Characterization of Finite and One-Sided Infinite Fixed Points of Morphisms on Free Monoids

David Hamm and Jeffrey Shallit*
Department of Computer Science
University of Waterloo
Waterloo, Ontario, Canada N2L 3G1
dhamm@graceland.uwaterloo.ca
shallit@graceland.uwaterloo.ca

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Abstract

Let Σ be a finite alphabet, and let $h : \Sigma^* \rightarrow \Sigma^*$ be a morphism on the free monoid. We give new proofs of the characterization of the finite and one-sided infinite fixed points of h, i.e., those words w for which $h(w) = w$. We also estimate the size of the minimal non-empty finite fixed point.

1 Introduction and Definitions

Let Σ be a finite alphabet, and let $h : \Sigma^* \rightarrow \Sigma^*$ be a morphism on the free monoid, i.e., a map satisfying $h(xy) = h(x)h(y)$ for all $x, y \in \Sigma^*$. Head [4] and Head and Lando [5] characterized the finite and one-sided infinite *fixed points* of h, i.e., those words w for which $h(w) = w$. In this paper we give new proofs for these facts (our Theorems 3 and 5), which are more "fixed point" in flavor than previous ones. (We cover the case of two-sided infinite words in a later paper [8].) We also deduce some new consequences.

We first introduce some notation, some of which is standard and can be found in [6]. For single letters, that is, elements of Σ, we use the lower case letters a, b, c, d. For finite words, we use the lower case letters u, v, w, x, y, z. For infinite words, we use bold-face letters $\mathbf{t}, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}$. We let ϵ denote the empty word. If $w \in \Sigma^*$, then by $|w|$ we mean the length of, or number of symbols in w. If S is a set, then by Card S we mean the number of elements of S. We say $x \in \Sigma^*$ is a *subword* of $y \in \Sigma^*$ if there exist words $w, z \in \Sigma^*$ such that $y = wxz$.

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If $h(a) \neq \epsilon$ for all $a \in \Sigma$, then h is *non-erasing*. If there exists an integer $j \geq 1$ such that $h^j(a) = \epsilon$, then the letter a is said to be *mortal*. The set of mortal letters associated with a morphism h is denoted by M_h. The *mortality exponent* of a morphism h is defined to be the least integer $t \geq 0$ such that $h^t(a) = \epsilon$ for all $a \in M_h$. Note that $M_h = \emptyset$ iff h is non-erasing. In this case, we take $t = 0$. We write the mortality exponent as $\exp(h) = t$. It is easy to prove that $\exp(h) \leq \text{Card } M_h$.

We let Σ^ω denote the set of all one-sided right-infinite words over the alphabet Σ. Most of the definitions above extend to Σ^ω in the obvious way. For example, if $\mathbf{w} = c_1 c_2 c_3 \cdots$, then $h(\mathbf{w}) = h(c_1)h(c_2)h(c_3)\cdots$. If $L \subseteq \Sigma^+$ is a set of nonempty words, then we define

$$L^\omega = \{w_1 w_2 w_3 \cdots \ : \ w_i \in L \text{ for all } i \geq 1\}.$$

Perhaps slightly less obviously, we can also define the word $\overrightarrow{h^\omega}(a)$ for a letter a, provided $h(a) = wax$ and $w \in M_h^*$. In this case, there exists $t \geq 0$ such that $h^t(w) = \epsilon$. Then we define

$$\overrightarrow{h^\omega}(a) = h^{t-1}(w) \cdots h(w)\, w\, a\, x\, h(x)\, h^2(x) \cdots ,$$

which is infinite iff $x \notin M_h^*$.

Infinite fixed points of morphisms have received a great deal of attention in the literature. The "usual way" to generate infinite fixed points is to take a morphism h and a letter a such that $h(a) = ax$ for some $x \notin M_h^*$. In this case, h is said to be "prolongable" on a [7], and

$$\overrightarrow{h^\omega}(a) = a\, x\, h(x)\, h^2(x) \cdots$$

is clearly an infinite fixed point of h. As we will see in Section 3, however, this approach does not necessarily generate all the infinite fixed points of h.

