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A TRIANGLE FOR THE BELL NUMBERS

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The Bell, or exponential, numbers B_n are defined by

$$B_n = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!} = \frac{1}{e}\left(\frac{0^n}{0!} + \frac{1^n}{1!} + \frac{2^n}{2!} + \cdots\right) \tag{1}$$

The first twelve Bell numbers are given in the following table:

TABLE 1. Bell Numbers

n	B_n
0	1
1	1
2	2
3	5
4	15
5	52
6	203
7	877
8	4140
9	21147
10	115975
11	678570

The Bell numbers also appear in the Maclaurin expansion of e^{e^x}:

$$e^{e^x} = e\sum_{k=0}^{\infty}\frac{B_k x^k}{k!} = e\left(1 + \frac{x}{1!} + \frac{2x^2}{2!} + \frac{5x^3}{3!} + \frac{15x^4}{4!} + \cdots\right) \tag{2}$$

The Bell numbers can be generated recursively by an interesting method described in [2]. If we take the array described in this article and "flip" it about and then reorient it, the following triangle appears. This triangle is similar in form to Pascal's triangle. We shall call it the "Bell Triangle," and denote each element by $B'(n,r)$. This notation is similar to $C(n,r)$ for Pascal's triangle. There are three rules of formation for this triangle.

(3) $$B'(0,0) = 1$$

(4) $$B'(n,0) = B'(n-1,n-1) \qquad (n \geq 1)$$

(5) $$B'(n,r) = B'(n,r-1) + B'(n-1,r-1) \qquad (1 \leq r \leq n)$$

$B'(3,2) + B'(4,2) = B'(4,3)$

Row								
0	1							
1	1	2						
2	2	3	5					
3	5	7	10	15				
4	15	20	27	37	52			
5	52	67	87	114	151	203		
6	203	255	322	409	523	674	877	
7	877	1080	1335	1657	2066	2589	3263	4140

The Bell numbers form the left and right sides of the triangle. In fact,

(6) $$B'(n,n) = B_{n+1}$$

(7) $$B'(n,0) = B_n$$

Equations (6) and (7) follow from the two equivalent identites for Bell numbers:

(8) $$B_n = \binom{n}{0}B_{n+1} - \binom{n}{1}B_n + \binom{n}{2}B_{n-1} - \cdots \pm \binom{n}{n}B_1$$

(9) $$B_n = nB_{n-1} - \binom{n-1}{2}B_{n-2} + \binom{n-1}{3}B_{n-3} - \cdots \pm \binom{n-1}{n-1}B_1$$

The Bell triangle has many interesting properties. Here we present several new identities:

(10) $$\sum_{k=a}^{b} B'(n,k) = B'(n+1,b+1) - B'(n+1,a).$$

For $a = 0$ and $b = n$, this reduces to

(11) $$\sum_{k=0}^{n} B'(n,k) = B'(n+1,n+1) - B'(n+1,0) = B'(n+1,n) = B_{n+2} - B_{n+1},$$

(12) $$\sum_{k=x}^{n} B'(k+a,k) = B'(n+a,n+1) - B'(x+a-1,x),$$

(13) $$\sum_{k=x}^{n} (-1)^{n-k}\binom{n-x}{k-x}B'(k+a,k) = B'(n+a,x).$$

For $a = 0$ and $x = 0$, equation (13) reduces to (8).

(14) $$\sum_{k=x}^{n} \binom{n-x}{k-x}B'(k,a) = B'(n,a+n-x).$$

For $a = 0$ and $x = 0$, the following identity results:

(15) $$\sum_{k=0}^{n} \binom{n}{k}B'(k,0) = B'(n,n).$$

This is equivalent to

(16) $$\sum_{k=0}^{n} \binom{n}{k}B_k = B_{n+1}.$$

To my knowledge, identities (10)-(14) were heretofore unknown.

If we ignore the restricting inequality in (4), and substitute $n = 0$, we get $1 = B'(0,0) = B'(-1,-1)$. From this value, we may obtain values of $B'(n,-1)$ for $n \geq -1$ (see Table 2).

Note the following identity:

(17) $$B'(n-1,-1) + B'(n,-1) = B'(n,0) = B_n.$$

TABLE 2. Values of $B'(n,-1)$

n	$B'(n,-1)$
-1	1
0	0
1	1
2	1
3	4
4	11
5	41
6	162
7	715

Apparently the Bell triangle cannot be extended further because $B(-1,0) = B_{-1}$ which is undefined, by equation (1). Epstein [3] drops the term $0^n/0!$ in equation (1) without explanation and therefore gets $B_0 = 1 - 1/e$, in contradiction with Williams [5], Bell [1], and Rota [4].

The Bell numbers have combinatoric significance in that B_n is the number of ways of factoring a product of n distinct primes. Whether the rest of the numbers in the Bell triangle have any such significance remains to be seen.

REFERENCES

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