Simple Continued Fractions for Some Irrational Numbers

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Abstract

It is proved that the simple continued fractions for the irrational numbers defined by

$$\sum_{k=0}^{\infty} \frac{1}{u^{2^k}} \quad (u \geq 3, \text{ an integer})$$

and related quantities are predictable, that is, have a definite pattern. The proof uses only elementary properties of continued fractions. The nature of the partial quotients is discussed.

1 Introduction

The continued fraction for a real number x is an expansion of the form

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

where the a_i's are positive integers, except for a_0, which is an integer. The a's are called the partial denominators[1] of the continued fraction.

It is well-known that the continued fraction for x terminates if and only if x is rational. On the other hand, if the continued fraction is infinite, and the a's are periodic after some point, then x is a quadratic irrational.

There are also well-known patterns in the expansions for e, e^2, $\tanh 1/k$, etc.

The purpose of this paper is to announce a new result concerning continued fractions; namely, that the continued fraction expansions for the irrational numbers defined by

$$\sum_{k=0}^{\infty} \frac{1}{u^{2^k}} \quad (u \geq 3, \text{ an integer})$$

[1] Readers may be puzzled by the use of the idiosyncratic term "partial denominators" in place of the much more familiar "partial quotients". This term was not my choice, but was imposed on me by the referee.

and related quantities are predictable, that is, have a definite pattern.

2 Elementary properties of continued fractions

We write

$$\begin{aligned} p_n/q_n &= a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n}}} \\ &= [a_0, a_1, a_2, \ldots, a_n] \end{aligned}$$

We call p_n/q_n the n'th convergent.

Now we recall some of the elementary properties of continued fractions, which are well-known and easily proved (for example, see Perron [1] or Hardy and Wright [2, p. 129]).

CF1: Let $p_n/q_n = [a_0, a_1, a_2, \ldots, a_n]$. Then

$$\begin{aligned} p_n &= a_n p_{n-1} + p_{n-2}, \quad (n \geq 1), \quad p_{-1} = 1,\ p_0 = a_0 \\ q_n &= a_n q_{n-1} + q_{n-2}, \quad (n \geq 1), \quad q_{-1} = 0,\ q_0 = 1 \end{aligned}$$

CF2: $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$

CF3: The continued fraction for a real rational number x is unique, apart from the fact that if $a_n \geq 2$, then

$$x = [a_0, a_1, \ldots, a_n] = [a_0, a_1, \ldots, a_n - 1, 1].$$

CF4: The convergents are always in lowest terms.

CF5: If $p_n/q_n = [a_0, a_1, \ldots, a_n]$ and $r_m/s_m = [b_0, b_1, \ldots, b_m]$ then

$$[a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_m] = \frac{p_{n-1} s_m + p_n r_m}{q_{n-1} s_m + q_n r_m}$$

CF6: If $p_n/q_n = [a_0, a_1, \ldots, a_n]$ then

$$[a_n, a_{n-1}, \ldots, a_2, a_1] = q_n/q_{n-1}$$

3 A Theorem

Theorem 1 *Let*

$$B(u, v) = \sum_{0 \leq k \leq v} \frac{1}{u^{2^k}} = \frac{1}{u} + \frac{1}{u^2} + \frac{1}{u^4} + \cdots + \frac{1}{u^{2^v}}$$

($u \geq 3$, an integer). Then

(A) $B(u,0) = [0,u]$;

$B(u,1) = [0,u-1,u+1]$.

(B) *If* $B(u,v) = [a_0,a_1,\ldots,a_n] = p_n/q_n$ *then* $B(u,v+1) = [a_0,a_1,\ldots,a_{n-1},a_n+1,a_n-1,a_{n-1},a_{n-2},\ldots,a_2,a_1]$.