The classical example of a fixed point of a prolongable morphism is the Thue-Morse word [9, 1]

$$\begin{aligned} \mathbf{t} &= t_0 t_1 t_2 \cdots \\ &= 0110100110010110 \cdots \end{aligned}$$

where t_i is the sum of the bits in the binary representation of n, taken modulo 2. Then $\mathbf{t}$ is a fixed point of the morphism μ which sends $0 \to 01$ and $1 \to 10$; in fact, $\mathbf{t} = \overrightarrow{\mu^\omega}(0)$. The infinite word $\mathbf{t}$ is of interest in part because it is cube-free, that is, it contains no nonempty subword of the form www. Similarly, the morphism $2 \to 210$, $1 \to 20$, and $0 \to 1$ has as a fixed point the infinite word

$$210201210120 \cdots$$

which is square-free (contains no nonempty subword of the form ww).

2 Finite Fixed Points

In this section we give a new proof of Head's characterization [4] of the finite fixed points of a morphism. We start with a general lemma that appears to be new.

Lemma 1 *Let $h : \Sigma^* \to \Sigma^*$ be a morphism. Let $w \in \Sigma^+$ be a finite nonempty word such that w is a subword of $h(w)$. Then there exists a letter $a \in \Sigma$ occurring in w such that a occurs in $h(a)$.*

Proof. Let $w = c_1 c_2 \cdots c_n$, where $c_i \in \Sigma$ for $1 \le i \le n$. For $0 \le i \le n$ define $s_w(i) = |h(c_1 c_2 \cdots c_i)|$. (If the word w is clear, we omit the subscript.) In particular, $s(0) = 0$.

Let $h(w) = d_1 d_2 \cdots d_{s(n)}$, where $d_i \in \Sigma$ for $1 \le i \le s(n)$. Hence

$$h(c_i) = d_{s(i-1)+1} \cdots d_{s(i)}$$

for $1 \le i \le n$. Since w is a subword of $h(w)$, we know there must exist an integer t, $0 \le t \le s(n) - n$, such that $w = d_{t+1} \cdots d_{t+n}$. Hence $c_i = d_{t+i}$ for $1 \le i \le n$.

Consider the least index $j \ge 1$ for which $s(j) \ge t + j$. Such an index must exist, since the inequality holds for $j = n$. There are now two cases to consider.

Case 1: $j = 1$: Then $s(1) \ge t+1$. Hence $h(c_1) = d_1 d_2 \cdots d_{s(1)}$ contains $d_{t+1} = c_1$. Let $a = c_1$.

Case 2: $j > 1$: Then by the definition of j we must have $s(j-1) < t + j - 1$. Hence $s(j-1)+1 < t+j$, and since $h(c_j) = d_{s(j-1)+1} \cdots d_{s(j)}$, we know $h(c_j)$ contains $d_{t+j-1} d_{t+j} = c_{j-1} c_j$ as a subword. Let $a = c_j$. ■

As a consequence, we deduce the following useful corollary.

Corollary 2 *If $w \in \Sigma^+$ is a nonempty finite word with $h(w) = w$, then there exist words $w_1, w_2, w_3, w_4 \in \Sigma^*$ and a letter $a \in \Sigma$ such that $w = w_1 w_2 a w_3 w_4$, $h(w_1 w_2) = w_1$, $h(a) = w_2 a w_3$, and $h(w_3 w_4) = w_4$.*

Proof. If $h(w) = w$, then, using Lemma 1, we have $t = 0$ and $s(n) = n$. Let

$$\begin{aligned} w_1 &= d_1 \cdots d_{s(j-1)}; \\ w_2 &= d_{s(j-1)+1} \cdots d_{j-1}; \\ a &= d_j; \\ w_3 &= d_{j+1} \cdots d_{s(j)}; \\ w_4 &= d_{s(j)+1} \cdots d_n. \end{aligned}$$

The verification is straightforward. ■

Now define

$$A_h = \{a \in \Sigma \; : \; \exists \; x, y \in \Sigma^* \text{ such that } h(a) = xay \text{ and } xy \in M_h^*\}$$

and

$$F_h = \{h^t(a) \; : \; a \in A_h \text{ and } t = \exp(h)\}.$$

Note that there is at most one way to write $h(a)$ in the form xay with $xy \in M_h^*$. Furthermore, note that if h is non-erasing, then the only letters a in A_h are those for which $h(a) = a$. In this case $F_h = A_h$.

