1/x_N) \\ &= a_0 + 1/(a_1 + \ldots 1/(a_{N-1} + 1/a_N) \\ &= \mathrm{val}(a_0, a_1, a_2, \ldots a_N). \end{aligned}$$

Theorem II.1.5.

If x is irrational, then $|cf(x)| = \infty$.

Proof.

Asssume that $|cf(x)| = N < \infty$. Then $\mathrm{val}(cf(x)) = x$ by the previous theorem. But $\mathrm{val}(a_0, a_1, \ldots a_n)$ is clearly rational since we are doing a finite number of field operations on $\mathbb{Q}$. Hence x is rational, contrary to assumption. Thus we have a contradiction and $|cf(x)| = \infty$.

Theorem II.1.6.

$|cf(x)| < \infty$ iff x is rational.

$|cf(x)| = \infty$ iff x is irrational.

Proof.

This is just a combination of Theorems II.1.3 and II.1.5.

Now we will derive some of the traditional results on real continued fractions. For example, see Hardy and Wright [8].

Theorem II.1.7.

Let $p_n/q_n = \mathrm{val}(a_0, a_1, \ldots a_n)$.

Then if

$$p_{-2} = 0; \quad p_{-1} = 1$$
$$q_{-2} = 1; \quad q_{-1} = 0$$

we have

$$p_n = a_n p_{n-1} + p_{n-2}$$
$$q_n = a_n q_{n-1} + q_{n-2}$$

for $n \geq 0$.

Proof. (Induction)

The above formulas are easily checked for $n = 0$. Assume true for n. Then we find

$$\mathrm{val}(a_0, a_1, \ldots a_n, a_{n+1}) = \mathrm{val}(a_0, a, \ldots a_n + 1/a_{n+1})$$
$$= ((a_n + 1/a_{n+1})p_{n-1} + p_{n-2})/((a_n + 1/a_{n+1})q_{n-1} + q_{n-2})$$
$$= (a_{n+1}(a_n p_{n-1} + p_{n-2}) + p_{n-1})/(a_{n+1}(a_n q_{n-1} + q_{n-2}) + q_{n-1})$$
$$= (a_{n+1}p_n + p_{n-1})/(a_{n+1}q_n + q_{n-1}).$$

Theorem II.1.8.

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}.$$

Proof.

$$p_nq_{n-1} - p_{n-1}q_n = (a_np_{n-1} + p_{n-1})q_{n-1} - p_{n-1}(a_nq_{n-1} + q_{n-1})$$
$$= -(p_{n-1}q_{n-2} - p_{n-2}q_{n-1})$$

Repeating this argument gives

$$p_nq_{n-1} - p_{n-1}q_n = (-1)^{n-1}(p_1q_0 - p_0q_1)$$
$$= (-1)^{n-1}.$$

Theorem II.1.9.

$$\gcd(p_n, q_n) = 1.$$

By gcd we mean the greatest positive common divisor.

Proof.

For by Theorem II.1.8, if $x|p_n$ and $x|q_n$, then $x|p_nq_{n-1}$ and $x|p_{n-1}q_n$. Hence $x|p_nq_{n-1} - p_{n-1}q_n = (-1)^{n-1}$. Since x is positive, $x = 1$.

2. The Real GCD Algorithm.

The sequences in the proof of Theorem II.1.3. can be rewritten

$$r_0 = a_0 s_0 + s_1, \quad s_1 = res(s_0, r_0)$$
$$s_0 = r_1 = a_1 s_1 + s_2, \quad s_2 = res(s_1, r_1)$$
$$\vdots$$
$$s_{n-2} = r_{n-1} = a_{n-1} s_{n-1} + s_n, \quad s_n = res(s_{n-1}, r_{n-1})$$
$$s_{n-1} = r_n = a_n s_n + s_{n+1}, \quad s_{n+1} = 0.$$

Then we have the following

Theorem II.2.1.

$$s_n = gcd(r,s).$$

Proof.

$s_n | r_n$ since $r_n = a_n s_n$. But $s_{n-1} = r_n$; hence $s_n | s_{n-1}$. But then $s_n | r_{n-1}$ since $r_{n-1} = a_{n-1} s_{n-1} + s_n$. Continuing in this fashion, $s_n | r_{n-2}$, ... $s_n | r_1 = s_0 = s$, and $s_n | r_0 = r$. Hence s_n is a divisor of r and s. It remains to show it is a greatest divisor.

Suppose $t|r$ and $t|s$. Then $t|r_0$, $t|s_0 \implies t|s_1 \implies t|s_2 \implies \ldots \implies t|s_{n-2} \implies t|s_{n-1} \implies t|s_n$. Hence s_n is indeed the greatest common divisor.

Thus we see that the gcd and cf algorithms are essentially the same--just the focus is different. In one, we are interested in remainders; in the other, the quotients.

3. The Complex Continued Fraction Algorithm.

At this point, we would like to generalize the continued fraction algorithm of section II.1 to the complex plane.

We do this by modifying Definition II.1.3 to use the cfl (complex floor) function in place of fl. Since cfl(x) = fl(x) if x is real, this change does not alter the results we have obtained for real CF's.

The proof of Theorem II.1.3 remains essentially unchanged, with one small modification. From Theorem I.1.5 we see that

$$|res(w,z)| < |w|$$

so $\{|s_0|^2, |s_1|^2, \ldots\}$ is an infinite sequence of strictly decreasing positive integers, which leads to the desired contradiction.

Similarly, the proofs of Theorems II.1.4 - II.1.9 go through without change. Thus we have

Theorem II.3.1

(a) z is a rational complex number iff $|cf(z)|<\infty$.

(b) z is an irrational complex number iff $|cf(z)|=\infty$.

(c) z rational ==> val(cf(z)) = z.

(d) If $p_n/q_n = val(a_0, a_1, \ldots a_n)$, and we put $p_{-2} = 0$; $p_{-1} = 1$; $q_{-2} = 1$; $q_{-1} = 0$, then for $n \geq 0$, $p_n = a_n p_{n-1} + p_{n-2}$ and

$$q_n = a_n q_{n-1} + q_{n-2}.$$

(e) $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$.

(f) $\gcd(p_n, q_n) = 1$.

We would like to obtain a complete description of complex continued fractions; that is, we would like to be able to describe the outputs of the cf algorithm.

Our first step in this direction is the following

Theorem II.3.2.

Let $cf(x) = (a_0, a_1, \ldots)$. Then $Re(a_j) \geq Im(a_j)$ for $j \geq 1$. Also, $a_j \neq 0$ for $j \geq 1$.

Proof.

First we need the following two lemmas.

Lemma II.3.1.

$$cfl(z-cfl(z)) = 0.$$

Proof.

Let $n = cfl(z)$. Then we find

$$cfl(z-n) = cfl(z) - n = cfl(z) - cfl(z) = 0.$$

Lemma II.3.2.

If $cfl(z) = 0$, then $Re(1/z) \geq Im(1/z)$.

Proof.

By examining Figure 5, we see that if $cfl(z) = 0$, then $Re(z) + Im(z) \geq 0$, i.e. $Re(z) \geq -Im(z)$. Now if $z = a+bi$, then $1/z = (a-bi)/(a^2+b^2) = c+di$, where $c = a/(a^2+b^2)$, $d = -b/(a^2+b^2)$. But $a \geq -b$, so $c \geq d$. Hence $Re(1/z) \geq Im(1/z)$.

Now to return to the proof of Theorem II.3.2. By Lemma II.3.1, we see that $fl(x_j - cfl(x_j)) = 0$. Hence if $z' = x_j - cfl(x_j)$, then $Re(1/z') \geq Im(1/z')$. By Theorem I.5.8, $Re(cfl(1/z')) \geq Im(cfl(1/z'))$, i. e., $Re(a_j) \geq Im(a_j)$.

To show $a_j \neq 0$, we have $|x_{j-1} - cfl(x_{j-1})| < 1$; hence we have $|1/(x_{j-1} - cfl(x_{j-1}))| < 1$. Thus $|x_j| > 1$. But $a_j = cfl(x_j)$. Assume $a_j = 0$. Then $|x_j - a_j| > 1$. But in fact we have $|x_j - cfl(x_j)| < 1$. Hence we have a contradiction and $a_j \neq 0$.

Theorem II.3.2 is useful because it immediately lets us say, for example, that

$$(i,\ 2i,\ 3i,\ 4i,\ \ldots.)$$

can never be generated by the cf algorithm.

However, it does not go far enough. For example, I claim that the cf algorithm will never generate an expansion that begins

$$(1,\ 2+2i,\ -3i,\ \ldots.).$$

For assume such an expansion is possible. Then $a_0 = 1$, $a_1 = 2+2i$, and $a_2 = -3i$. Then since $a_0 = cfl(x_0)$, x_0 lies in

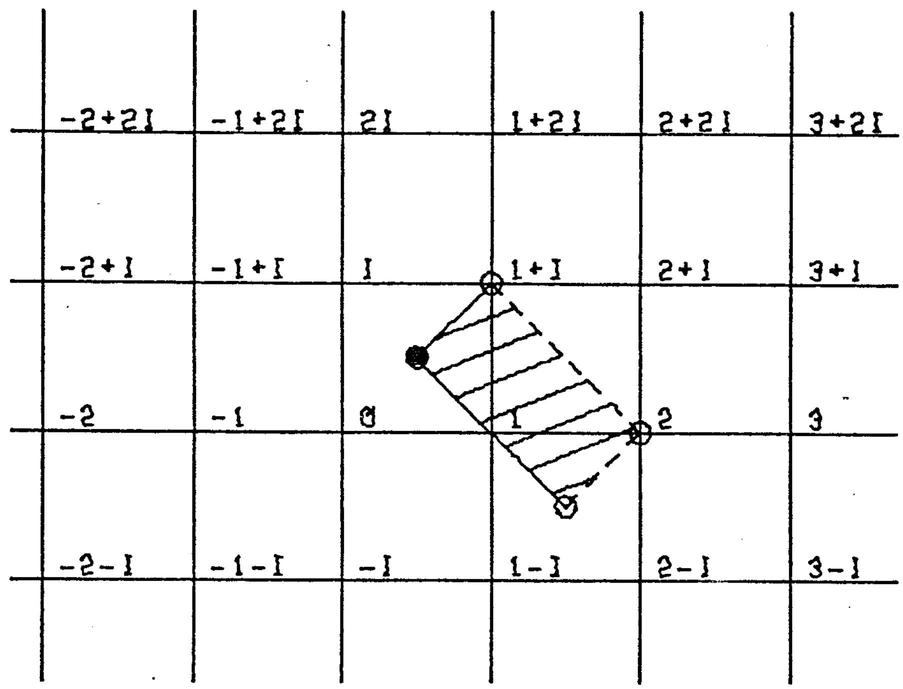

Figure 7

the shaded region sketched in Figure 7. Hence $x_0 - a_0$ lies in the shaded region of Figure 8. Hence $x_1 = 1/(x_0 - a_0)$ must lie in the region formed by taking the reciprocal of each element in the shaded region of Figure 8. Thus x_1 lies in the shaded area of Figure 9.

Now if $a_1 = fl(x_1) = 2+2i$, then x_1 must lie in the shaded area of Figure 10. Hence $x_1 - a_1$ lies in the shaded area of Figure 11. Hence $x_2 = 1/(x_1 - a_1)$ lies in the shaded area of