Proof. Part (A) is easily verified by a short computation. Let us prove part (B). We have

$$\begin{aligned}
[a_0,a_1,a_2,\ldots,a_{n-1}] &= p_{n-1}/q_{n-1} \\
[a_0,a_1,a_2,\ldots,a_{n-1},a_n+1] &= (p_n+p_{n-1})/(q_n+q_{n-1}) \quad (CF1) \\
[a_0,a_1,a_2,\ldots,a_{n-1},a_n+1,a_n-1] &= \frac{(a_n-1)(p_n+p_{n-1})+p_{n-1}}{(a_n-1)(q_n+q_{n-1})+q_{n-1}} \quad (CF1) \quad (1) \\
[a_{n-1},a_{n-2},\ldots,a_2,a_1] &= q_{n-1}/q_{n-2} \quad (CF6) \quad (2)
\end{aligned}$$

Applying (CF5) to equations (1) and (2), we find

$$\begin{aligned}
&[a_0,a_1,\ldots,a_{n-1},a_n+1,a_n-1,a_{n-1},a_{n-2},\ldots,a_1] \\
&\quad = \frac{(p_n+p_{n-1})(q_{n-2})+((a_n-1)(p_n+p_{n-1})+p_{n-1})(q_{n-1})}{(q_n+q_{n-1})(q_{n-2})+((a_n-1)(q_n+q_{n-1})+q_{n-1})(q_{n-1})} \\
&\quad = \frac{p_nq_{n-2}+p_{n-1}q_{n-2}+a_np_nq_{n-1}+a_np_{n-1}q_{n-1}-p_nq_{n-1}}{q_nq_{n-2}+q_{n-1}q_{n-2}+a_nq_nq_{n-1}+a_nq_{n-1}^2-q_nq_{n-1}}.
\end{aligned} \quad (3)$$

From (CF1) it follows that

$$\begin{aligned}
(p_n-p_{n-2})q_{n-1} &= a_np_{n-1}q_{n-1} \quad (4) \\
(q_n-q_{n-2})p_n &= a_np_nq_{n-1} \quad (5) \\
(q_n-q_{n-2})q_n &= a_nq_nq_{n-1} \quad (6) \\
(q_n-q_{n-2})q_{n-1} &= a_nq_{n-1}^2 \quad (7)
\end{aligned}$$

Substituting equations (4)–(7) in the right side of (3), we obtain

$$\begin{aligned}
&[a_0,a_1,\ldots,a_{n-1},a_n+1,a_n-1,a_{n-1},a_{n-2},\ldots,a_1] \\
&\quad = \frac{p_{n-1}q_{n-2}-p_{n-2}q_{n-1}+p_nq_n}{q_n^2} \quad (8)
\end{aligned}$$

At this point, let us assume that n is even — an assumption which will later be verified by induction. Since n is even,

$$p_{n-1}q_{n-2}-p_{n-2}q_{n-1} = (-1)^{n-2} = 1 \quad \text{(CF2)} \quad (9)$$

Substituting (9) in the right side of (8), we find

$$[a_0,a_1,\ldots,a_{n-1},a_n+1,a_n-1,a_{n-1},a_{n-2},\ldots,a_1] = \frac{p_nq_n+1}{q_n^2}. \quad (10)$$

We now show that $q_n = u^{2^v}$. We have

$$\begin{aligned} p_n/q_n &= B(u,v) \\ &= \sum_{0\le k\le v} \frac{1}{u^{2^v}} \\ &= \frac{R}{u^{2^v}} \end{aligned}$$

where $R = \sum_{0\le k\le v} u^{2^v-2^k}$. Now R is not divisible by u (and therefore not by u^{2^v}) since

$$R = 1 + u \sum_{0\le k\le v-1} u^{2^v-2^k-1}.$$

Hence $p_n/q_n = R/u^{2^v}$ in lowest terms. Applying (CF4), we conclude that $q_n = u^{2^v}$.