We now state Head's result [4]:

Theorem 3 *Let $h : \Sigma^* \to \Sigma^*$ be a morphism. Then a finite word $w \in \Sigma^*$ has the property that $w = h(w)$ if and only if $w \in F_h^*$.*

Proof. ($\Longleftarrow$): Suppose $w \in F_h^*$. Then we can write $w = w_1 w_2 \cdots w_r$, where each $w_i \in \Sigma^*$, and there exist letters $a_1, a_2, \ldots, a_r \in A_h$ such that $w_i = h^t(a_i)$, with $t = \exp(h)$.

Since $a_i \in A_h$, we know that there exist x_i, y_i with $x_i y_i \in M_h^*$ such that $h(a_i) = x_i a_i y_i$. Since $t = \exp(h)$, we have $h^t(x_i) = h^t(y_i) = \epsilon$. Hence

$$h^{t+1}(a_i) = h^t(x_i)\, h^t(a_i)\, h^t(y_i) = h^t(a_i).$$

Thus $h(w_i) = w_i$ for $1 \leq i \leq r$, and so $h(w) = w$.

($\Longrightarrow$): We prove the result by contradiction. Suppose $h(w) = w$, and assume w is the shortest such word with $w \notin F_h^*$. Clearly $w \neq \epsilon$.

By Corollary 2 there exist w_1, w_2, w_3, w_4, a such that $w = w_1 w_2 a w_3 w_4$, $h(w_1 w_2) = w_1$, $h(a) = w_2 a w_3$, and $h(w_3 w_4) = w_4$.

Now a is a subword of w, so $h(a)$ is a subword of $h(w) = w$, and hence by an easy induction, it follows that

$$h^i(a) \text{ is a subword of } w \text{ for all } i \geq 0. \tag{1}$$

Then we must have $w_2 w_3 \in M_h^*$, since otherwise the length of

$$h^i(a) = h^{i-1}(w_2) \cdots h(w_2)\, w_2\, a\, w_3\, h(w_3) \cdots h^{i-1}(w_3)$$

would grow without bound as $i \to \infty$, contradicting (1). It follows that $h^t(w_2 w_3) = \epsilon$, where $t = \exp(h)$.

Now we have $w_1 = h(w_1 w_2)$, so by applying h^t to both sides, we see

$$h^t(w_1) = h^{t+1}(w_1 w_2) = h^{t+1}(w_1)\, h^{t+1}(w_2) = h^{t+1}(w_1).$$

Hence, defining $y_1 = h^t(w_1)$, we have $h(y_1) = y_1$. In a similar fashion, if we set $y_2 = h^t(w_4)$, then $h(y_2) = y_2$. Since $|y_1|, |y_2| < |w|$, it follows by the minimality of w that $y_1, y_2 \in F_h^*$. Now

$$w = h^t(w) = h^t(w_1)\, h^t(w_2)\, h^t(a)\, h^t(w_3)\, h^t(w_4) = y_1\, h^t(a)\, y_2,$$

and hence $w \in F_h^*$, a contradiction. ∎

We now examine the following question. Suppose h possesses a nonempty finite fixed point w. How long can the shortest w be, as a function of the description of h?