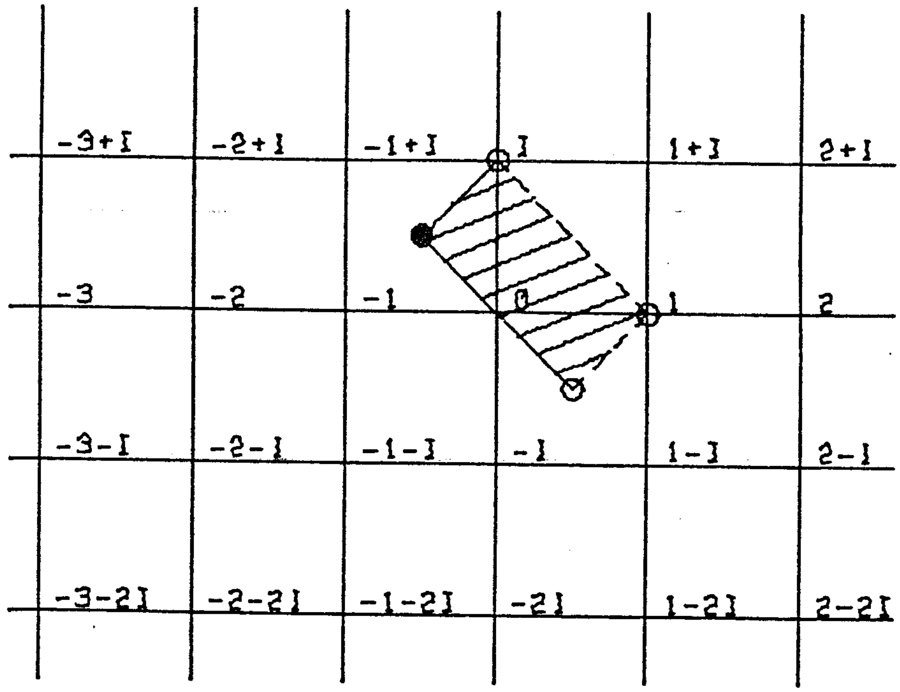

Figure 8

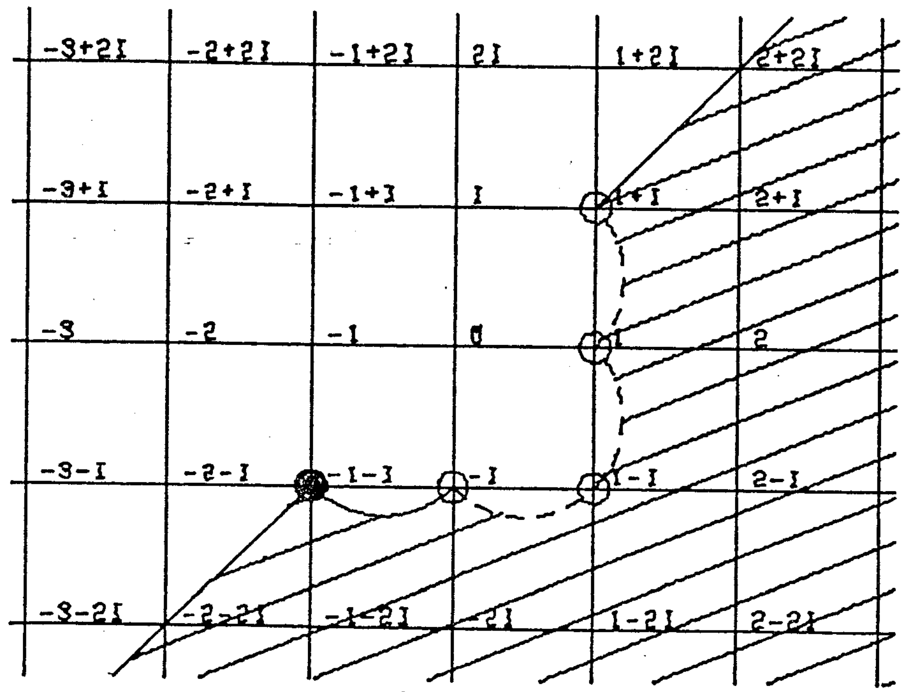

Figure 9

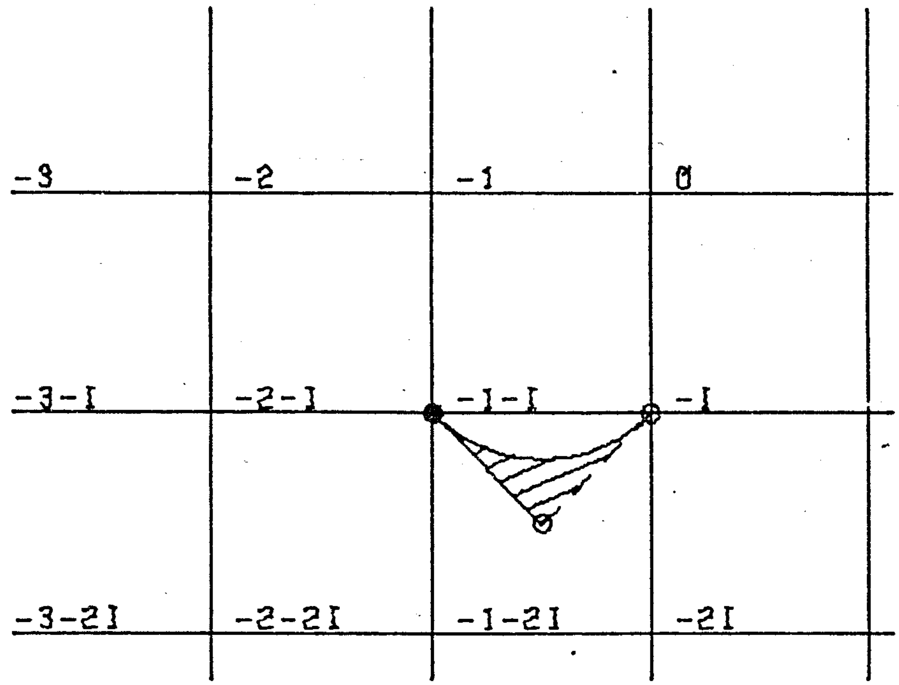

Figure 10

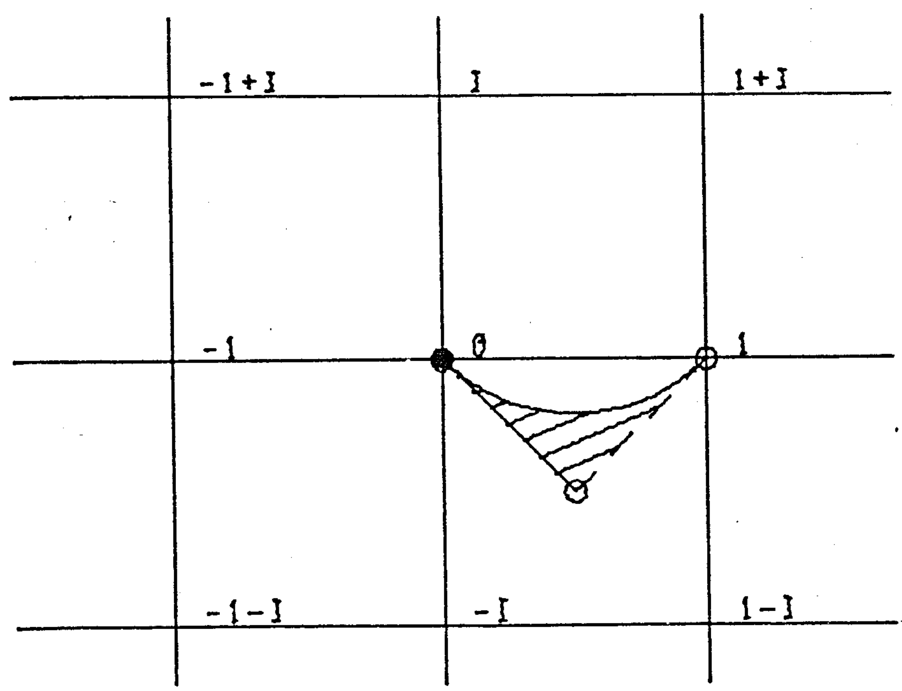

Figure 11

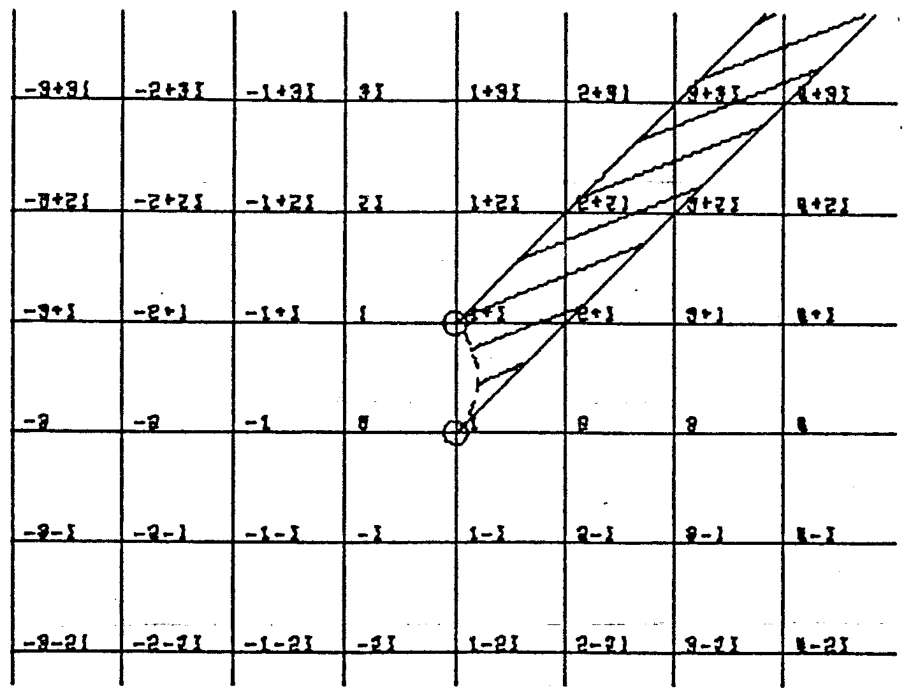

Figure 12

Figure 12. But now we see that -3i cannot equal $fl(x_2)$ for any x_2!

If we do this type of analysis in a very detailed fashion, we can eventually describe the possible results of the cf algorithm in a precise manner. Our goal is to develop a set of "rules" that allow us to say precisely when a given expansion can be generated by the cf algorithm. These "rules" should also allow us to construct expansions that are "legal", i. e., are generated by the algorithm.

To formalize this concept of "rules", we introduce the idea of context-free grammars.

A string is a sequence of primitive symbols chosen from some set called an alphabet. For example, we can consider words in the English language to be strings over the alphabet {a, b, c, ... z}. One useful function on strings is concatenation. The concatenation of two strings is a new string formed by the juxtaposition of the two strings.

For example, if x = 'the ' and y = 'dog', then xy = 'the dog'. Note that concatenation is written like multiplication. This leads to the notation a^n to represent the concatenation of a with itself n times. Thus x^3 = 'the the the '. We define $a^0 = \emptyset$, where $\emptyset$ is a symbol denoting the "empty string" satisfying

$$w\emptyset = \emptyset w = w$$

for all strings w.

A language is a finite or infinite set of strings. For example,

$$L = \{1^n 0 2^k : n,k \geq 0\} = \{0, 10, 02, 102, 110, 022, 1102, ...\}$$

is an infinite language.

If A and B are two languages, then by AB we understand the set $\{xy : x \in A, y \in B\}$.

The Kleene closure, A^*, of a language A is the set

$$\bigcup_{k \geq 0} A^k.$$

A context-free grammar (CFG) is a set $G = (V_N, V_T, P, S)$ where V_N is a set of symbols called variables, V_T is a set of symbols called terminals, $S \in V_N$ is a distinguished symbol called the start symbol, and P is a set of expressions of the form a-->b, called productions.

If a-->b is a production in P and c and d are any strings, then the production a-->b is applied to the string cad to obtain the string cbd. This is written

cab ===> cbd

and we say cad is derived from cbd.

If a_1 ===> a_2 ===> ... ===> a_m, i. e., a_m is derived from a_1 after a finite number of productions, we write

$$a_1 \overset{*}{===>} a_m.$$

We define L(G), the language generated by the CFG G to be the set

$$L(G) = \{w \mid w \in V_T^*, S \overset{*}{===>} w\}.$$

Example.

If G = ({S,A,B}, {0,1,2}, P, S) where P is the set of productions {S-->A0B, A-->Ø|1A, B-->Ø|2B}, then L(G) = $\{0^n 1 2^k : n,k \geq 0\}$. Note: we use the symbol | to simplify our writing of productions, as above. For example, A-->Ø|1A is shorthand for the two separate productions A-->Ø, A-->1A.

For more information on languages and grammars, see Hopcroft and Ullman [9].

Now we'll return to our description of complex continued fractions. We can think of any complex CF as a finite string where the symbols are chosen from the alphabet $\mathbb{Z}[i]$. With this sort of correspondence, we have

Theorem II.3.3.

Let $cf(z) = (a_0, a_1, \ldots, a_n)$. Let CFE be the set of all strings corresponding to CF expansions generated by the cf algorithm on rational complex numbers; that is,

$$CFE = \{a_0 a_1 a_2 \ldots a_n \mid cf(z) = (a_0, a_1, \ldots a_n),\ z \in \mathbb{Q}[i]\}.$$

Then $L(G) = CFE$ where G is the context-free grammar

$$G = (\{D_0, D_1, \ldots D_{24}\}, \mathbb{Z}[i], P, D_0)$$

and P is the set of productions below.

$D_0 \dashrightarrow x_1 \mid x_1 D_1 \qquad (x_1 \in E_0)$

$D_1 \dashrightarrow \emptyset \mid (1)D_2 \mid (1-i)D_2 \mid (-i)D_3 \mid (-1-i)D_4 \mid (1+i)D_5 \mid x_1 D_6 \mid x_2 D_1$

$(x_1 \in E_4,\ x_2 \in E_1 - (\{1, 1-i, -1-i, -i, 1+i\} \cup E_4))$

$D_2 \dashrightarrow (1)D_2 \mid (1-i)D_2 \mid (1+i)D_5 \mid (-i)D_7 \mid x_1 D_6 \mid x_2 D_8 \mid x_3 D_1$

$(x_1 \in E_4,\ x_2 \in E_5,\ x_3 \in (E_1 \cap E_{12}) - (\{1, 1-i, 1+i, -i\} \cup E_4 \cup E_5))$

$D_3 \dashrightarrow (1)D_9 \mid (1+i)D_5 \mid x_1 D_6 \mid x_2 D_{10}$

$(x_1 \in E_4,\ x_2 \in E_6)$

$D_4 \dashrightarrow \emptyset \mid D_{16}$

$D_5 \dashrightarrow (1+i)D_5 \mid x_1 D_6 \mid (1)D_2 \mid (1-i)D_2 \mid x_2 D_1$

$(x_1 \in E_4,\ x_2 \in (E_1 \cap E_3) - (\{1+i, 1, 1-i\} \cup E_4)$

$D_6 \dashrightarrow \emptyset \mid D_5$

$D_7 \dashrightarrow (1)D_9 | x_1 D_{14} | x_2 D_{15}$

$(x_1 \in E_4 \cup \{1+i\},\ x_2 \in E_6)$

$D_8 \dashrightarrow (1)D_2 | (1-i)D_2 | (-i)D_3 | (-1-i)D_{16} | x_1 D_{14} | x_2 D_{17} | x_3 D_1$

$(x_1 \in E_4 \cup E_8 \cup \{1+i\},\ x_2 \in E_6 \cup E_9,$

$x_3 \in E_1 - (\{1, 1-i, -i, -1-i, 1+i\} \cup E_4 \cup E_8 \cup E_6 \cup E_9))$

$D_9 \dashrightarrow (-i)D_7 | x_1 D_8 \qquad (x_1 \in E_5)$

$D_{10} \dashrightarrow (-1-i)D_4 | (-i)D_3 | x_1 D_6 | x_2 D_1$

$(x_1 \in E_8,\ x_2 \in (E_1 \cap E_2) - (E_8 \cup \{-1-i, -i\}))$

$D_{11} \dashrightarrow (-i)D_7 | (-1-i)D_{18} | x_1 D_8 | x_2 D_{19}$

$(x_1 \in E_5,\ x_2 \in E_{10})$

$D_{12} \dashrightarrow \emptyset | (-1-i)D_4 | (-i)D_3 | (1-i)D_{18} | x_1 D_{19} | x_2 D_6 | x_3 D_1$

$(x_1 \in E_{10},\ x_2 \in E_8,\ x_3 \in (E_1 \cap E_{11}) - (\{-1-i, -i, 1-i\} \cup E_{10} \cup E_8)$

$D_{13} \dashrightarrow (-1-i)D_{20}$

$D_{14} \dashrightarrow (1+i)D_5 | x_1 D_6 | (1)D_2 | (1-i)D_{21} | x_2 D_8 | x_3 D_1$

$(x_1 \in E_4,\ x_2 \in E_{10},\ x_3 \in (E_1 \cap E_3) - (\{1+i, 1, 1-i\} \cup E_4 \cup E_{10}))$

$D_{15} \dashrightarrow (-1-i)D_3 | (-i)D_3 | x_1 D_6 | x_2 D_1$

$(x_1 \in E_8,\ x_2 \in (E_1 \cap E_2) - (\{-1-i, -i\} \cup E_8))$

$D_{16} \dashrightarrow (1)D_{11} | (1+i)D_5 | x_1 D_6 | x_2 D_{12} | x_3 D_{13}$

$(x_1 \in E_4,\ x_2 \in E_6,\ x_3 \in E_7)$

$D_{17} \dashrightarrow \emptyset | (-1-i)D_3 | (-i)D_3 | (1-i)D_2 | (1)D_2 | (1+i)D_5 | x_1 D_6 | x_2 D_1$

$(x_1 \in E_8 \cup E_4,\ x_2 \in E_1 - (E_8 \cup E_4 \cup \{-1-i, -i, 1-i, 1, 1+i\}))$

$D_{18} \dashrightarrow (1+i)D_{22} | x_1 D_{23} | x_2 D_{13}$

$(x_1 \in E_4,\ x_2 \in E_6)$

$D_{19} \dashrightarrow \emptyset | (1+i)D_{22} | (-1-i)D_{24} | x_1 D_{13} | x_2 D_{23}$

$(x_1 \in E_6 \cup E_9,\ x_2 \in E_4 \cup E_8)$

$D_{20} \dashrightarrow \emptyset | (1)D_{22} | x_1 D_{13} | x_2 D_{23}$

$(x_1 \in E_7,\ x_2 \in E_6)$

$D_{21} \to (1)D_2|(1-i)D_2|(-i)D_7|x_1D_{14}|x_2D_{17}|x_3D_8|x_4D_1$

$(x_1 \in E_4 \cup \{1+i\},\ x_2 \in E_6,\ x_3 \in E_5,$

$\quad x_4 \in (E_1 \cap E_{12}) - (E_4 \cup E_5 \cup E_6 \cup \{1, 1-i, -i, 1+i\}))$

$D_{22} \to (1-i)D_{18}|x_1D_{19} \qquad (x_1 \in E_{10})$

$D_{23} \to \emptyset|D_{22}$

$D_{24} \to \emptyset$

The sets E_1 through E_{12} consist of Gaussian integers and are described below:

$E_0 = \mathbb{Z}[i]$

$E_1 = \{z: Re(z) \geq Im(z),\ z \neq 0\}$

$E_2 = \{z: Re(z) \leq -(1+Im(z)), z \neq 0\}$

$E_3 = \{z: Re(z) \geq -Im(z), z \neq 0\}$

$E_4 = \{z: z=k(1+i), k \geq 2\}$

$E_5 = \{z: z=k-(k+1)i, k \geq 1\}$

$E_6 = \{z: z=k+1+ki, k \geq 1\}$

$E_7 = \{z: z=k+2+ki, k \geq 0\}$

$E_8 = \{z: z=k(1+i), k \leq -2\}$

$E_9 = \{z: z=k+1+ki, k \leq -2\}$

$E_{10} = \{z: z=k(1-i), k \geq 2\}$

$E_{11} = \{z: Re(z) \leq -Im(z), z \neq 0\}$

$E_{12} = \{z: Re(z) \geq -(1+Im(z)), z \neq 0\}$

This theorem is difficult to state (to say the least) and certainly seems far from intuitive. However, it follows quite naturally from considerations similar to those following Theorem II.3.2.

We define a sequence of regions similar to that in Figures 7-12, as follows:

$B_0 = ¢$

$C_k = \{z : z = cfl(w),\ w \in B_k\}$

The set C_k is partitioned into disjoint subsets $c_k(n_j)$ such that

$cfl^{-1}(a) \cap B_k = A_{n_j}$ for each $a \in c_k(n_j)$.

$B_k = \{1/z : z \in A_k - \{0\}\}$

What is surprising is that there are only a finite number of distinct regions A_k and B_k. We do not list the partitions of the C_k explicitly, as they can be inferred from Figures 13-60. We state that the context-free grammar G of the theorem contains a production of the form D_k --> $a_j D_n$ iff $a_j \in c_k(n)$. Also, productions of the form D_k --> Ø appear in G iff $0 \in A_k$. This can be verified (in a very tedious fashion) by examining Figures 13-60.