Therefore,

$$\begin{aligned} [a_0, a_1, \ldots, a_{n-1}, a_n+1, a_n-1, a_{n-1}, a_{n-2}, \ldots, a_1] &= \frac{p_n q_n + 1}{q_n^2} \\ &= \frac{p_n}{q_n} + \frac{1}{q_n^2} \\ &= B(u,v) + \frac{1}{(u^{2^v})^2} \\ &= B(u,v) + \frac{1}{u^{2^{v+1}}} \\ &= B(u, v+1), \end{aligned}$$

as was to be shown. (CF3) ensures the uniqueness of the result. Note that the continued fraction for $B(u, v+1)$ given in (10) has a total of $2n+1$ partial denominators while the continued fraction for $B(u,v)$ has $n+1$ partial denominators.

We may now justify our assumption that n is even: the assumption that the continued fraction for $B(u,v)$ has an odd number of partial denominators (n even) leads to the proof of part (B) and the fact that the continued fraction for $B(u, v+1)$ also has an odd number of partial denominators. But the continued fraction for $B(u,1)$ has 3 partial denominators, so the proof of part (B) of the theorem is now complete, by induction.

At this point it should be stated that the conclusions of Theorem 1 essentially hold for $u = 2$. However, we run into the difficulty that some of the partial denominators may be 0. When this occurs, we can transform the continued fraction using the following equation

$$[a_0, a_1, \ldots, a_k, 0, a_{k+1}, \ldots] = [a_0, a_1, \ldots, a_k + a_{k+1}, a_{k+2}].$$

4 Further results

Theorem 2 *The continued fraction for $B(u,v)$ has 2^v+1 partial denominators.*

Proof. This follows immediately from the remarks in the last paragraph of the proof of Theorem 1.

Theorem 3 $B(u, \infty) = \sum_{0 \le k < \infty} \frac{1}{u^{2^k}}$ *is irrational for integer* $u \ge 2$.

Proof. We write the base-u expansion of $B(u, \infty)$ as

$$.1101000100000001 \cdots_{(u)} ,$$

with 1's in the first, second, fourth, etc., places. This expansion neither terminates nor repeats. Thus $B(u, \infty)$ is irrational, and its continued fraction does not terminate.

Theorem 4 *The first* 2^v *partial denominators of the continued fraction for* $B(u, v)$ *are identical with those of the continued fraction for* $B(u, \infty)$.

Proof. Examination of part (B) of Theorem 1 shows that the first 2^v partial denominators of the continued fraction for $B(u, v)$ are identical with those of the continued fraction for $B(u, v + 1)$, which are identical with those of the continued fraction for $B(u, v + 2)$, etc.

We observe that repeated application of part (B) of Theorem 1 thus generates the partial denominators of the continued fraction for $B(u, \infty)$.

For example, we find for $u = 3$:

$$\begin{aligned} B(3,0) &= [0,3] \\ B(3,1) &= [0,2,4] \\ B(3,2) &= [0,2,5,3,2] \\ B(3,3) &= [0,2,5,3,3,1,3,5,2] \\ &\vdots \\ B(3,\infty) &= [0,2,5,3,3,1,3,5,3,1,5,3,1,\ldots] \end{aligned}$$

For $u = 4$ we find

$$B(4, \infty) = [0, 3, 6, 4, 4, 2, 4, 6, 4, 2, 6, 4, 2, \ldots].$$

Comparison of the two preceding continued fractions leads to the following theorem.

Theorem 5 *If* $B(u, \infty) = [a_0, a_1, a_2, \ldots]$ *then* $B(u + b, \infty) = [a_0, a_1 + b, a_2 + b, \ldots]$, $(u \ge 3, b \ge 0)$.

Proof. The proof follows easily by induction.

Note that this theorem implies that once the continued fraction for $B(3, \infty)$ is determined, it is trivial to calculate the continued fractions for $B(4, \infty)$, $B(5, \infty)$, etc.