Theorem 4 *If a morphism h possesses a nonempty finite fixed point, then there exists such a fixed point w with $|w| \leq m^{n-1}$, where $n = \text{Card } \Sigma$ and $m = \max_{a\in\Sigma} |h(a)|$. Furthermore, this bound is best possible.*

Proof. As we have seen in Theorem 3, a word w is a finite fixed point iff $w \in F_h^*$. Hence, if there exists a nonempty finite fixed point, the shortest such must lie in F_h. But

$$F_h = \{h^t(a) \ : \ a \in A_h \text{ and } t = \exp(h)\}.$$

Since $a \in A_h$, we have $h(a) = xay$ with $xy \in M_h^*$. Hence $a \notin M_h$ and so $\exp(h) \leq \text{Card } M_h \leq n-1$. If $m = \max_{a\in\Sigma} |h(a)|$, then clearly $|h^i(a)| \leq m^i$ for all $i \geq 0$. It follows that $|w| = |h^t(a)| \leq m^{n-1}$.

To see that the bound is best possible, consider the morphism h defined on $\Sigma = \{a_1, a_2, \ldots, a_n\}$ as follows:

$$\begin{aligned} h(a_1) &= a_1 a_2^{m-1}; \\ h(a_i) &= a_{i+1}^m \text{ for } 2 \leq i \leq n-1; \\ h(a_n) &= \epsilon. \end{aligned}$$

Then

$$w = a_1 a_2^{m-1} a_3^{m(m-1)} \cdots a_n^{m^{n-2}(m-1)}$$

is a fixed point of h, and

$$|w| = 1 + (m-1) + m(m-1) + \cdots + m^{n-2}(m-1) = m^{n-1}.$$

■

3 One-Sided Infinite Fixed Points

Let $\mathbf{w} = c_1 c_2 c_3 \cdots$ be an infinite (one-sided) word over Σ, and let h be a morphism. Head and Lando [5] characterized those $\mathbf{w}$ for which $h(\mathbf{w}) = \mathbf{w}$. We now give a different proof of this characterization.

Theorem 5 *The infinite word $\mathbf{w}$ is a fixed point of h if and only if at least one of the following two conditions holds:*

(a) $\mathbf{w} \in F_h^\omega$*; or*

(b) $\mathbf{w} \in F_h^* \ \overrightarrow{h^\omega}(a)$ *for some $a \in \Sigma$, and there exist $x \in M_h^*$ and $y \notin M_h^*$ such that $h(a) = xay$.*

Note that there is at most one way to write $h(a) = xay$ with $x \in M_h^*$ and $y \notin M_h^*$.

Proof. ($\Longleftarrow$): First, suppose condition (a) holds. Then we can write $\mathbf{w} = w_1 w_2 w_3 \cdots$, where each $w_i \in F_h$. Then by Theorem 3 we have $h(w_i) = w_i$. It follows that $h(\mathbf{w}) = \mathbf{w}$.

Second, suppose condition (b) holds. Then we can write $\mathbf{w} = v\,\mathbf{z}$, where $v \in F_h^*$ and $\mathbf{z} = \overrightarrow{h^\omega}(a)$, where $h(a) = xay$ for some $x \in M_h^*$, $y \notin M_h^*$. Then from Theorem 3, we have $h(v) = v$.

Since $x \in M_h^*$, we have $h^t(x) = \epsilon$, and hence

$$\mathbf{z} = \overrightarrow{h^\omega}(a) = h^{t-1}(x) \cdots h(x)\, x\, a\, y\, h(y)\, h^2(y)\, h^3(y) \cdots .$$

Since $y \notin M_h^*$, it follows that $|h^i(y)| \geq 1$ for all $i \geq 0$, and hence $\mathbf{z}$ is indeed an infinite word. We then have

$$h(\mathbf{z}) = h^t(x) \cdots h(x)\, x\, a\, y\, h(y)\, h^2(y)\, h^3(y) \cdots = \mathbf{z}$$

and so $h(\mathbf{w}) = h(v\mathbf{z}) = v\mathbf{z} = \mathbf{w}$.

($\Longrightarrow$): Now suppose $\mathbf{w} = c_1 c_2 c_3 \cdots$ is an infinite word, with $c_i \in \Sigma$ for $i \geq 1$, and $h(\mathbf{w}) = \mathbf{w}$. As before, we define $s_{\mathbf{w}}(i) = |h(c_1 c_2 \cdots c_i)|$ for $i \geq 0$. There are several cases to consider.