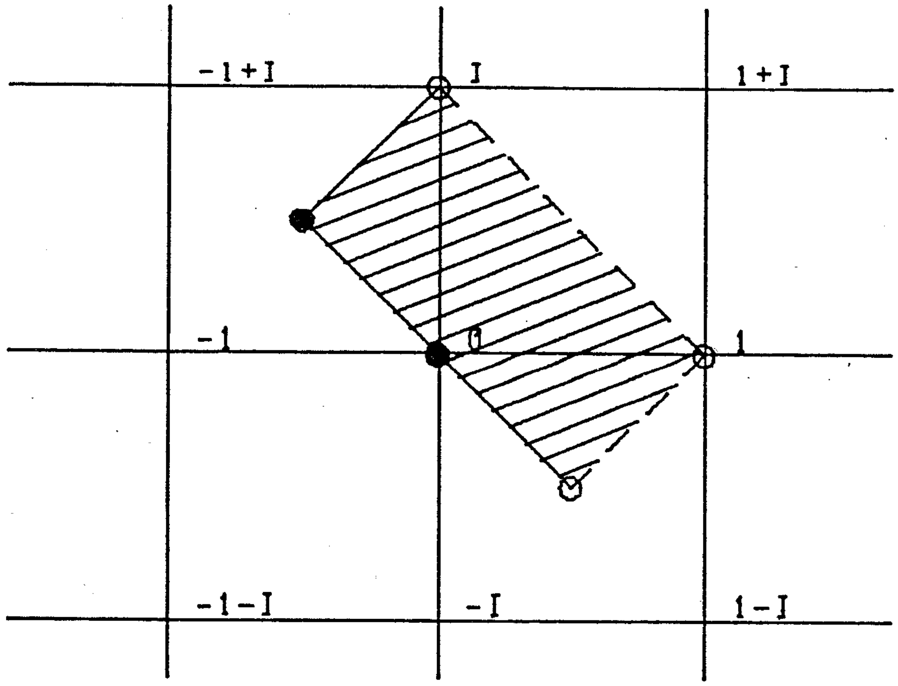

Figure 13: Region A_1

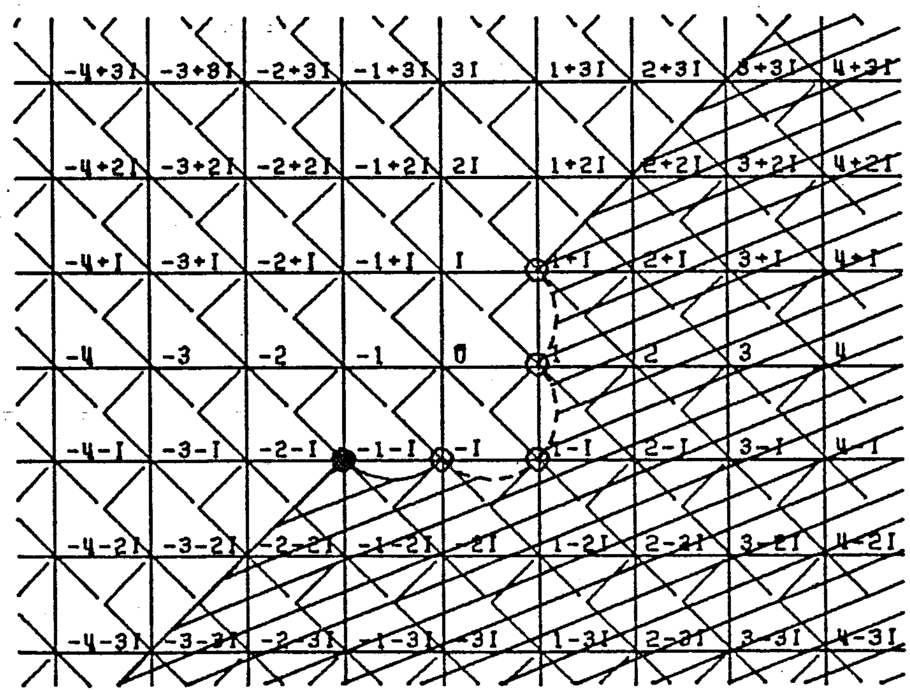

Figure 14: Region B_1

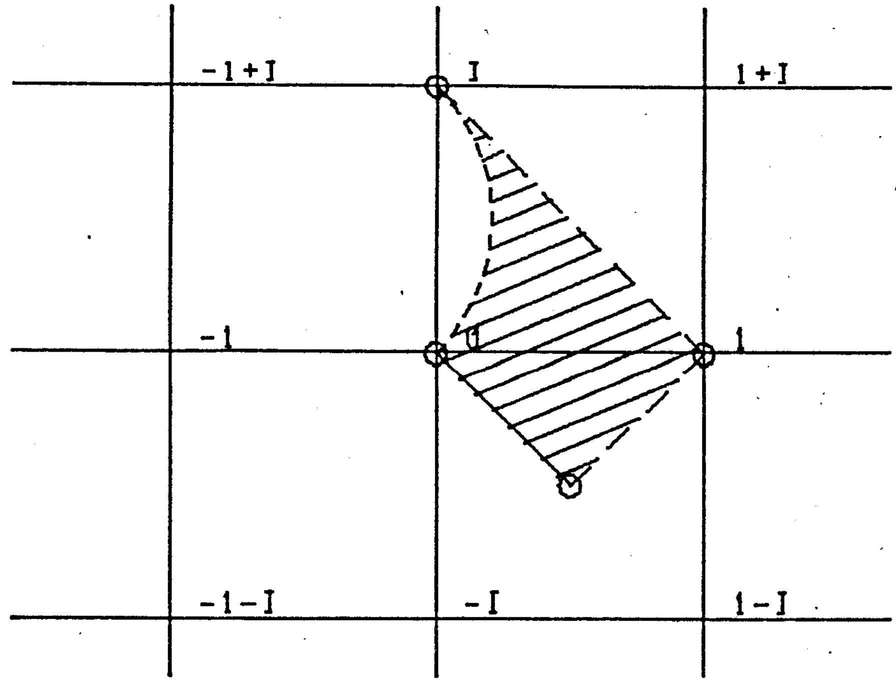

Figure 15: Region A_2

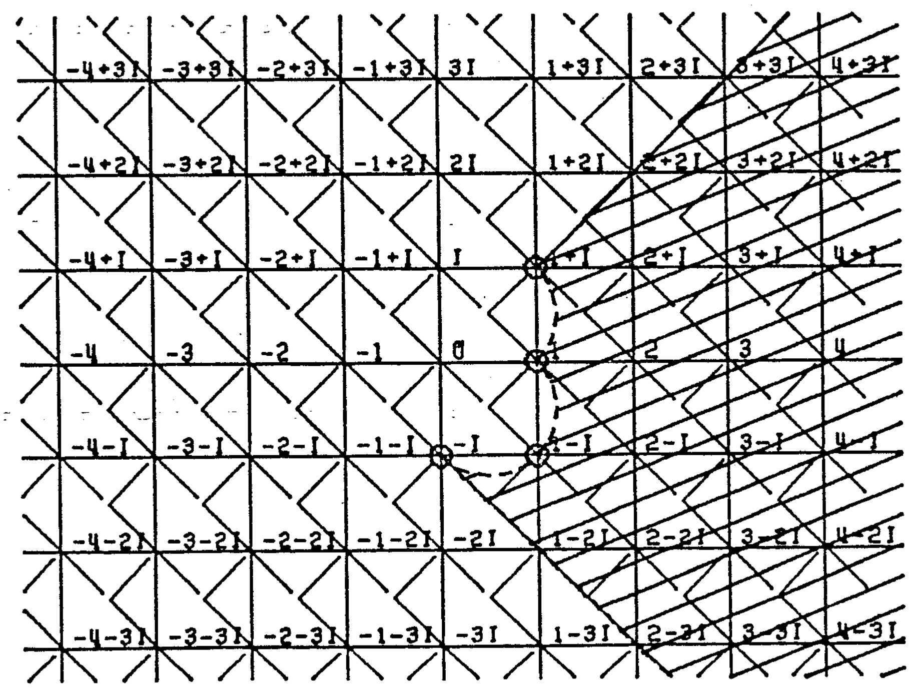

Figure 16: Region B_2

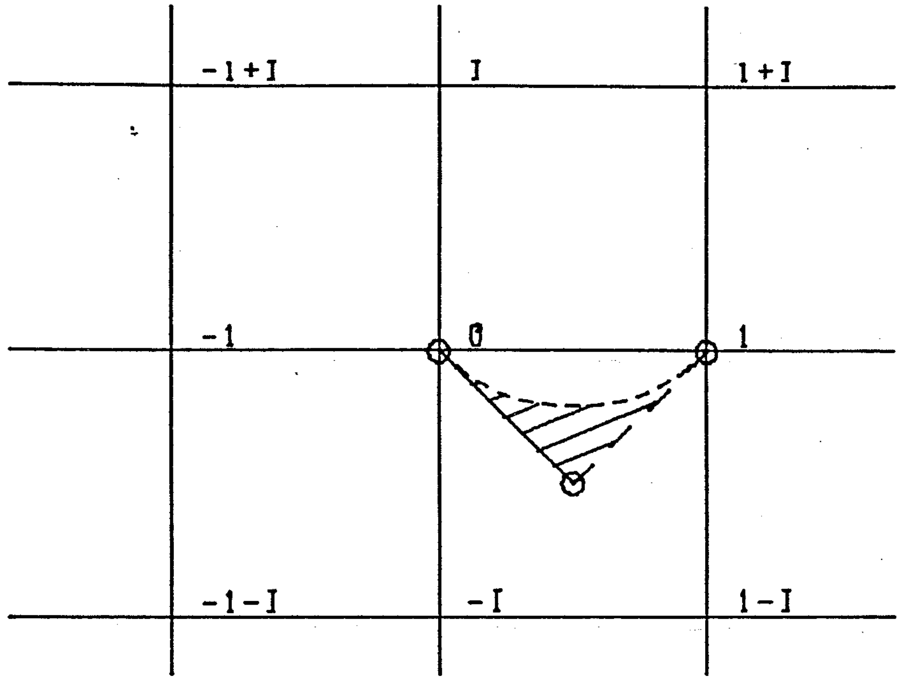

Figure 17: Region A_3

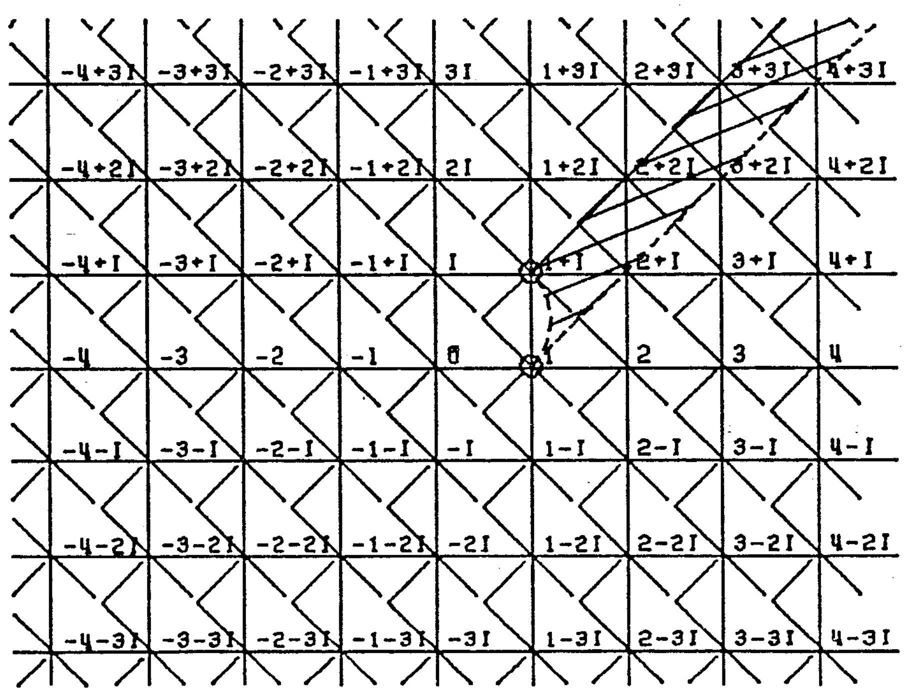

Figure 18: Region B_3

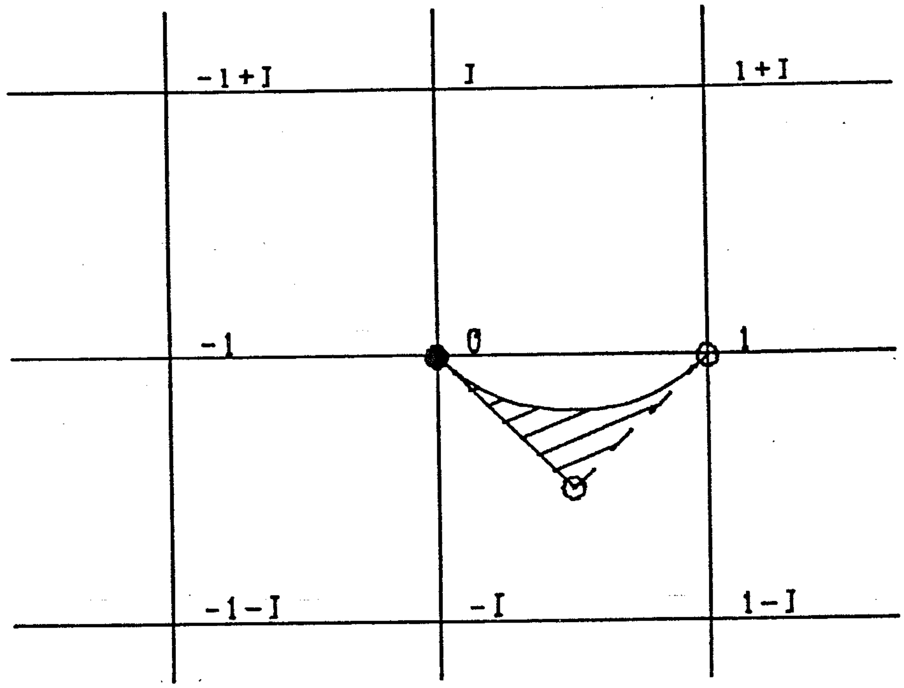

Figure 19: Region A_4

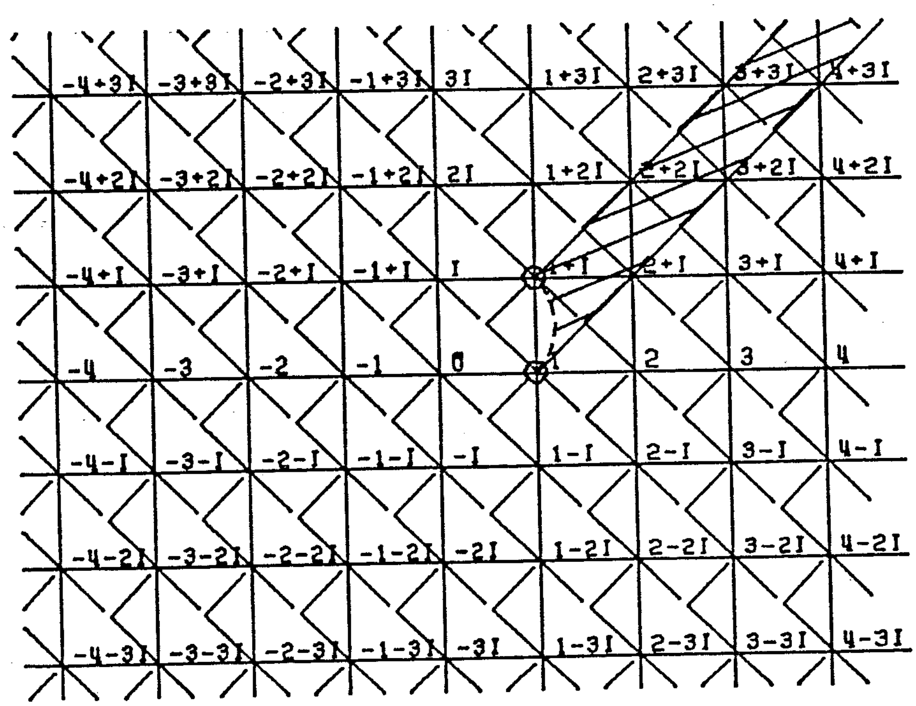

Figure 20: Region B_4

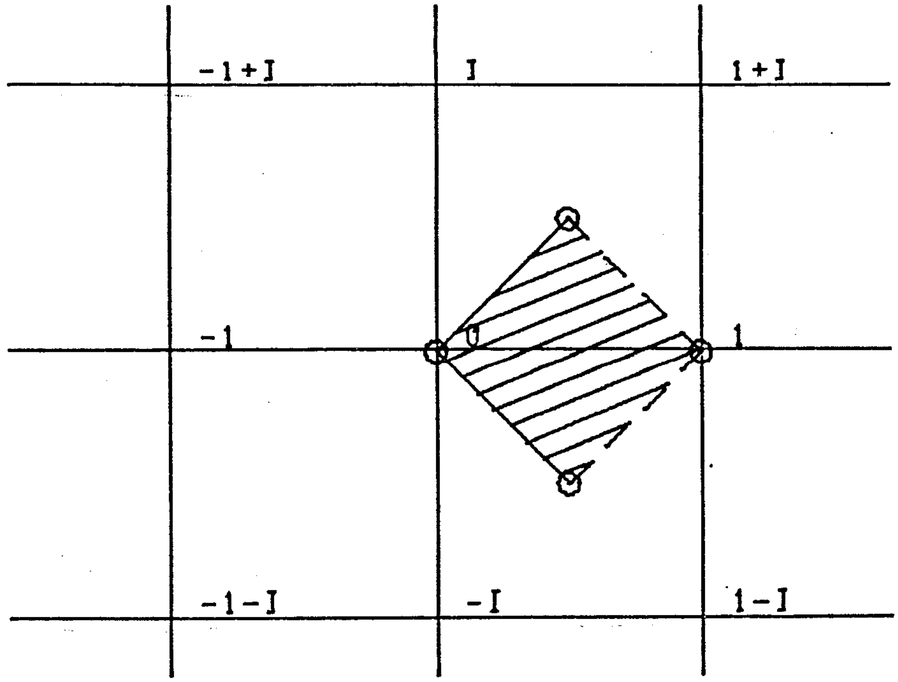

Figure 21: Region A_5

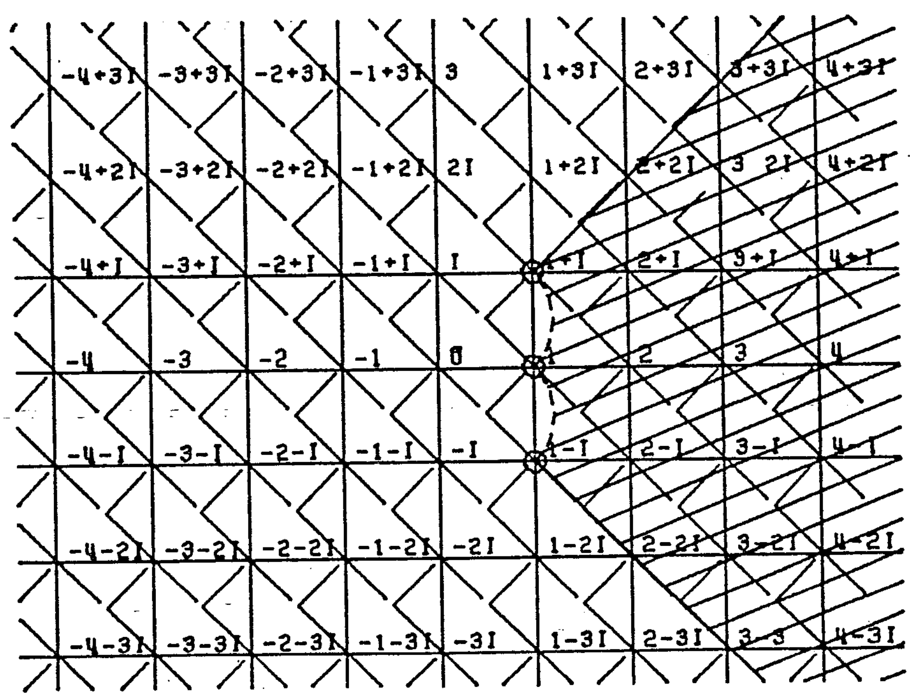

Figure 22: Region B_5

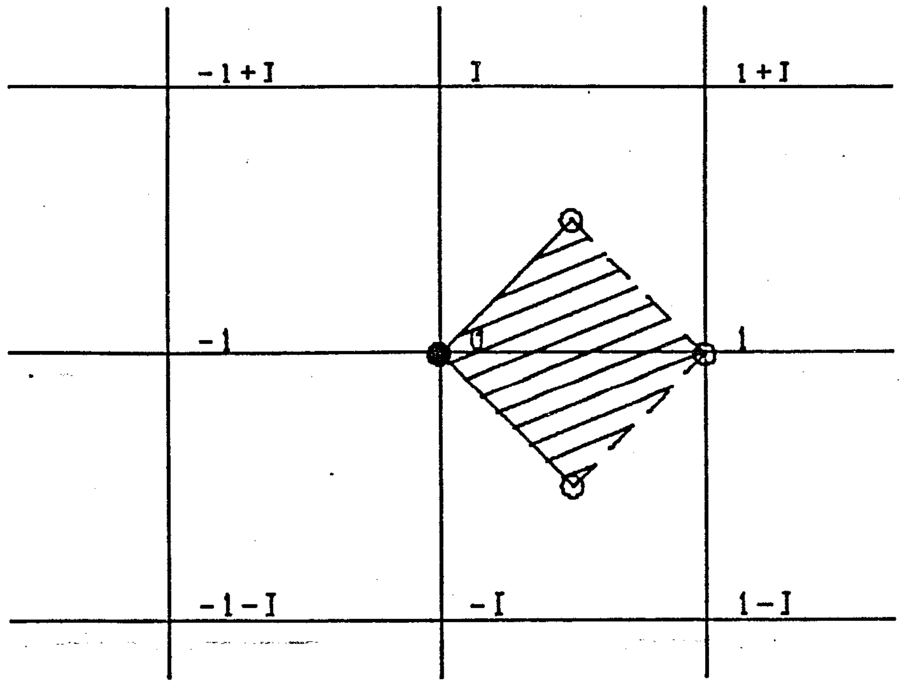

Figure 23: Region A_6

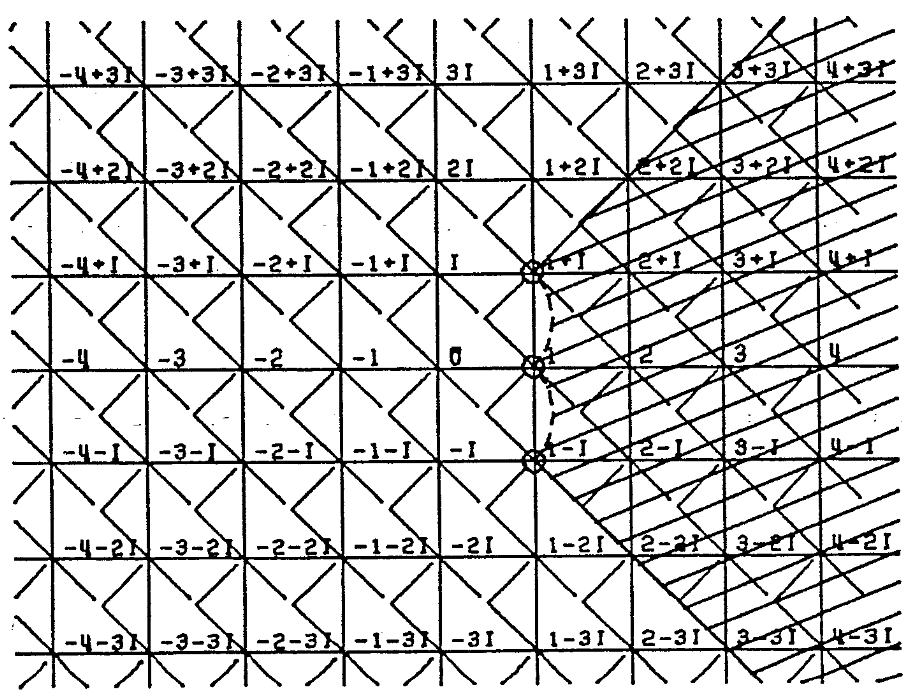

Figure 24: Region B_6

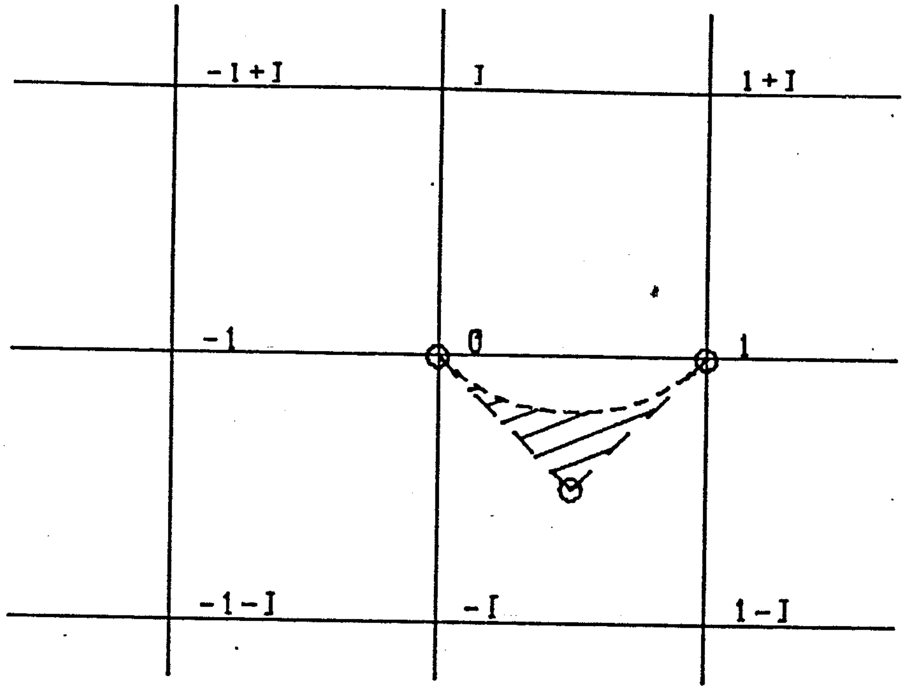

Figure 25: Region A_7

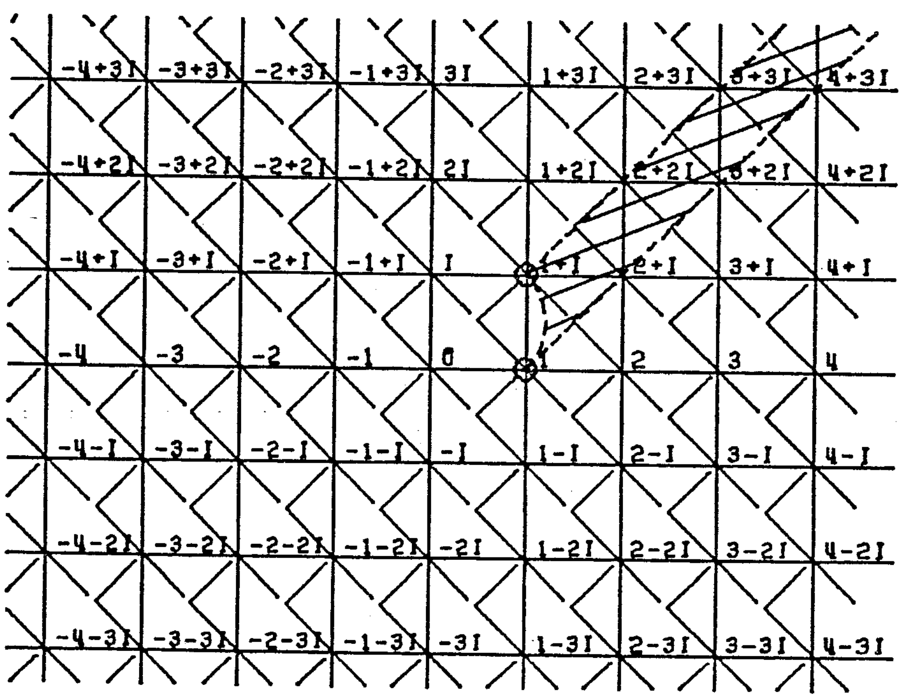

Figure 26: Region B_7

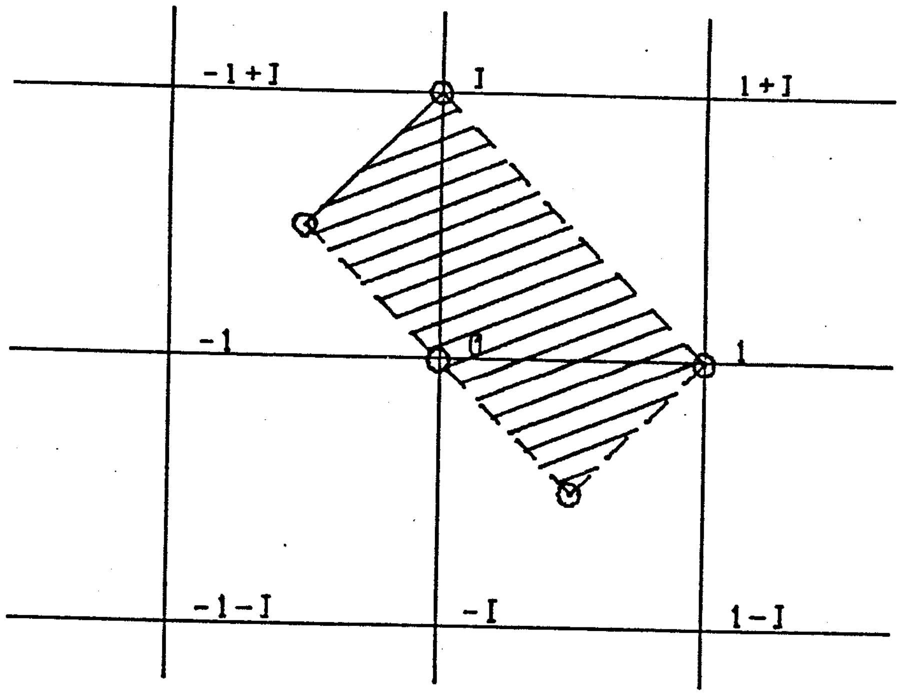

Figure 27: Region A_8

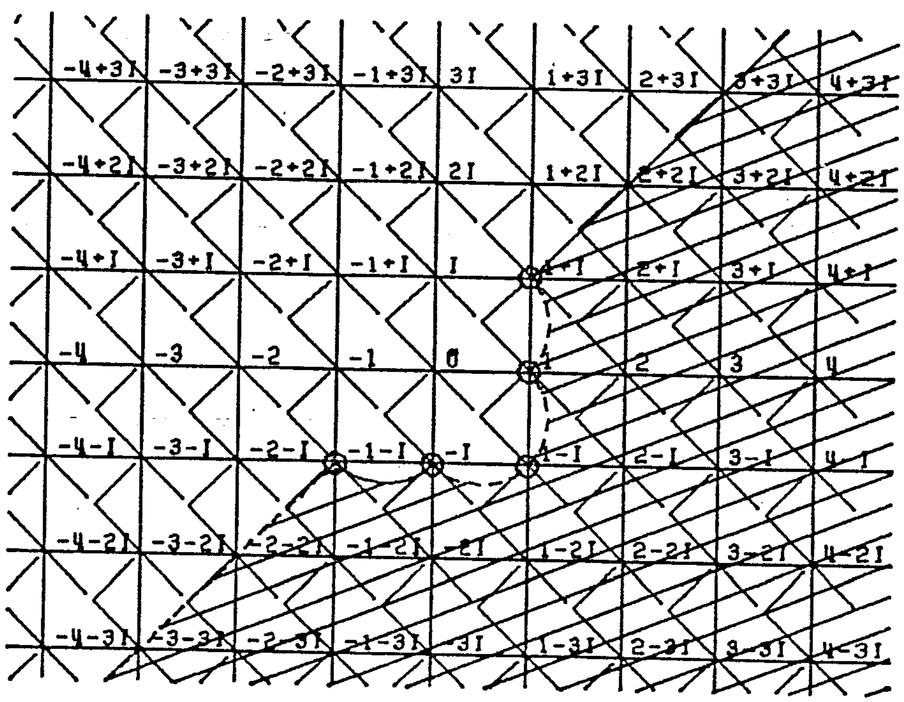

Figure 28: Region B_8

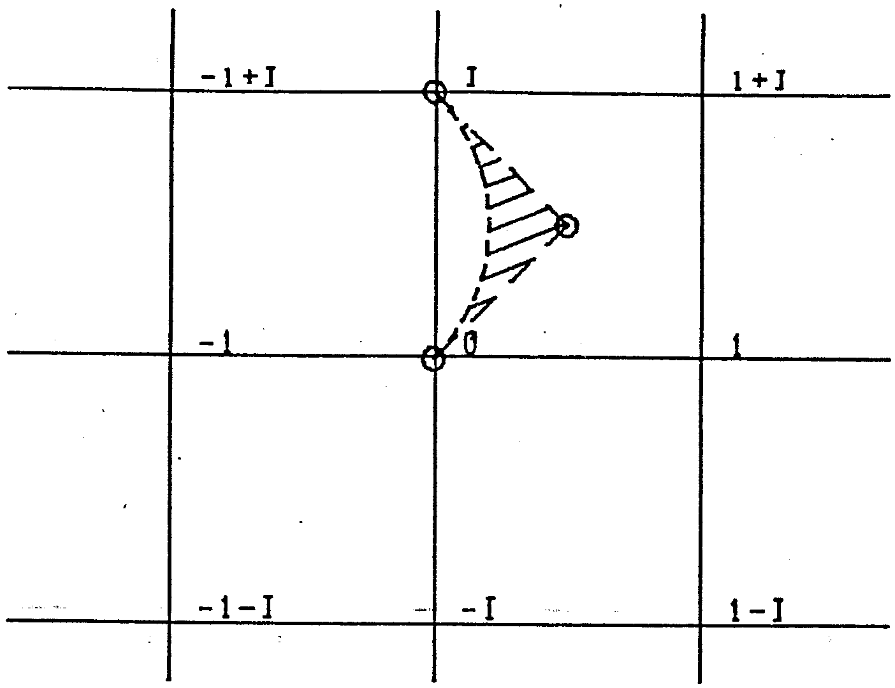

Figure 29: Region A_9

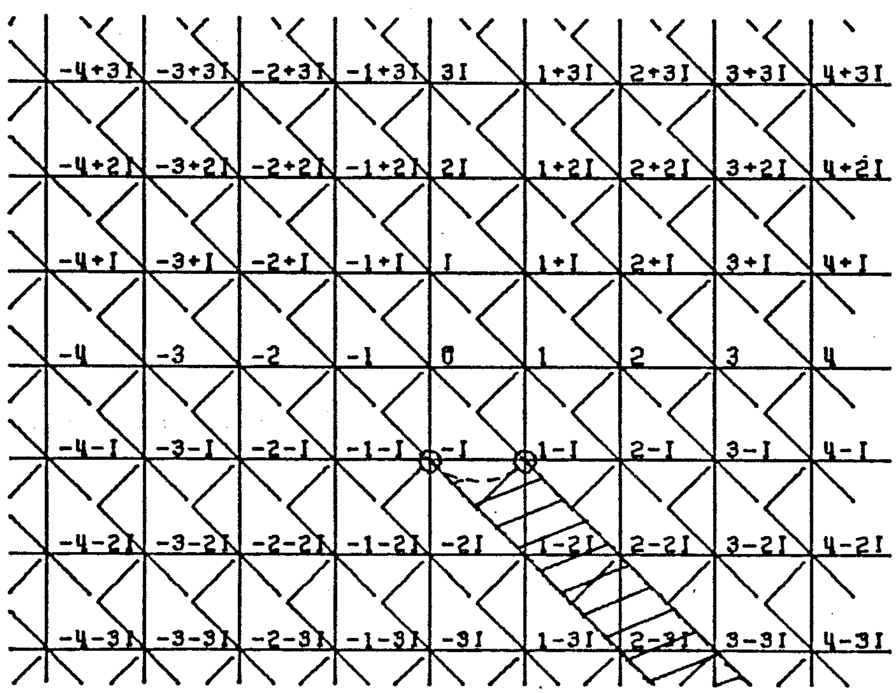

Figure 30: Region B_9

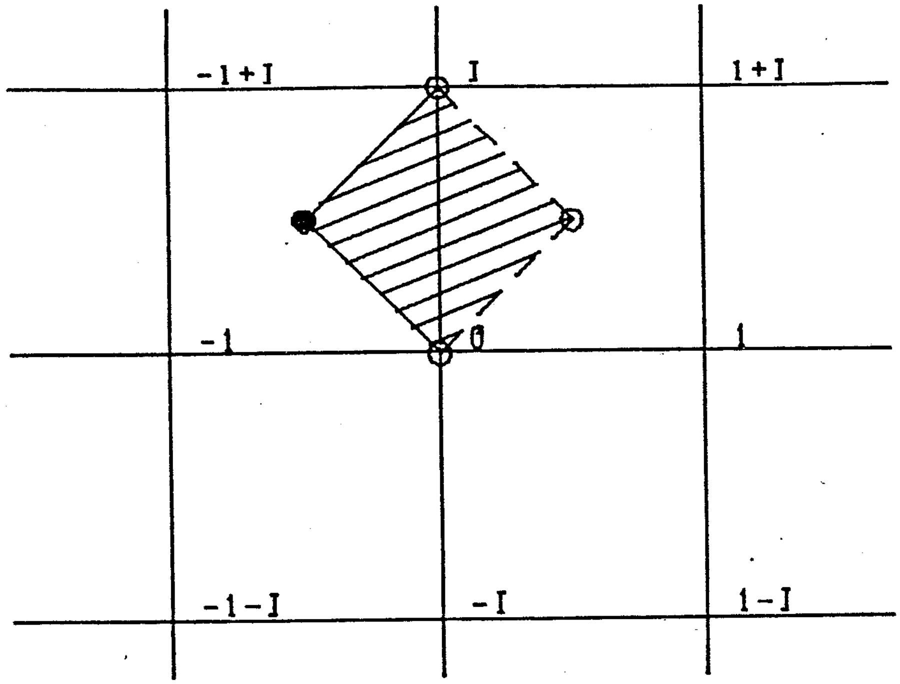

Figure 31: Region A_{10}

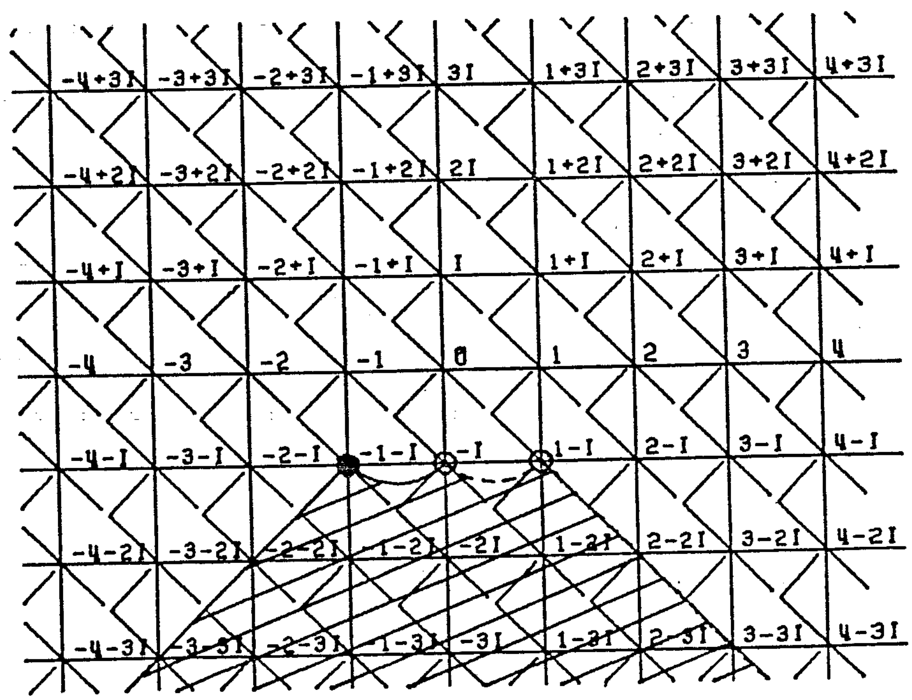

Figure 32: Region B_{10}

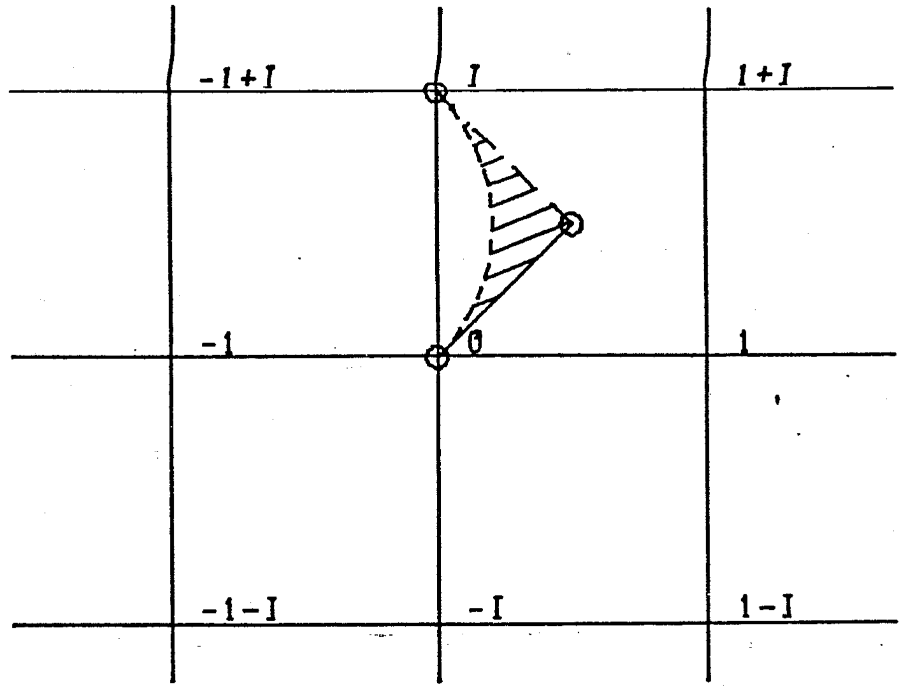

Figure 33: Region A_{11}

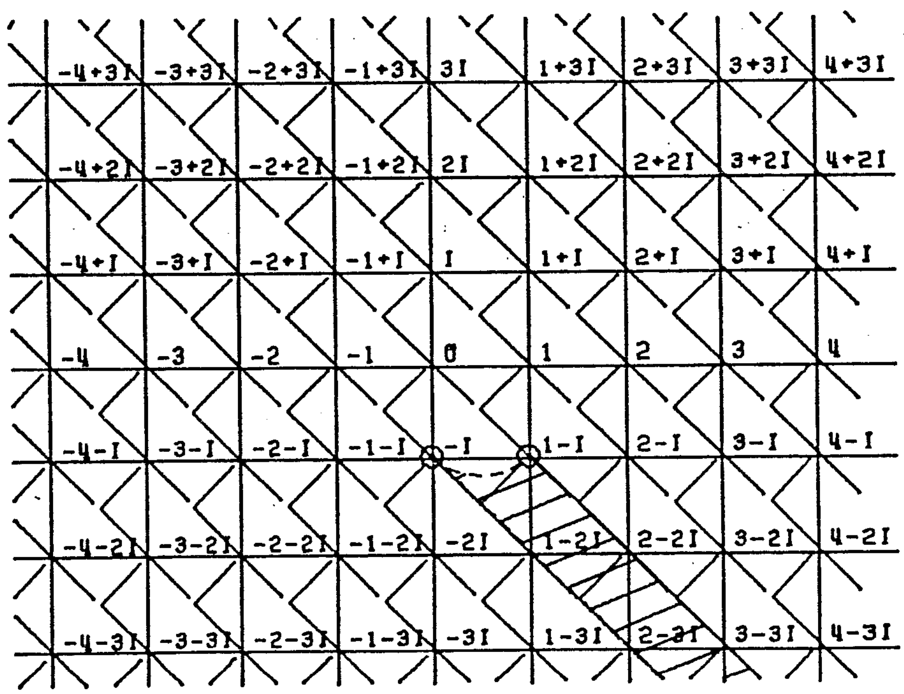

Figure 34: Region B_{11}

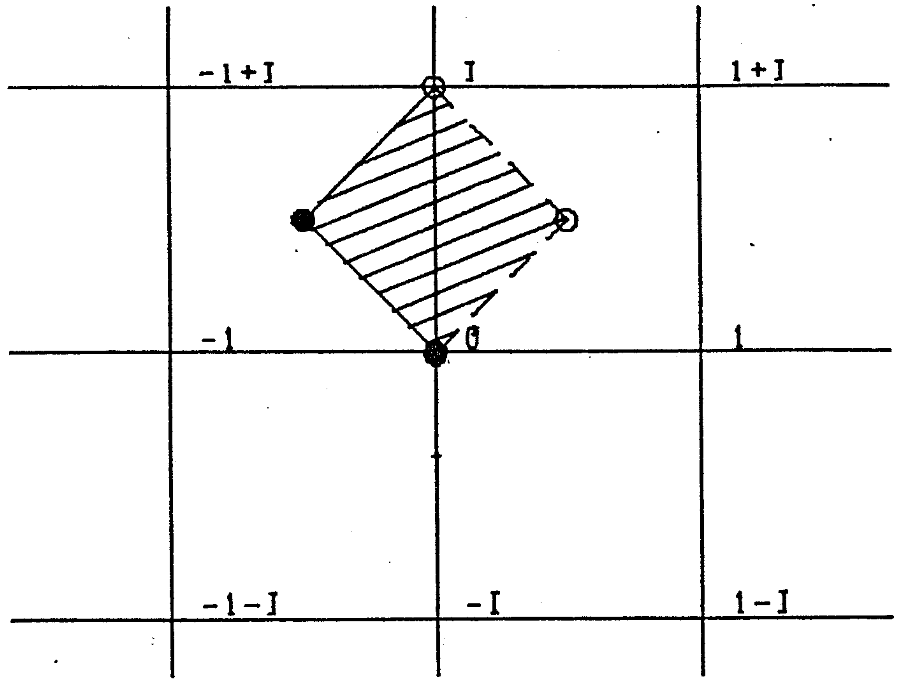

Figure 35: Region A_{12}

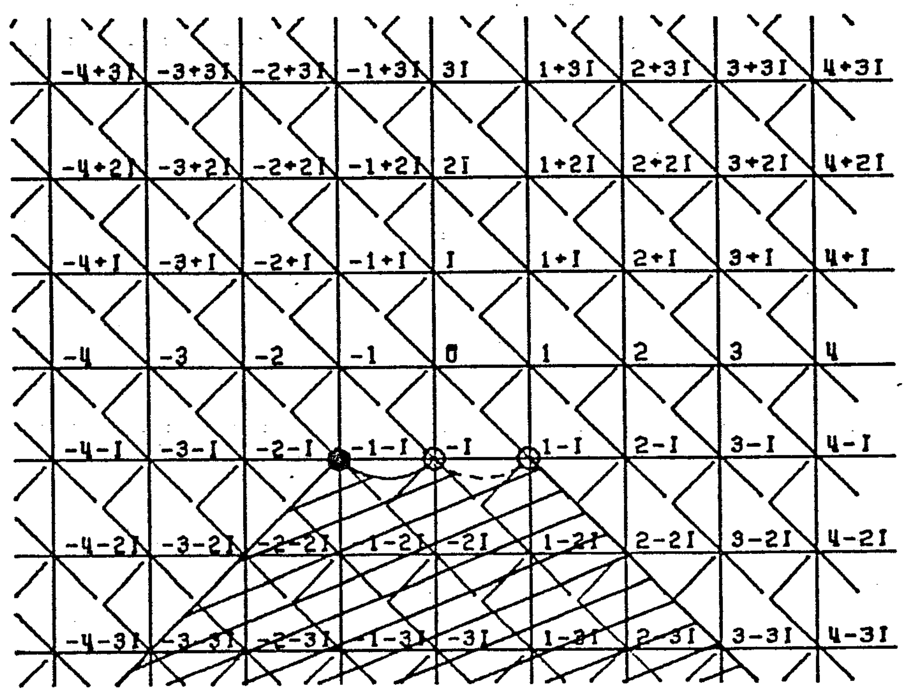

Figure 36: Region B_{12}

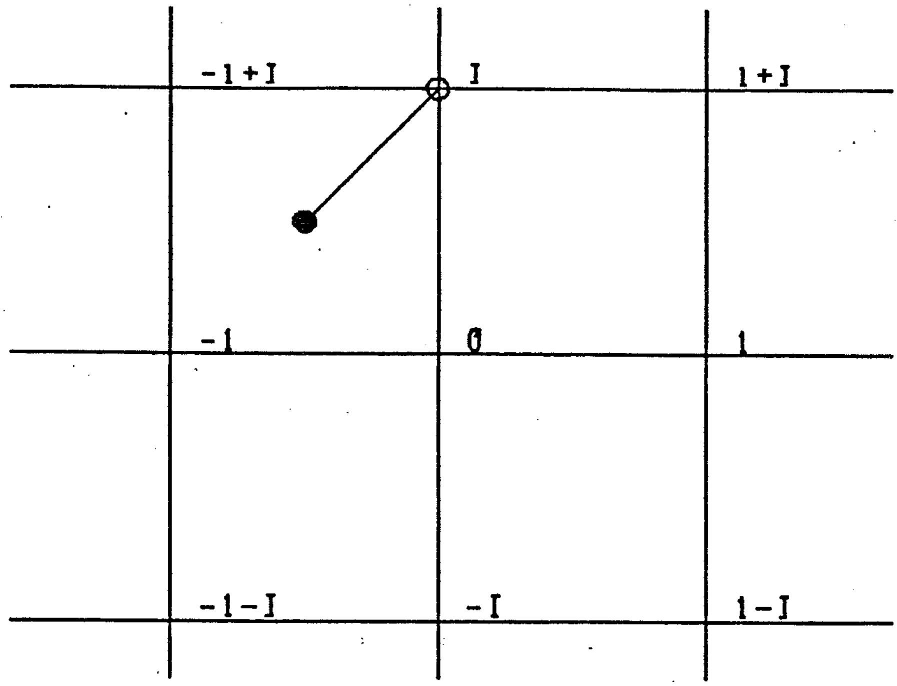

Figure 37: Region A_{13}

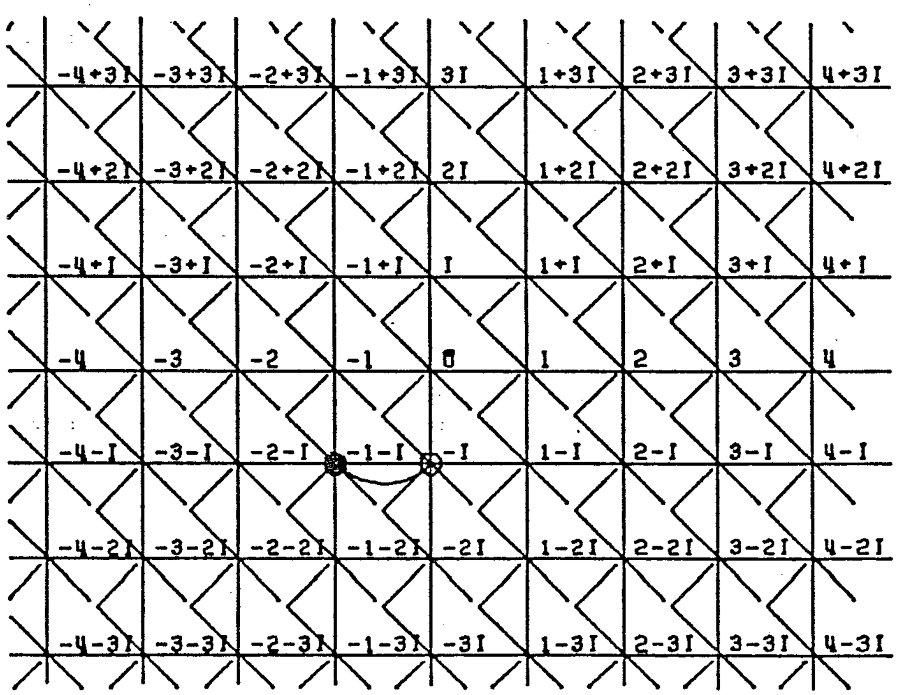

Figure 38: Region B_{13}

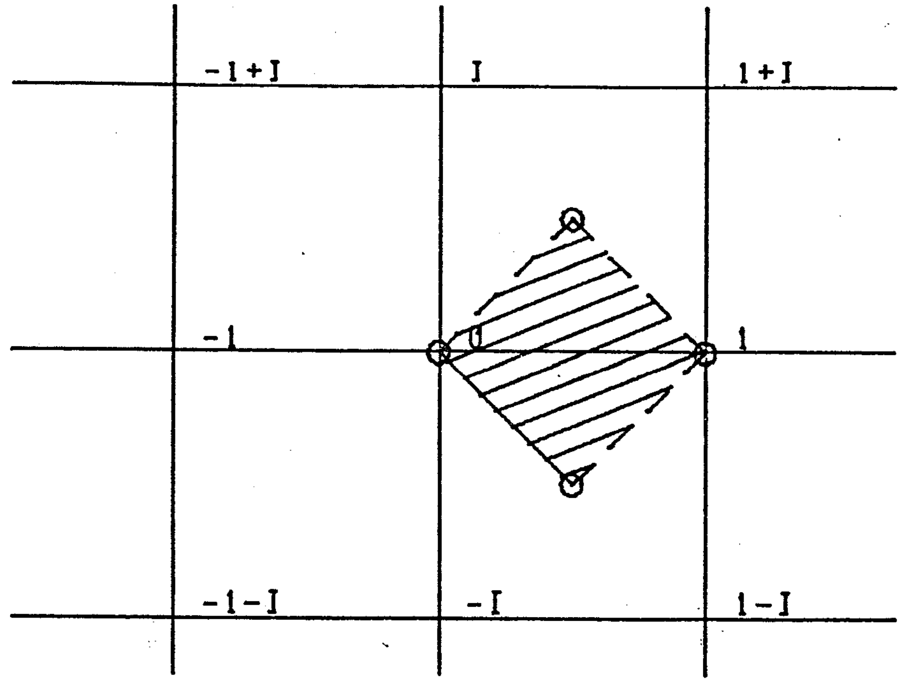

Figure 39: Region A_{14}

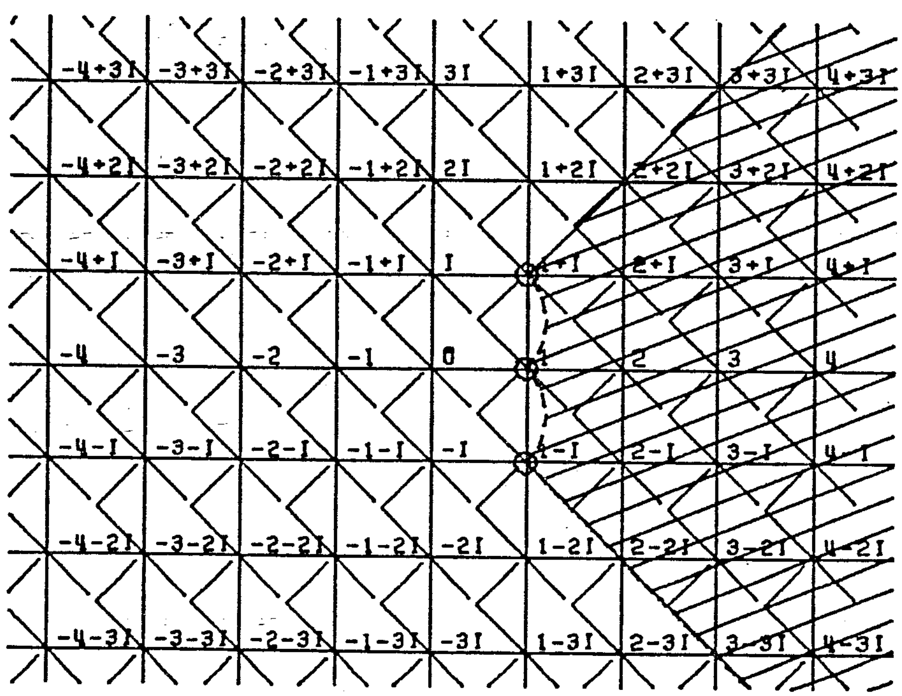

Figure 40: Region B_{14}

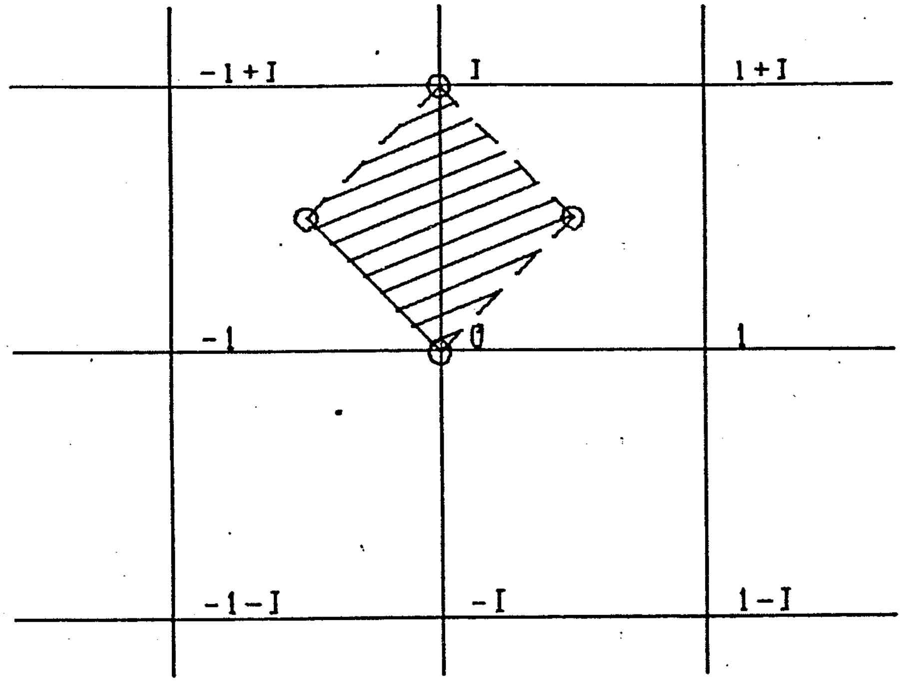

Figure 41: Region A_{15}

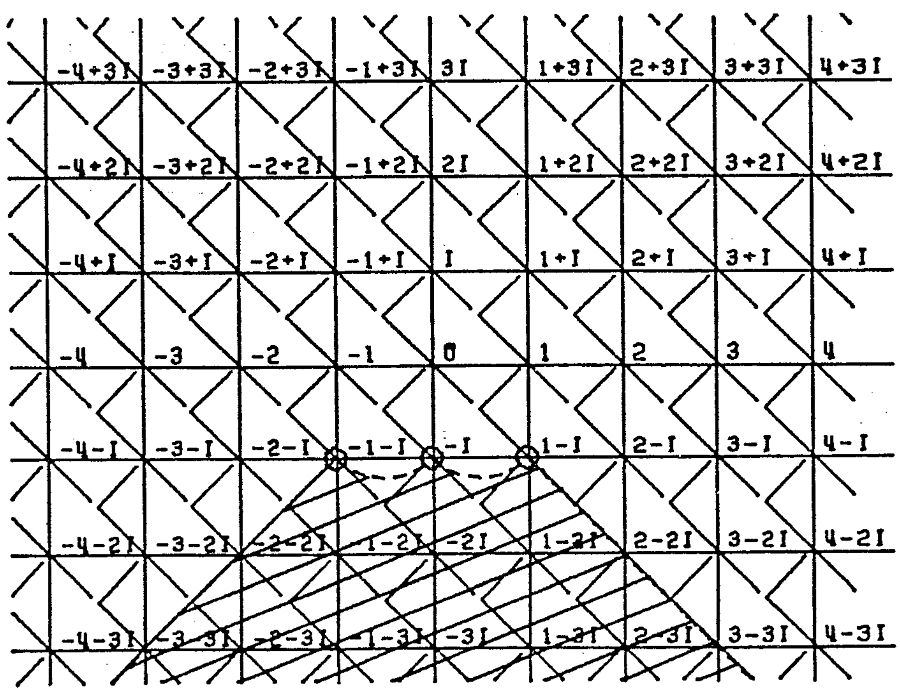

Figure 42: Region B_{15}

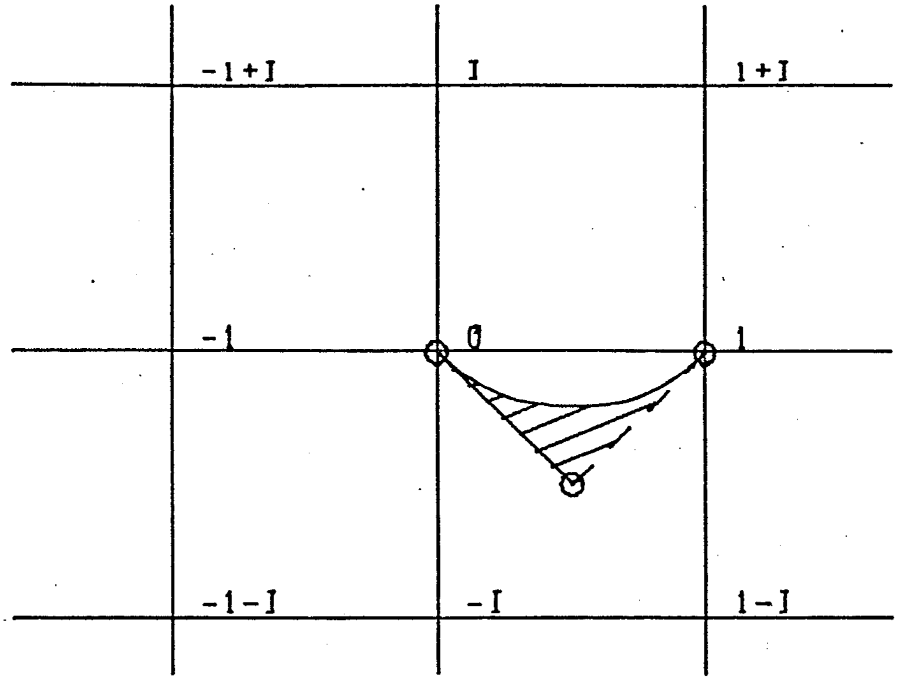

Figure 43: Region A_{16}

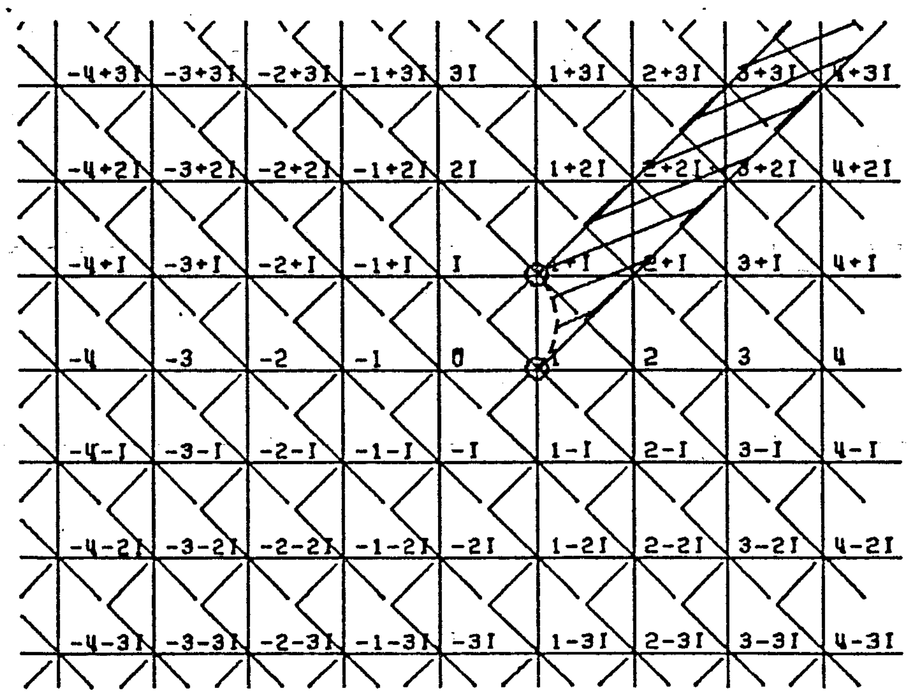

Figure 44: Region B_{16}

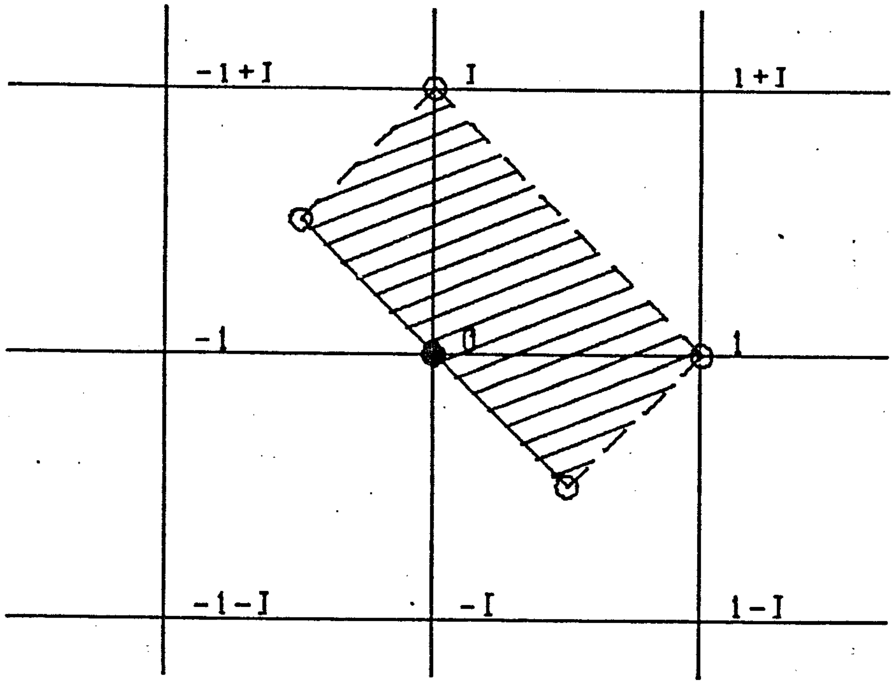

Figure 45: Region A_{17}

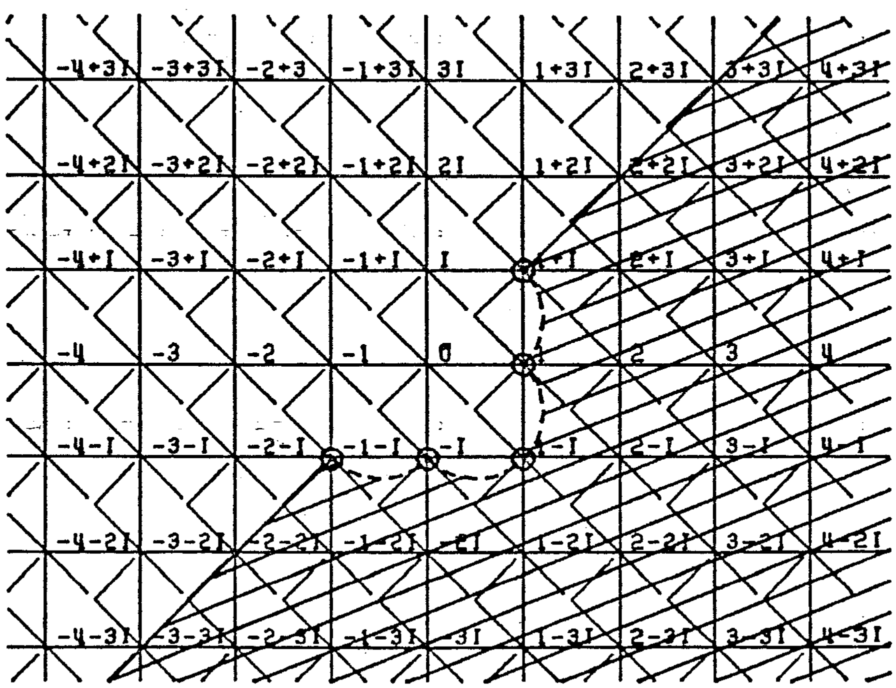

Figure 46: Region B_{17}

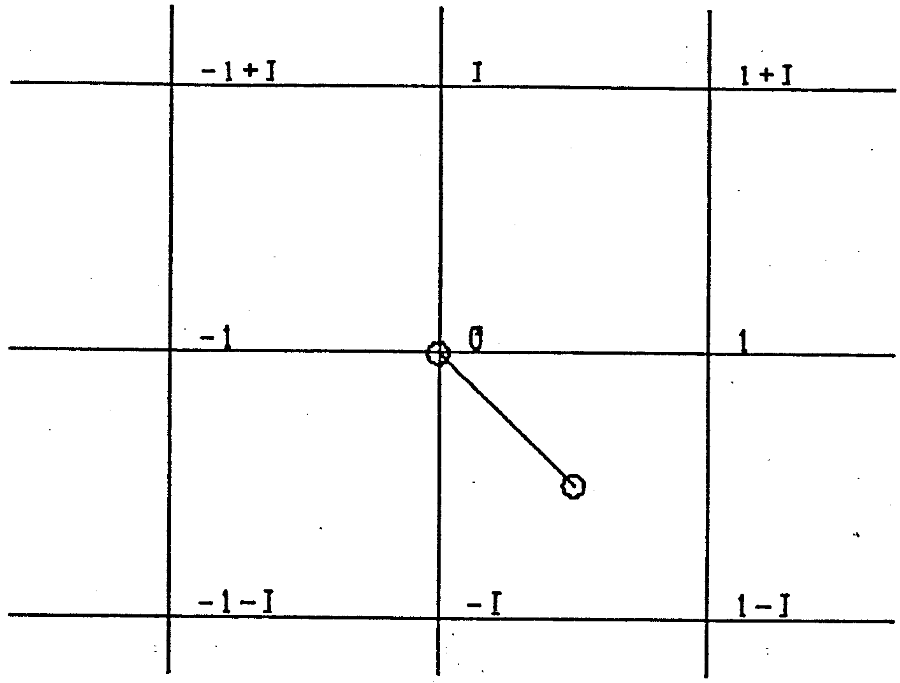

Figure 47: Region A_{18}

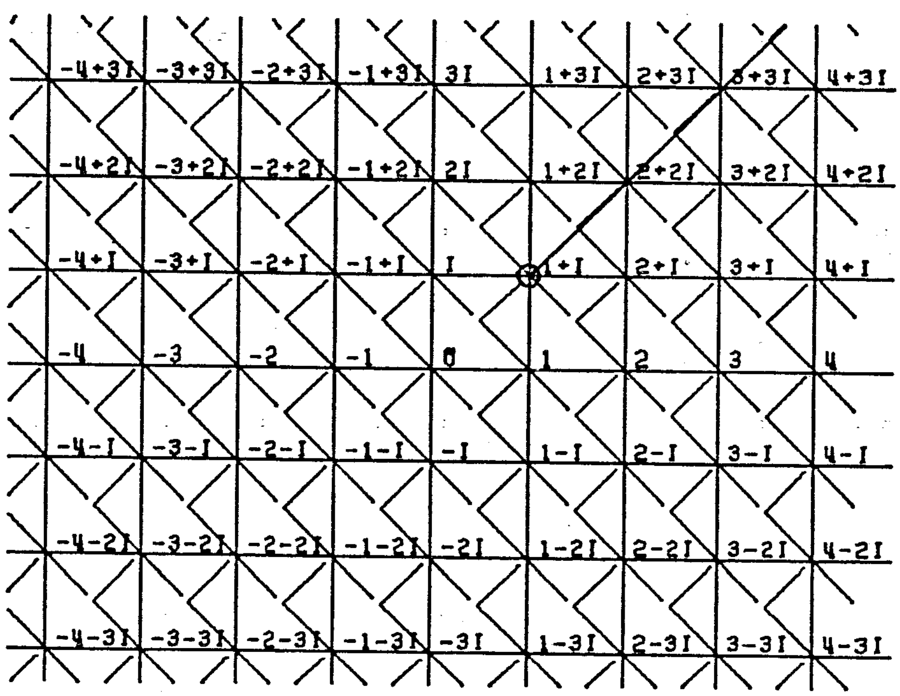

Figure 48: Region B_{18}