Using the terminology of Maurice Shrader-Frechette [3], we define the mass $M(x)$ of a rational number x, as the sum of the partial denominators of the continued fraction for x. That is, if $x = [a_0, a_1, a_2, \ldots, a_n]$, then $M(x) = \sum_{0 \le k \le n} a_k$. Then

Theorem 6 $M(B(u,v)) = u \cdot 2^v$.

Proof. From part (A) of Theorem 1, we see that

$$\begin{aligned} M(B(u,0)) &= u; \\ M(B(u,1)) &= 2u. \end{aligned}$$

Part (B) of Theorem 1 implies that $M(B(u,v+1)) = 2M(B(u,v))$ since $a_0 = 0$. The desired conclusion follows by induction.

Looking at the continued fraction expansions after Theorem 4 leads one to ask if these expansions ever repeat. In fact, they do not, as is shown in the following theorem.

Theorem 7 *$B(u,\infty)$ is not a quadratic irrational.*

Proof. We know that a number is a quadratic irrational if and only if its continued fraction expansion is infinite and periodic after some point. We will show that the assumption that the continued fraction for $B(u,\infty)$ is periodic after some point leads to a contradiction. Assume that the length of the repeating portion is r terms. We may also assume without loss of generality that the repeating portion begins with the partial denominator a_{2^n+1} with $r \le 2^n$. Thus we have

$$a_{jr+s} = a_s \quad (jr+s, s \ge 2^n+1) \tag{11}$$

It is easily verified that the following two equations are consequences of part (B) of Theorem 1:

$$\begin{aligned} a_{2^{n+1}+1} &= a_{2^{n+1}} - 2 && (12)\\ a_{2^{n+1}+x+1} &= a^{2^{n+1}-x}, \quad (1 \le x \le 2^{n+1}-1). && (13) \end{aligned}$$

The length of the repeating period, r, must be at least 2 since the middle terms of the derived continued fraction given in part (B) of Theorem 1

$$\ldots, a_n + 1, a_n - 1, \ldots$$

are evidently different. Thus let us substitute $x = r - 1$ in equation (13) to obtain

$$a_{2^{n+1}+r} = a_{2^{n+1}-r+1}. \tag{14}$$

Putting $s = 2^{n+1}+1$, $j = -1$ in equation (11), we obtain

$$a_{2^{n+1}-r+1} = a_{2^{n+1}+1}. \tag{15}$$

Combining equations (14) and (15), we find

$$a_{2^{n+1}+r} = a_{2^{n+1}+1}. \tag{16}$$

Again, putting $s = 2^{n+1}$, $j = 1$ in equation (11), we see

$$a_{2^{n+1}+r} = a_{2^{n+1}}. \tag{17}$$

Combining equations (16) and (17), we find

$$a_{2^{n+1}} = a_{2^{n+1}+1}. \tag{18}$$

We see that equation (18) contradicts equation (12). Thus, no such repeating portion can exist and $B(u, \infty)$ cannot be a quadratic irrational.

In fact, $B(u, \infty)$ is transcendental, as is shown in Schneider [4, p. 35].

Theorem 8 *The continued fraction for $B(u, \infty)$ consists only of five unique partial denominators: 0, $u-2$, $u-1$, u, and $u+2$. The distribution of the partial denominators for $B(u, v)$ is as follows ($u \geq 3$):*

Partial Denominator	*Number of Occurrences*
0	*1*
$u-2$	$2^{v-2}-1$
$u-1$	2
u	$2^{v-1}-1$
$u+2$	2^{v-2}

Proof. The proof follows easily by induction from part (B) of Theorem 1.

Theorem 8 immediately implies the following:

Theorem 9 *The partial denominators of the continued fraction for $B(u, \infty)$ are bounded.*

A theorem of Khintchine [5, p. 69] states that the set of all numbers in $(0, 1)$ whose continued frcations have bounded partial denominators is of measure zero, so Theorem 9 is a little surprising.