Case 1: $s_{\mathbf{w}}(i) = i$ for infinitely many integers $i \geq 1$. Suppose $s(i) = i$ for $i = i_0, i_1, i_2, \ldots$. Clearly we may take $i_0 = 0$. Then we can write

$$\mathbf{w} = y_1 y_2 y_3 \cdots$$

where $y_j = c_{i_{j-1}+1} \cdots c_{i_j}$ and $h(y_j) = y_j$ for $j \geq 1$. It follows that $\mathbf{w} \in F_h^\omega$.

Case 2: $s_{\mathbf{w}}(i) = i$ for finitely many $i \geq 1$, and at least one such i. Let $s(i) = i$ for $i = i_0, i_1, \ldots, i_r$, and again take $i_0 = 0$. Then for some integer $r \geq 1$ we can write

$$\mathbf{w} = y_1 y_2 y_3 \cdots y_r\, \mathbf{x}$$

where $y_j = c_{i_{j-1}+1} \cdots c_{i_j}$ and $h(y_j) = y_j$ for $1 \leq j \leq r$, and $h(\mathbf{x}) = \mathbf{x}$. Furthermore, if we write $\mathbf{x} = d_1 d_2 d_3 \cdots$ for $d_i \in \Sigma$, $i \geq 1$, then

$$s_{\mathbf{x}}(i) \neq i \text{ for all } i \geq 1. \tag{2}$$

If we can show that (2) implies that $\mathbf{x} = \overrightarrow{h^\omega}(a)$, where $h(a) = xay$ for some $x \in M_h^*$, $y \notin M_h^*$, we will be done. This leads to Case 3.

Case 3: $s_{\mathbf{w}}(i) \neq i$ for all $i \geq 1$. Suppose there exist i, j with $1 \leq i < j$ and

$$s(i) > i \text{ but } s(j) < j. \tag{3}$$

Among all pairs (i, j) with $1 \leq i < j$ satisfying (3), let j_0 be the smallest such j. Next, among all pairs (i, j_0) satisfying (3), let i_0 be the largest such i. Suppose there exists an integer k with $i_0 < k < j_0$. If $s(k) < k$, then j_0 is not minimal, while if $s(k) > k$, then i_0 is not maximal. Hence $s(k) = k$. But this is impossible by our assumption. It follows that $j_0 = i_0 + 1$. Then $s(i_0) > i_0$, but $s(i_0 + 1) < i_0 + 1$, a contradiction, since $s(i_0) \leq s(i_0 + 1)$.

It follows that either (a) $s(i) < i$ for all $i \geq 1$, or (b) there exists an integer $r \geq 1$ such that $s(i) < i$ for $1 \leq i < r$ and $s(i) > i$ for all $i \geq r$.

Case 3a: $s_{\mathbf{w}}(i) < i$ for all $i \geq 1$. Since this is true for $i = 1$, in particular we see that $h(c_1) = \epsilon$. Now let j_1 be the least index such that

$$h(c_{j_1}) \text{ contains } c_1; \tag{4}$$

such an index must exist since $h(\mathbf{w}) = \mathbf{w}$. We then have $h(c_2) = h(c_3) = \cdots = h(c_{j_1 - 1}) = \epsilon$, so the first occurrence of c_{j_1} in $\mathbf{w}$ is at position j_1.

Now inductively assume that we have constructed a strictly increasing sequence $j_0 = 1 < j_1 < \cdots < j_t$ such that the first occurrence of c_{j_i} in $\mathbf{w}$ is at position j_i, for $1 \leq i \leq t$.

Let j_{t+1} be the least index such that $h(c_{j_{t+1}})$ contains c_{j_t}. Assume $j_t \geq j_{t+1}$. Since $s(i) < i$ for all i, we have $h(c_{j_{t+1}}) = c_k \cdots c_l$ with $l < j_{t+1} \leq j_t$. Since $h(c_{j_{t+1}})$ contains c_{j_t}, this implies that c_{j_t} occurs to the left of position j_t, a contradiction. Hence $j_t < j_{t+1}$.

