1387. *Proposed by Kenneth Stolarsky, University of Illinois at Urbana-Champaign, Urbana, Illinois.*

Given $\varepsilon > 0$ and a function $f(x)$, continuous on $(-\infty, \infty)$, must there exist a function $g(x)$, continuous on $[0,1]$, such that

$$\inf_{-\infty < y < \infty} \left(\max_{0 \leqslant x \leqslant 1} |g(x) - f(x+y)| \right) \geqslant \varepsilon ?$$

Quickies

Answers to the Quickies are on page 357.

Q784. *Proposed by Michael Golomb, Purdue University, West Lafayette, Indiana.*

Let Δ be a triangle subdivided into triangles $\Delta_1, \Delta_2, \ldots, \Delta_n$. Let $A, A_1, A_2, \ldots, A_n$ denote the areas of the ellipses of maximum area inscribed in $\Delta, \Delta_1, \Delta_2, \ldots, \Delta_n$, respectively. Show that

$$A = A_1 + A_2 + \cdots + A_n .$$

Q785. *Proposed by Jeffrey Shallit, University of Waterloo, Waterloo, Canada.*

Prove that every real number in the interval $[0,2]$ can be written as the sum of two elements of the Cantor set.

Solutions

Generalized Pentagonal Numbers **December 1990**

1358. *Proposed by Peter Ross, Santa Clara University, Santa Clara, California.*

Which positive integers are representable in the form $\binom{k}{2} + kn$, $k > 1$, $n \geqslant 1$?

Solution by Jack V. Wales, Jr., The Thacher School, Ojai, California.

A positive integer x is representable in the given form if and only if x is not a power of 2, that is, if and only if x has an odd factor greater than 2. Further, any such x has as many such representations as there are distinct factorizations of $2x$ into two distinct factors, one of which is odd.

Let x be a number of the form $\binom{k}{2} + nk$, $k > 1$, $n \geqslant 1$. Then $x = k(k-1)/2 + nk$, or

$$2x = k(k + 2n - 1).$$

We see that $2x$ must have an odd factor greater than 2, and therefore so must x.

we have equivalently that

$$AP^2 + BP^2 + CP^2 + DP^2 = 4R^2, \quad (R = \text{radius of the circle}).$$

The latter corresponds to the known result (*Crux Mathematicorum* 15 (1989) 293, #1) that the sum of the areas of the four circles whose diameters are AP, BP, CP, and DP is equal to the area of the given circle. In this result it is assumed that P lies within the circle. But the above proof shows that it is valid if P is outside the circle. This four-circle result apparently has been generalized (*Crux Mathematicorum* 16 (1990), p. 109, #1535) to a result concerning two intersecting chords in an ellipse. However, the ellipse result can be shown to follow from the circle result by an affine transformation.

Answers

Solutions to the Quickies on page 351.

A784. By an affine transformation, one can transform a given triangle T into an equilateral triangle. Such a transformation transforms inscribed ellipses into inscribed ellipses and preserves ratios of areas. Therefore, if $A(T)$ is the area of T and $A(E)$ is the maximal area of an inscribed ellipse,

$$A(E) = \gamma A(T),$$

for some constant γ independent of T. The claim follows. (For the equilateral triangle, the inscribed ellipse of maximal area is the inscribed circle, hence $\gamma = \pi\sqrt{3}/9$.)

A785. Recall that the Cantor set C is the set of all real numbers in $[0,1]$ containing only the digits 0 and 2 in their base-3 representation. Thus we can state the result as $[0,2] \subseteq C + C$.

Let D be the set of all real numbers containing only the digits 0 and 1 in their base-3 representation. It is easy to see that $[0,1] \subseteq D + D$. For example, if $x \in [0,1]$ then we write its base-3 representation as

$$x = .x_1x_2x_3x_4\dots.$$

Then we put

$$y_i = \begin{cases} 1, & \text{if } x_i \geqslant 1, \\ 0 & \text{otherwise}. \end{cases}$$

$$z_i = \begin{cases} 1, & \text{if } x_i \geqslant 2, \\ 0 & \text{otherwise}. \end{cases}$$

Then $x_i = y_i + z_i$, so $x = y + z$ (with no carries!).

To prove $[0,2] \subseteq C + C$, we take an $x \in [0,2]$. Express $x/2 = y + z$, with $y, z \in D$. Then $x = 2y + 2z$. But $2y$ and $2z$ are numbers containing only the digits 0 and 2 in their base-3 representation. (Note: Since $C + C \subseteq [0,2]$, it follows that $C + C = [0,2]$.)

This result is not new (but the proof seems to be). It forms the basis of *Marshall Hall's* result in his paper "On the sum and product of continued fractions" (*Ann. Math.* (2) 48 (1947), 966–993). Also, it is a theorem of *E. Borel* (*Éléments de la Théorie des Ensembles, Paris*, 1949, Note V). Also, see *M. Pavone*, The Cantor set and a geometric construction (*L'enseignement Math.* 35 (1989), 41–49).

It can also be found in Steinhaus's collected papers, the English translation of a paper published in Wektor in 1917, and a related result in Randolph, J. F. (1940). Distances between points of the Cantor set. Amer. Math. Monthly. 47(8):549–551.