A theorem of Kuzmin [5, p. 101] says that for almost all real numbers,

$$\lim_{k\to\infty} (a_1 a_2 \cdots a_k)^{1/k} = K,$$

where $K \doteq 2.68545$. This theorem fails to hold $B(u, \infty)$, since Theorem 8 gives

$$(a_1 a_2 \cdots a_k)^{1/k} = [(u-1)^2 (u-2)^{k/4-1} u^{k/2-1} (u+2)^{k/4}]^{1/k}$$

for $k = 2^v$. Letting $k \to \infty$, we see that

$$\lim_{k\to\infty} (a_1 a_2 \cdots a_k)^{1/k} = [u^2(u-2)(u+2)]^{1/4} \neq K.$$

Although Theorem 7 showed that there is no repeating portion in the partial denominators for $B(u, \infty)$, nevertheless, *certain* partial denominators occur with regularity, as shown in the following theorem.

Theorem 10 *If $B(u, \infty) = [a_0, a_1, \ldots, a_n, \ldots]$ then $a_n = u + 2$ if $n \equiv 2$ or $7 \pmod 8$, and $a_n = u$ if $n \equiv 3$ or $6 \pmod 8$.*

Proof. The proof follows by induction from part (B) of Theorem 1.

Similar theorems can be proved if the mod (8) in Theorem 10 is replaced by mod (greater powers of 2).

The following generalization of Theorem 1 will be stated without proof, although the proof is virtually identical to that for Theorem 1.

Let us consider the continued fraction for $u^t B(u, \infty)$ where $t \geq 0$. Let v' be the least non-negative integer such that $2^{v'} > t$.

Theorem 11 *Let $c = u^t B(u, v' - 1)$ (put $c = 0$ for $v' = 0$), and let $d = 2^{v'} - t$. Then*

(A)

$$\begin{aligned} u^t B(u, v') &= [c, u^d]; \\ u^t B(u, v' + 1) &= [c, u^d - 1, 1, u^t - 1, u^d]. \end{aligned}$$

and

(B) for all $v \geq v'+1$, if $u^t B(u, v) = [a_0, a_1, \ldots, a_n]$, then $u^t B(u, v+1) = [a_0, a_1, \ldots, a_n, u^t - 1, 1, a_n - 1, a_{n-1}, a_{n-2}, \ldots, a_2, a_1]$.

For example, repeated application of Theorem 11 gives

$$4^5 B(4, \infty) = [324, 63, 1, 1023, 64, 1023, 1, 63, 1023, 1, 63, \ldots.$$

Theorem 11 implies statements about $u^t B(u, v)$ similar to those about $B(u, v)$ given in Theorems 2-10. One particular interesting consequence of Theorem 11 is obtained for $t = 1$, $u = 2$. We find

$$2B(2, \infty) = [1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, \ldots]$$

and the continued fraction consists solely of 1's and 2's.

The following theorem, stated without proof, says that the continued fractions for

$$C(u, v) = \sum_{0 \leq k \leq v} \frac{(-1)^k}{u^{2^k}} = \frac{1}{u} - \frac{1}{u^2} + \frac{1}{u^4} - \cdots + \frac{(-1)^v}{u^{2^v}}$$

are similar to those in Theorem 1.

Theorem 12 *(A)*

$$\begin{aligned} C(u, 0) &= [0, u]; \\ C(u, 1) &= [0, u + 1, u - 1]. \end{aligned}$$

(B) If $C(u, v) = [a_0, a_1, \ldots, a_n]$ then $C(u, v + 1) = [a_0, a_1, \ldots, a_{n-1}, a_n - (-1)^v, a_n + (-1)^v, a_{n-1}, a_{n-2}, \ldots, a_1]$.

Thus, for example,

$$C(3, \infty) = [0, 4, 3, 1, 3, 5, 1, 3, 5, 3, 3, 1, 5, 3, 1, \ldots].$$

Theorem 12 has consequences similar to those stated in Theorems 2–10.

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