Thus we can construct an infinite strictly increasing sequence $j_0 < j_1 < \cdots$ such that the first occurrence of c_{j_i} in $\mathbf{w}$ is at position j_i. It follows that the letters $c_{j_0}, c_{j_1}, \ldots$ in Σ are all distinct. But Σ is finite, a contradiction. Hence this case cannot occur.

Case 3b: There exists an integer $r \geq 1$ such that

$$s_{\mathbf{w}}(i) < i \text{ for } 1 \leq i < r \text{ and } s_{\mathbf{w}}(i) > i \text{ for all } i \geq r. \tag{5}$$

Put $a = c_r$. Then $h(a) = c_{s(r-1)+1} \cdots c_{s(r)}$. If $r = 1$, then (5) implies that $s(r) > r$, so $h(a) = xay$ for $x = \epsilon$ and some $y \in \Sigma^+$. If $r > 1$, then (5) implies that $s(r-1) + 1 < r$ and $s(r) > r$, so $h(a) = xay$ for some $x, y \in \Sigma^+$. More precisely, the conditions (5) imply that we can write $\mathbf{w} = u\, a\, \mathbf{v}$ for some $u \in \Sigma^*$, $\mathbf{v} \in \Sigma^\omega$, and $h(\mathbf{w}) = h(u)\, x\, a\, y\, h(\mathbf{v})$ such that $u = h(u)x$. An easy induction now gives

$$h^i(\mathbf{w}) = h^i(u)\, h^{i-1}(x) \cdots h(x)\, x\, a\, y\, h(y) \cdots h^{i-1}(y)\, h^i(\mathbf{v}) \tag{6}$$

and

$$u = h^i(u)\, h^{i-1}(x) \cdots h(x)\, x \tag{7}$$

for all $i \geq 0$. Since $|u| < \infty$, it follows from letting $i \to \infty$ in Eq. (7) that there exists an integer $j \geq 0$ such that $h^j(x) = \epsilon$. Hence $x \in M_h^*$, and so $h^t(x) = \epsilon$, where $t = \exp(h)$.

Now $u = h(u)x$, so $h^t(u) = h^{t+1}(u)h^t(x) = h^{t+1}(u)$. Define $u' = h^t(u)$; then $h(u') = u'$. Hence, putting $j = |u'|$, it follows that $s(j) = j$. Hence $j = 0$ and $u' = \epsilon$.

Now, to get a contradiction, suppose that $y \in M_h^*$. Then $h^t(y) = \epsilon$. Define $z = h^t(a)$. Then

$$h(z) = h^{t+1}(a) = h^t(h(a)) = h^t(xay) = h^t(x)\, h^t(a)\, h^t(y) = h^t(a) = z.$$

Hence, putting $j = |z|$, we see that $s(j) = j$, a contradiction since $|z| \geq 1$. Hence $y \notin M_h^*$.

Now, letting $i \to \infty$ in (6), we see that $\mathbf{w} = \overrightarrow{h^\omega}(a)$. ■

We stated Theorem 5 for right-infinite words, but of course the same arguments work for left-infinite words. Let $\Sigma^{-\omega}$ denote the set of all left-infinite words, which are of the form $\mathbf{w} = \cdots c_{-2}c_{-1}c_0$. We write $h(\mathbf{w}) = \cdots h(c_{-2})h(c_{-1})h(c_0)$. If $L \subseteq \Sigma^+$ is a set of nonempty words, we define $L^{-\omega}$ to be the set of left-infinite words formed by concatenating infinitely many words from L, that is,

$$L^{-\omega} = \{\cdots w_{-2}w_{-1}w_0 \ : \ w_i \in L \text{ for all } i \leq 0\}.$$

If $h(a) = wax$, and $w \notin M_h^*$, $x \in M_h^*$, then by $\overleftarrow{h^\omega}(a)$ we mean the left-infinite word

$$\cdots h^2(w)\, h(w)\, w\, a\, x\, h(x) \cdots h^{t-1}(x),$$

where $h^t(x) = \epsilon$. Again, if the factorization of $h(a)$ as wax exists, with $w \notin M_h^*$, $x \in M_h^*$, then it is unique. Then we have