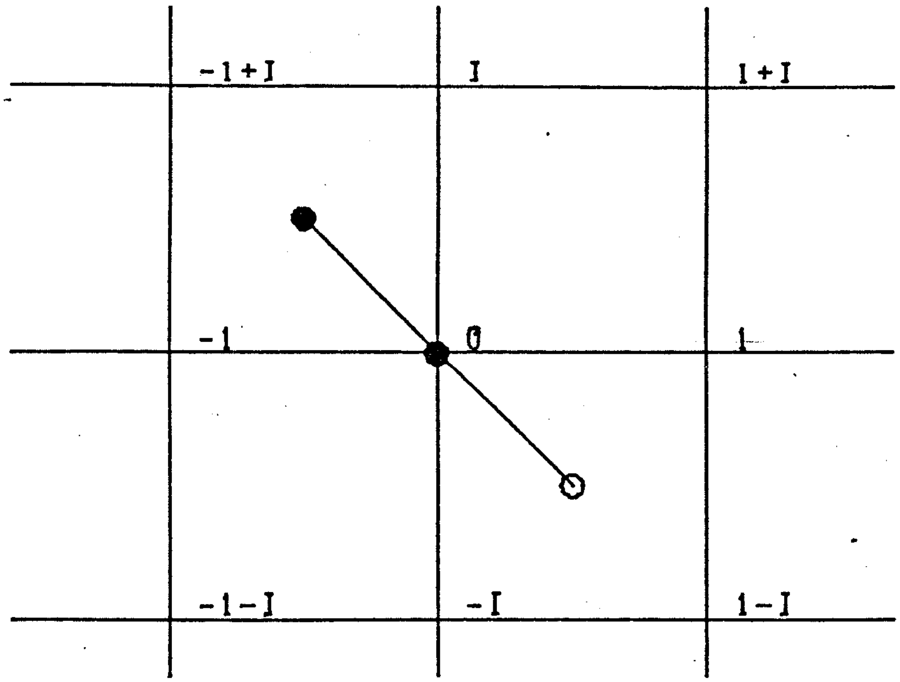

Figure 49: Region A_{19}

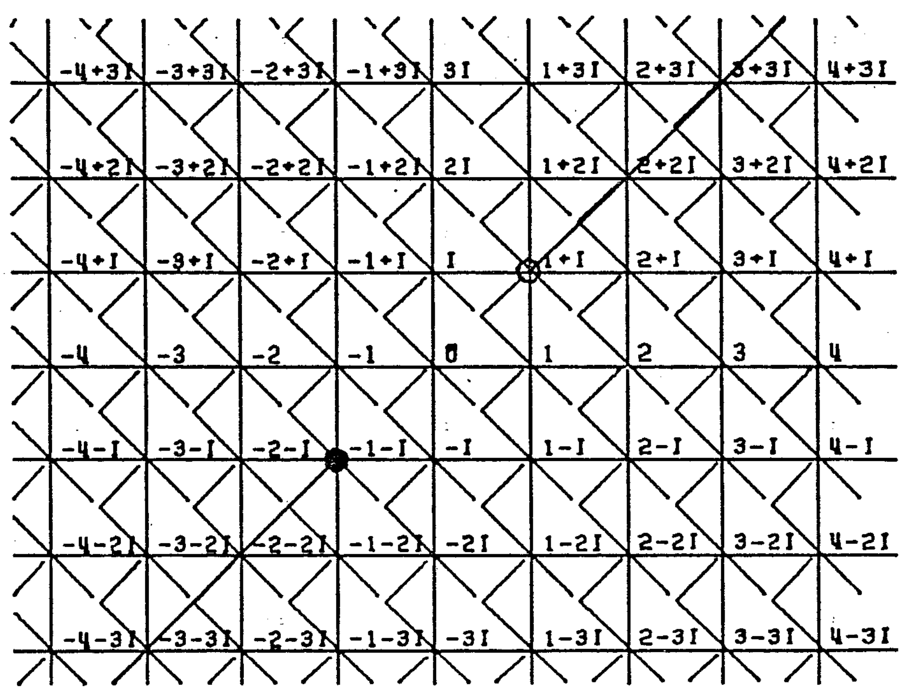

Figure 50: Region B_{19}

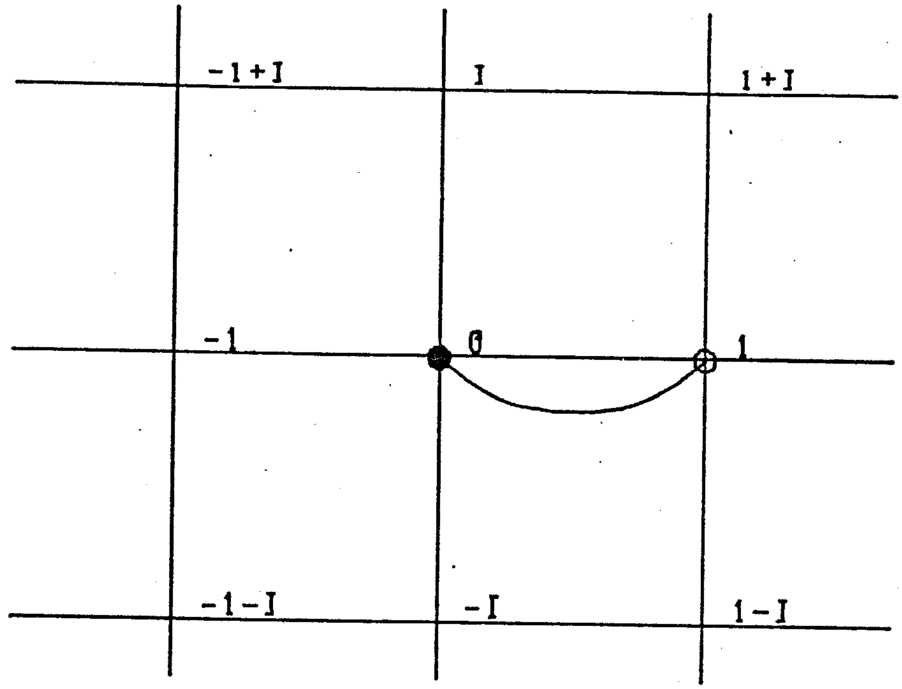

Figure 51: Region A_{20}

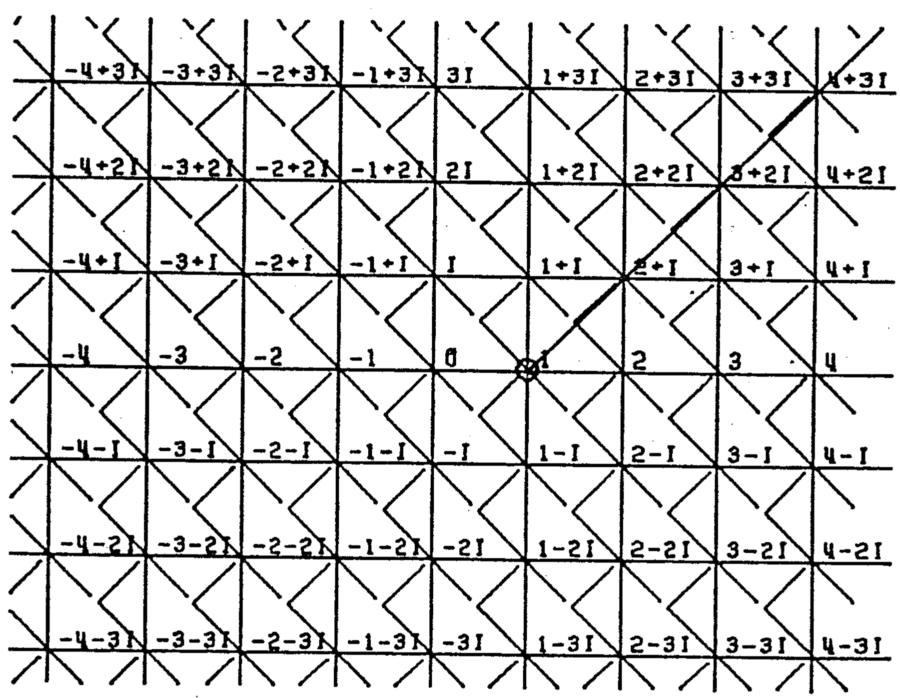

Figure 52: Region B_{20}

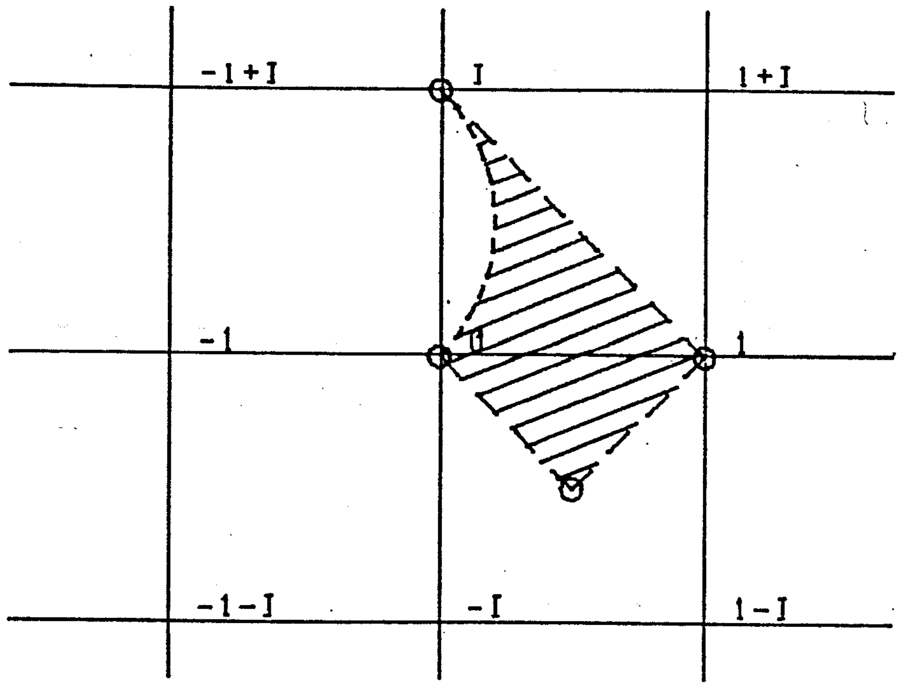

Figure 53: Region A_{21}

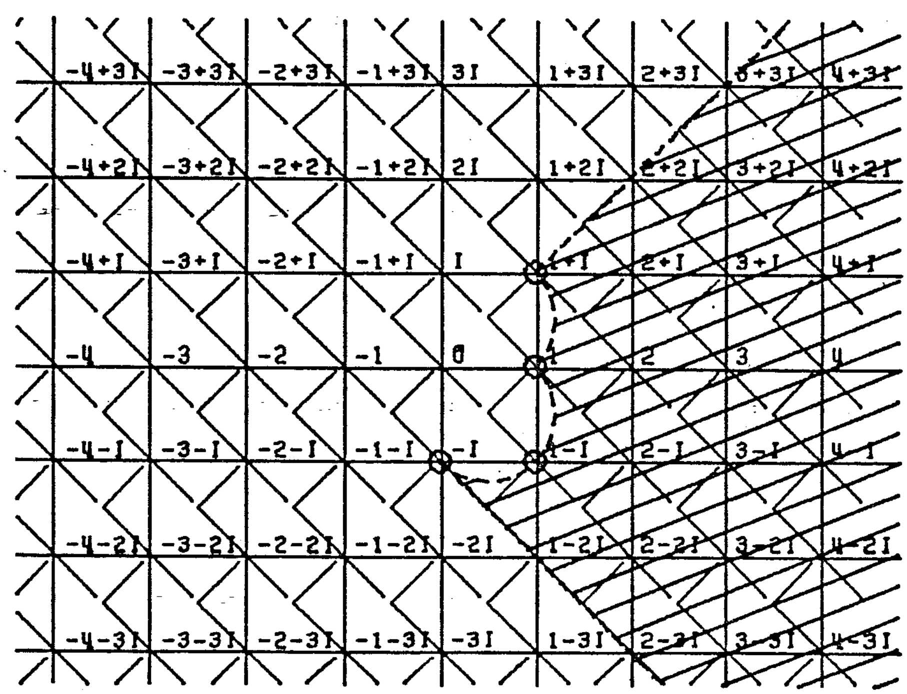

Figure 54: Region B_{21}

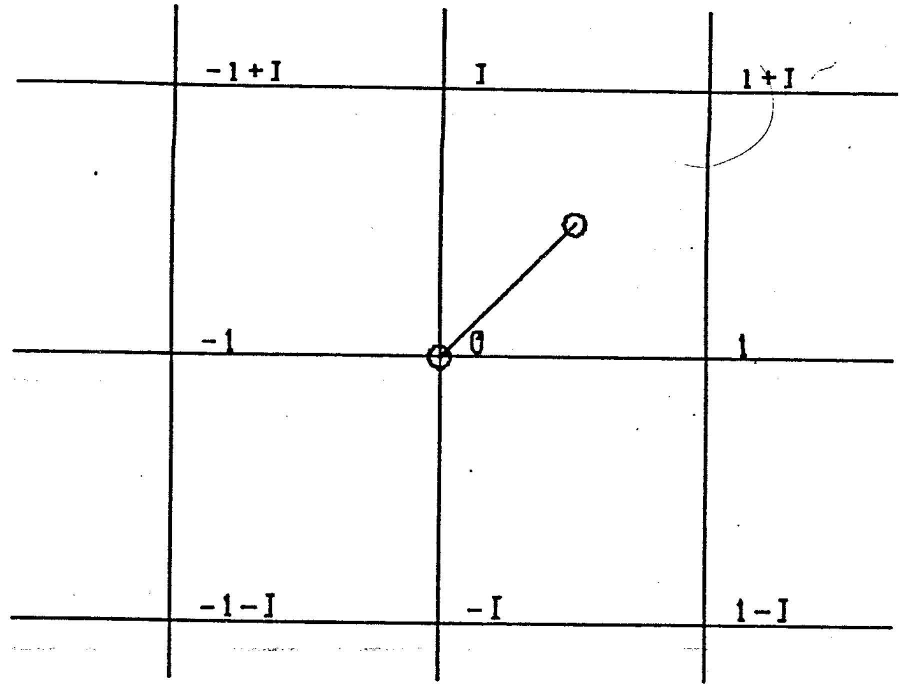

Figure 55: Region A_{22}

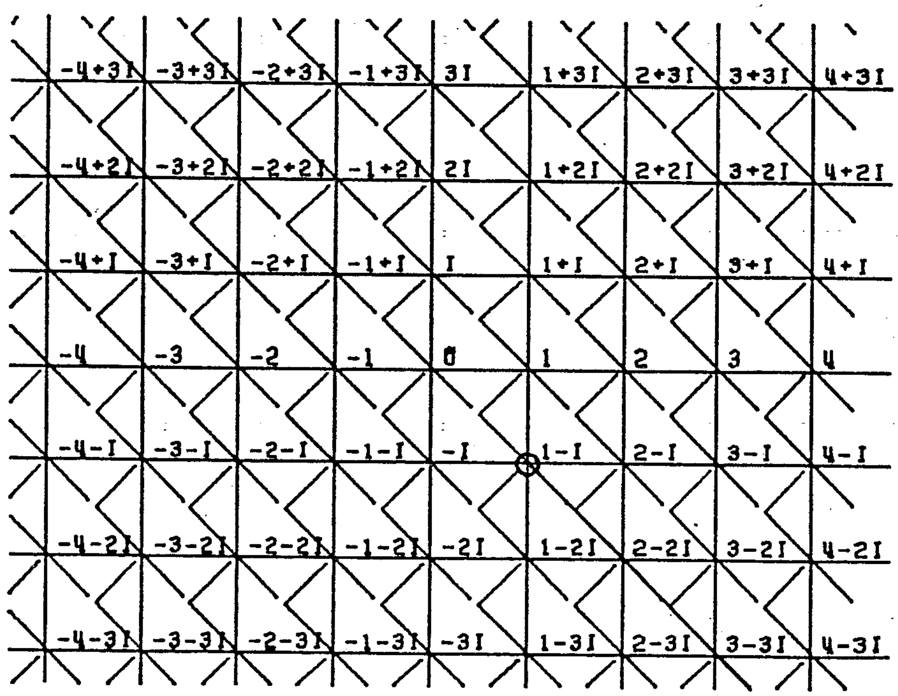

Figure 56: Region B_{22}

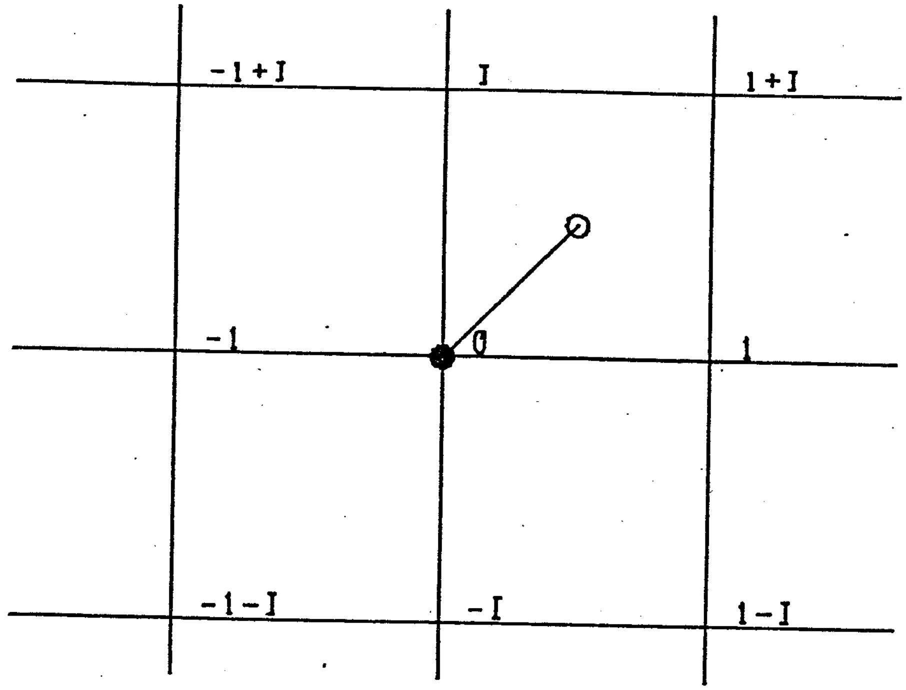

Figure 57: Region A_{23}

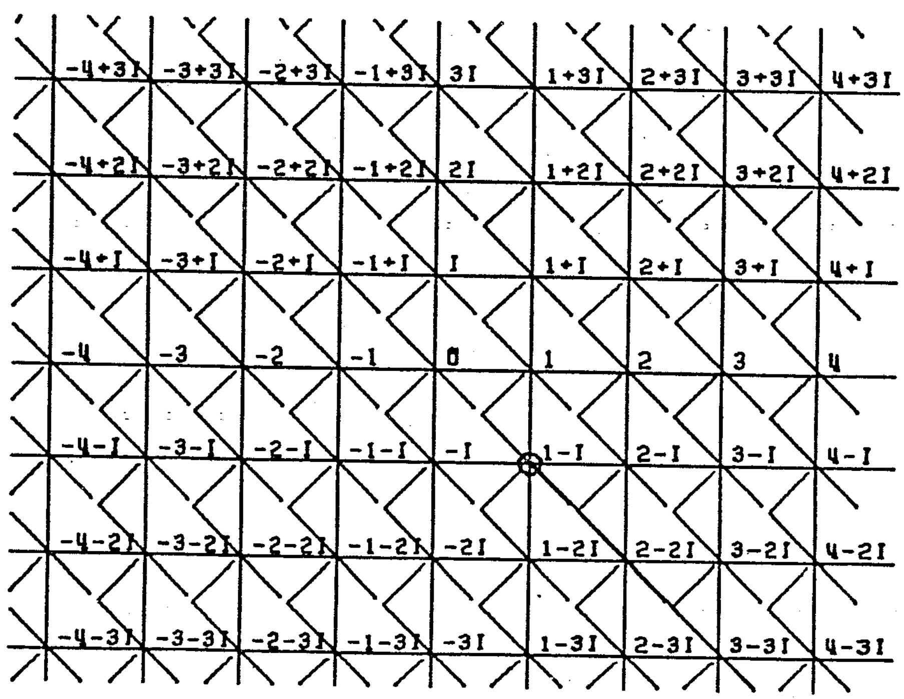

Figure 58: Region B_{23}

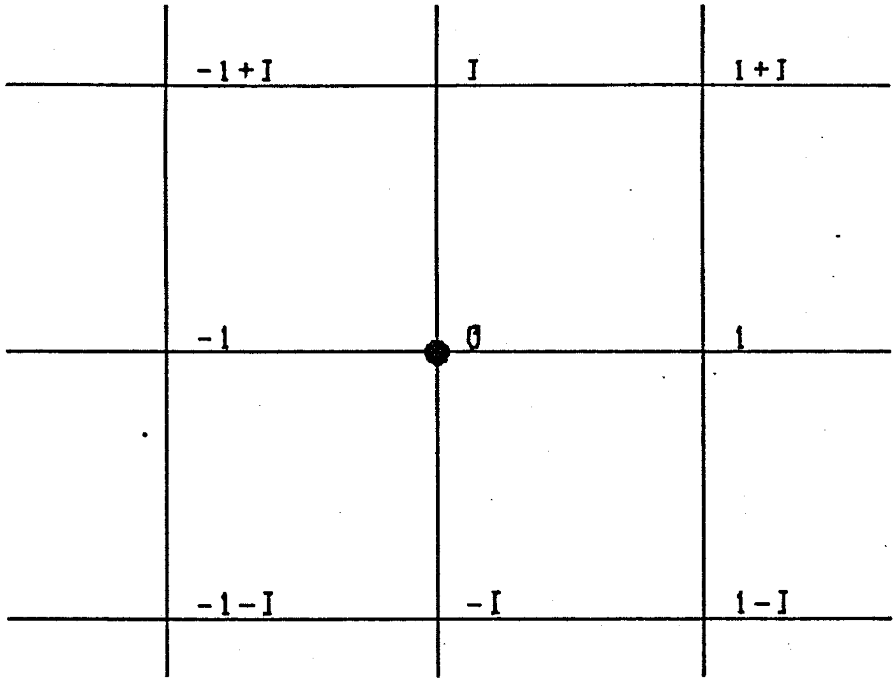

Figure 59: Region A_{24}

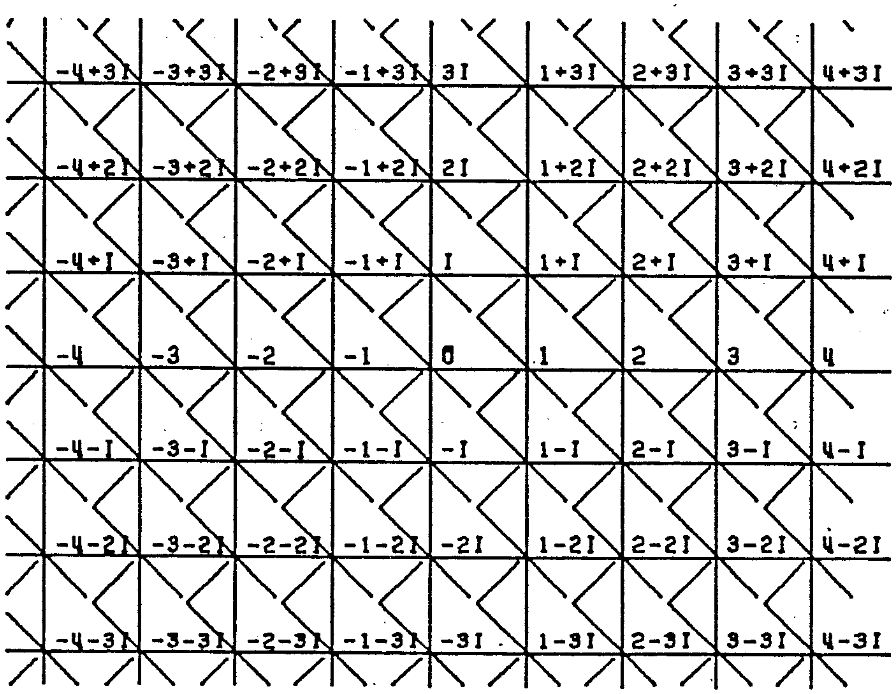

Figure 60: Region B_{24}

We will now prove first that CFE $\subset$ L(G), i.e., if $(a_0, a_1, \ldots a_n) = cf(z)$ for some $z \in ¢$, then the string $a_0a_1\ldots a_n$ is in L(G).

The proof is by induction. We assert the existence of the productions

$$D_0 \dashrightarrow a_0D_{i_0}$$
$$D_{i_0} \dashrightarrow a_1D_{i_1}$$
$$D_{i_1} \dashrightarrow a_2D_{i_2}$$
$$\vdots$$
$$D_{i_{j-2}} \dashrightarrow a_{j-1}D_{i_{j-1}}$$

and also that $x_j \in B_{i_{j-1}}$.

For we find $x_0 \in B_0$ and $a_0 \in C_0 \implies a_0 \in c_0(1) \implies x_0 - a_0 \in A_1$ and we have the corresponding production $D_0 \dashrightarrow D_1$. Since $x_0 - a_0 \in A_1$, $x_1 = 1/(x_0 - a_0) \in B_1$.

Now assume true for j. We find $x_j \in B_{i_{j-1}} \implies a_j = cfl(x_j) \in C_{i_{j-1}}$. Since the set $C_{i_{j-1}}$ is partitioned into the $c_{i_{j-1}}(k)$, there exists some i_j such that $a_j \in c_{i_{j-1}}(i_j)$. Then $x_j - a_j = x_j - cfl(x_j) \in A_{i_j}$. Hence $x_{j+1} = 1/(x_j - a_j) \in B_{i_j}$. But since $a_j \in c_{i_{j-1}}(i_j)$, there exists a production $D_{i_{j-1}} \dashrightarrow a_jD_{i_j}$.

By induction, we have the productions

$$D_0 \dashrightarrow a_0 D_{i_0}$$
$$D_{i_0} \dashrightarrow a_1 D_{i_1}$$
$$\vdots$$
$$D_{i_{n-1}} \dashrightarrow a_n D_{i_n}$$

and $x_n \in B_{i_{n-1}}$.

Now $x_n \in B_{i_{n-1}} \Longrightarrow a_n = cfl(x_n) \in C_{i_{n-1}}$. Since the set $C_{i_{n-1}}$ is partitioned into the $c_{i_{n-1}}(k)$, there exists i_n such that $a_n \in c_{i_{n-1}}(i_n)$. Then $x_n - a_n \in A_{i_n}$. But $x_n = a_n$ by the definition of the cf algorithm. Hence $0 \in A_{i_n}$. Hence there exists a production $D_{i_n} \dashrightarrow \emptyset$.