Theorem 6 *The left-infinite word* $\mathbf{w}$ *is a fixed point of* h *if and only if at least one of the following two conditions holds:*

(a) $\mathbf{w} \in F_h^{-\omega}$*; or*

(b) $\mathbf{w} \in \overleftarrow{h^\omega}(a)F_h^*$ *for some* $a \in \Sigma$*, and there exist* $x \notin M_h^*$ *and* $y \in M_h^*$ *such that* $h(a) = xay$*.*

4 Non-Trivial Infinite Fixed Points

Call an infinite fixed point *trivial* if it is in F_h^ω. Our last result shows that, up to application of a coding (i.e., a letter-to-letter morphism), all non-trivial infinite fixed points can be generated in the "usual way", i.e., by iterating a morphism f on a letter b such that $f(b) = b\,u$ with $u \notin M_f^*$.

Theorem 7 *Suppose* $h : \Sigma^* \to \Sigma^*$ *is a morphism and* $\mathbf{w} \in \Sigma^\omega$ *is an infinite word such that* $h(\mathbf{w}) = \mathbf{w}$ *and* $\mathbf{w} \notin F_h^\omega$*. Then there exists an alphabet* Δ*, a non-erasing morphism* $f : \Delta^* \to \Delta^*$*, a coding* $g : \Delta \to \Sigma$*, a nonempty word* $u \in \Delta^+$ *and a letter* $b \in \Delta$ *such that* $f(b) = bu$ *and* $g(\overrightarrow{f^\omega}(b)) = \mathbf{w}$*.*

Proof. If $\mathbf{w} \notin F_h^\omega$, then by Theorem 5, there exists $a \in \Sigma$ such that $\mathbf{w} \in F_h^* \ \overrightarrow{h^\omega}(a)$, and $h(a) = xay$ with $x \in M_h^*$ and $y \notin M_h^*$. Thus, if $t = \exp(h)$, there exists $v \in F_h^*$ such that

$$\mathbf{w} = v\, h^{t-1}(x) \cdots h(x)\, x\, a\, y\, h(y)\, h^2(y) \cdots .$$

Define $z = vh^{t-1}(x)h^{t-2}(x)\cdots h(x)x$, and let $r = |z|$. If $r = 0$, then $v = x = \epsilon$, and the desired result follows by taking $f = h$ and $g =$ the identity map.

Hence assume $r > 0$ and write $z = b_1 b_2 \cdots b_r$ for $b_i \in \Sigma$, $1 \leq i \leq r$. Introduce $r+1$ new symbols $b, a_2, \ldots, a_r, a_{r+1}$, and set $\Delta = \Sigma \cup \{b, a_2, \ldots, a_r, a_{r+1}\}$.

For $d \in \Delta$ define

$$f(d) = \begin{cases} b\, a_2 & \text{if } d = b; \\ a_{i+1}, & \text{if } d = a_i \text{ with } 2 \leq i \leq r; \\ y, & \text{if } d = a_{r+1}; \\ h(d), & \text{if } d \in \Sigma. \end{cases}$$

Then we have

$$\overrightarrow{f^\omega}(b) = b\, a_2 \cdots a_r\, a_{r+1}\, y\, h(y)\, h^2(y) \cdots .$$

Finally, define the coding $g : \Delta \to \Sigma$ as follows:

$$g(d) = \begin{cases} b_1, & \text{if } d = b; \\ b_i, & \text{if } d = a_i \text{ with } 2 \leq i \leq r; \\ a, & \text{if } d = a_{r+1}; \\ d, & \text{if } d \in \Sigma. \end{cases}$$

It follows that

$$g(\overrightarrow{f^\omega}(b)) = b_1\, b_2 \cdots b_r\, a\, y\, h(y)\, h^2(y) \cdots = \mathbf{w},$$

as desired.

Note that f is non-erasing iff h is. In any event, by a theorem of Cobham [2], there exists a letter c, a non-erasing morphism f', and a coding g' such that $\mathbf{w} = g'(\overrightarrow{f'^\omega}(c))$. ■

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