Thus we have demonstrated the existence of the productions

$$D_0 \dashrightarrow a_0 D_{i_0}$$
$$\vdots$$
$$D_{i_{n-1}} \dashrightarrow a_n D_{i_n}$$
$$D_{i_n} \dashrightarrow \emptyset.$$

Hence we find $D_0 \Longrightarrow a_0 D_{i_0} \Longrightarrow a_0 a_1 D_{i_1} \Longrightarrow \ldots a_0 a_1 \ldots a_n D_{i_n} \Longrightarrow a_0 a_1 a_2 \ldots a_n$.

Now we prove $L(G) \subset CFE$.

Let $a_0 a_1 \ldots a_n$ be a string in $L(G)$. Then there exists a production $D_{i_n} \rightarrow \emptyset$. Hence $0 \in A_{i_n}$. If we put $x_n = a_n$, then $x_n - a_n = 0 \in A_{i_n}$. Now there exists a production $D_{i_{n-1}} \rightarrow a_n D_{i_n}$; hence there exists $c_{i_{n-1}}(i_n)$ for which $a_n \in c_{i_{n-1}}(i_n)$. Then $a_n \in C_{i_{n-1}}$. Also, $x_n = a_n \in B_{i_{n-1}}$. If we put $x_{n-1} = a_{n-1} + 1/x_n$, then clearly $x_{n-1} - a_{n-1} \in A_{i_{n-1}}$.

Proceeding in this manner, we define x_{n-2}, x_{n-3}, ... x_1, x_0 inductively. At each step we have

$$x_j - a_j \in A_{i_j}$$

and there exists a production $D_{i_{j-1}} \rightarrow a_j D_j$. Hence there exists $c_{i_{j-1}}(i_j)$ for which $a_j \in c_{i_{j-1}}(i_j)$. Then $a_j \in C_{i_{j-1}}$. If $x_j = a_j + 1/x_{j+1}$, then $x_j \in B_{i_{j-1}}$. Putting $x_{j-1} = a_{j-1} - 1/x_j$ results in $x_{j-1} - a_{j-1} \in A_{i_{j-1}}$.

Hence we have produced exactly the x_0, x_1, ... x_n and a_0, a_1, ... a_n which would have been produced by the cf algorithm and hence $L(G) \subset CFE$. We therefore have $cf(x_0) = (a_0, a_1, \ldots a_n)$. Incidentally, this shows that if a_0, a_1, ... a_n corresponds to a string in $L(G)$, then $cf(val(a_0, a_1, \ldots a_n)) = (a_0, a_1, \ldots a_n)$.

From this point on, we will call a sequence of complex numbers $(a_0, a_1, \ldots a_n)$ <u>legal</u> if the string $a_0 a_1 \ldots a_n \in L(G)$. Thus, we have also proven the following

<u>Theorem II.3.4.</u>

$(a_0, a_1, \ldots a_n) = cf(val(a_0, a_1, \ldots a_n))$ iff the sequence $(a_0, a_1, \ldots a_n)$ is legal.

Remarks.

Hurwitz [1] defined a continued fraction algorithm using the cmid function. He did not, however, give a rule to decide whether or not a given expansion was generated by his algorithm. An attempt to follow the same procedure as above leads to many more regions of strange shapes, and it is not clear, in fact, that there are only a finite number of such regions for Hurwitz's algorithm.

4. Continued Fractions for Pure Imaginary Numbers

In this section, we will examine the first of two special topics--the description of continued fractions for pure imaginary numbers.

In analogy with regular continued fractions, we may define similar entities called <u>reduced continued fractions</u>, as follows:

$val'(b_0, b_1, \ldots b_n) = b_0 - 1/(b_1 - 1/(\ldots b_n))$.

$rcf(y) = (b_0, b_1, \ldots b_n)$ where

$y_0 = y$, $b_n = ce(y_n)$, $y_{n+1} = 1/(b_n - y_n)$.

This expansion has properties similar to those of regular continued fractions. For example, see Tietze[10] and Sierpinski [11].

In particular, we have

(a) $b_j \geq 2$ for $j \geq 1$.

(b) The rcf expansion terminates iff x is rational.

(c) An infinite expansion that is not all 2's after some point represents an irrational number. Conversely, every irrational number has a non-terminating expansion that is not all 2's after some point.

(d) The expansion of the root of a quadratic equation with integer co-efficients is periodic after some point. The converse is also true.

We will now prove the following theorem relating the rcf expansion to the cf expansion for a pure imaginary number.

Theorem II.4.1.

Suppose $rcf(-a) = (b_0, b_1, b_2, \ldots)$, and a is real. Then $cf(ai) = (-ib_0, -ib_1, -ib_2, \ldots)$.

Proof.

We will use induction. First, for the cf algorithm, we find

$$x_0 = ai,\ a_0 = fl(ai) = ifl(a).$$

If $x_j = bi$, then $a_j = fl(bi) = ifl(b)$.

$$x_{j+1} = 1/(x_j - a_j) = 1/i(b-fl(b)) = -i/(b-fl(b))$$

$$a_{j+1} = fl(x_{j+1}) = ifl(-1/(b-fl(b)) = -ice(1/(b-fl(b))$$

For the rcf algorithm, we find

$$y_0 = -a;\ b_0 = ce(-a) = -fl(a)$$

If $y_j = -b$, then

$$b_j = ce(y_j) = -fl(b)$$

$$y_{j+1} = 1/(b_j - y_j) = 1/(-fl(b) - (-b)) = 1/(b-fl(b))$$

$$b_{j+1} = ce(y_{j+1}) = ce(1/(b-fl(b))).$$

By induction we see

$$-ib_j = a_j.$$

Hence the theorem follows.

Example.

It is well known (see Tietze [10]) that

$$rcf(\sqrt{2}) = (2, 2, 4, 2, 4, 2, 4, \ldots).$$

Applying the above theorem, we get

$$cf(-\sqrt{2}i) = (-2i, -2i, -4i, -2i, -4i, \ldots).$$

5. The Complex GCD. The Complex Linear Diophantine Equation.

The method of Section II.2 can now be extended to the complex plane to give a greatest common divisor algorithm for the complex numbers.

The proof of Theorem II.2.1 goes through without change.

As an example, let us calculate the gcd of r and s where $r = 44+17i$, $s = -22+79i$.

$$44+17i = (-i)(-22+79i) + -35-5i$$

$$-22+79i = (-2i)(-35-5i) + -12-9i$$

$$-35-5i = (2+i)(-12+9i) + -2-11i$$

$$-12+9i = (-1-i)(-2-11i) + -3-4i$$

$$-2-11i = (2+i)(-3-4i)$$

Hence $-3-4i$ is a gcd of r and s.

Now we will examine another topic of interest--the solution of the complex linear Diophantine equation. We have the following

Theorem II.5.1.

The complex linear Diophantine equation

$$Aw + Bz = C \qquad (A,B,C,z,w \in \mathbb{Z}[i])$$

has a solution (w,z) iff $\gcd(A,B) \mid C$.

Proof.

Suppose (w,z) is a solution but $x = \gcd(A,B) \nmid C$. Then $x|A$ and $x|B$; hence $x|Aw + Bz$. Hence we have a contradiction.

Now suppose $\gcd(A,B)|C$. We will construct a solution (w,z). If we expand A/B as a continued fraction, we find

$$cf(A/B) = (a_0, a_1, \dots a_n).$$

Let p_n/q_n be the n-th convergent to the CF. Then

$$p_n/q_n = A/B$$

Hence $A = xp_n$, $B = xq_n$ for some $x \in \mathbb{Z}[i]$. Then we have

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}.$$
$$xp_n q_{n-1} - xp_{n-1} q_n = (-1)^{n-1} x.$$
$$Aq_{n-1} - Bp_{n-1} = (-1)^{n-1} x.$$

Since $x|A$ and $x|B$, we have $x|C$. Thus we find

$$CA/x \cdot q_{n-1} - CB/x \cdot p_{n-1} = (-1)^{n-1} C$$

Thus $(Cq_{n-1}/x, \; -Cp_{n-1}/x)$ is a solution for n odd, and $(-Cq_{n-1}/x, \; Cp_{n-1}/x)$ is a solution if n is even.

Example.

Solve $(7+17i)w + (13-3i)z = 7-i$ in complex integers w,z.

Solution.

Let $A = 7+17i$, $B = 13-3i$, $C = 7-i$. We find $\gcd(A,B) = 1+i$ which divides $C = (1+i)(3-4i)$. Also,

$$cf(A/B) = (i, 1-2i, 4)$$

and $p_2/q_2 = (12+5i)/(5-8i)$; $p_1/q_1 = (3+i)/(1-2i)$.

Hence $x = A/p_n = 1+i$. Since n is even, a solution is $(w,z) = (5+10i, 13-9i)$.

6. Infinite Complex Continued Fractions

Not all infinite complex continued fractions converge. For example, the convergents to

$$(0, i, i, i, \ldots)$$

are periodic with period 12:

$$0/1,\ 1/i,\ i/0,\ 0/i,\ i/-1,\ -1/0,$$

$$0/-1,\ -1/-i,\ -i/0,\ 0/-i,\ -i/1,\ 1/0.$$

Hence $\lim_{n\to\infty} \mathrm{val}(0, \underbrace{i, i, i, \ldots, i}_{n})$ does not exist.

The following theorem, which is not best possible, is a simple criterion for convergence.

Theorem II.5.1.

All legal infinite CF expansions not involving the terms $1, 2, -i, -2i, 1+i, 1-i, -1-i$ converge.

Proof.

Since $|p_n/q_n - p_{n-1}/q_{n-1}| = |1/q_n q_{n-1}|$ by Theorem II.1.8, it suffices to show $|q_n| > (1+e)|q_{n-1}|$ for some $e > 0$. We can then use the Cauchy criterion to prove convergence.

Put $k_n = q_n/q_{n-1}$. Then we find

$$\begin{aligned} k_n &= (a_n q_{n-1} + q_{n-2})/q_{n-1} \\ &= a_n + q_{n-2}/q_{n-1} \\ &= a_n + 1/k_{n-1}. \end{aligned}$$

Since $k_1 = q_1/q_0 = a_1/1$, if $|k_1| \le 1+e$, then we have

$a_1 \in \{0, -1, i, 1, -i\}$. But the first three values are ruled out by Theorem II.3.2. The last two possibilities are ruled out by the hypothesis of the theorem.

Now we'll use induction. Assume $|k_{n-1}| > 1+e$ but $|k_n| \leq 1+e$. Then $a_n = k_n - 1/k_{n-1}$ and $|1/k_{n-1}| < 1/(1+e)$. But

$$\begin{aligned} |a_n| &= |k_n - 1/k_{n-1}| \\ &\leq 1+e + 1/(1+e) \\ &\leq 2+e'. \end{aligned}$$

Hence $a_n \in \{0, -1, -2, i, 2i, -1+i, 1, 2, -i, -2i, 1+i, 1-i, -1-i\}$. But the first six of these are ruled out by Theorem II.3.2; the hypothesis of the theorem rules out the last seven. Hence we have a contradiction and we see that $|k_n| > 1+e$, as desired.

7. Unsolved Problems and Conjectures

The theory of complex continued fractions as we have developed it here is by no means complete. For example, it would be nice to be able to say exactly what infinite complex continued fractions converge.

The following conjectures are supported by numerical evidence, but have not been proven.

1. Let $cf(x) = (a_0, a_1, \ldots)$ and assume $\lim_{n\to\infty} val(a_0, a_1, \ldots a_n)$ exists and $= y$. Then $y = x$.

2. Define an infinite expansion to be legal if there exists a production $D_{j_{n-1}} \to a_n D_{j_n}$ for all $n \geq 1$. Then $val(a_0, a_1, \ldots)$ converges to a limit x.

3. If $(a_0, a_1, \ldots)$ is legal, then $val(a_0, a_1, \ldots)$ is irrational unless there exists N such that $a_j = -2i$ for all $j \geq N$. Also, if $val(a_0, a_1, \ldots)$ is irrational, then

$$cf(val(a_0, a_1, \ldots)) = (a_0, a_1, \ldots).$$

4. Let z be an irrational root of

$$Az^2 + Bz + C,$$

where $A, B, C \in \mathbb{Z}[i]$, $A \neq 0$. Then $cf(z_1)$ is eventually periodic.

5. If $(a_0, a_1, \ldots)$ is legal and eventually periodic (and the period is not $-2i$), then $\mathrm{val}(a_0, a_1, \ldots)$ is a root of a quadratic equation with complex integer co-efficients.

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