Groups whose Word Problem is a Petri Net Language (DCFS2015)

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Overview

1. Motivation and definitions
   - Characterisations of word problems
   - Definitions
2. Results
   - Virtually abelian to PNL
   - PNL to Virtually abelian
3. Relations with other classes of languages

Languages to groups

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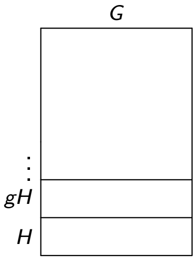
This question has been answered for many “nice” classes of languages: *cones*. These are classes closed under intersections with regular languages, (monoid) homomorphisms and inverse (monoid) homomorphisms.

Interlude: Virtually $\mathcal{P}$

G a group, H a subgroup. The sets gH for $g \in G$ are either equal or disjoint with each other. If there are finitely many, then we say H has *finite index* in G.



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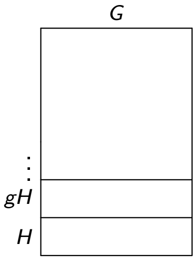

If H has property $\mathcal{P}$ and has finite index in G, we say that G is *virtually* $\mathcal{P}$.

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One which is not a cone:

- G has a word problem which is a finite intersection of one-counter languages $\Leftrightarrow$ G is virtually abelian (has a finite index subgroup which is abelian). [Holt, Owens, Thomas 2008]

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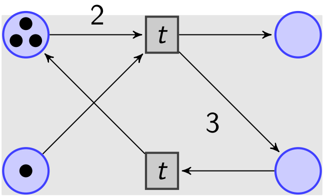
A finitely generated group has a word problem which is a terminal Petri net language if and only if it is virtually abelian.

Terminal languages of Petri nets



The distribution of tokens at any time is given by *markings* $m \in \mathbb{N}^S$ where S is the set of places.
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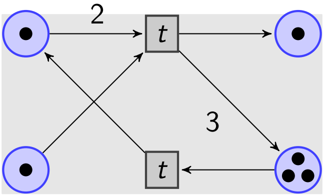

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Properties of the class PNL

Fortunately, the class *PNL* of terminal Petri net languages is almost a cone (closed under λ-free homomorphisms) and has other nice properties which imply the following properties for groups with a word problem in PNL (we will call this class of groups $\mathcal{PNL}$):

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- $\mathcal{PNL}$ is closed under taking finite extensions.

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A well-known result says

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Any finitely generated abelian group is expressible as a direct product

$$\mathbb{Z}^r \times \mathbb{Z}/a_1\mathbb{Z} \ldots \times \mathbb{Z}/a_m\mathbb{Z}$$

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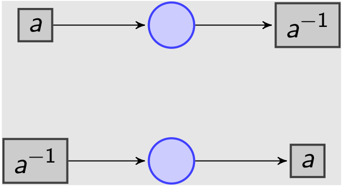
where $a_i = p_i^{n_i}$ for some prime p_i and some natural numbers $n_i \geqslant 1$, $r, m \in \mathbb{N}$.

The word problem of a finite cyclic group $\mathbb{Z}/a_i\mathbb{Z}$ is regular, and hence a *PNL*.

$\mathbb{Z} = \langle a \rangle$ is in $\mathcal{PNL}$

The word problem of $\mathbb{Z}$ is also a *PNL*:



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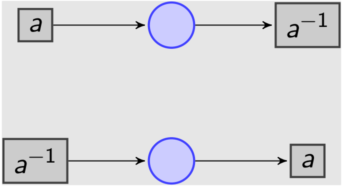

So the word problem of any finitely generated abelian group (a direct product of such groups) is a *PNL*.

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Now let's prove the converse.

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It turns out that groups in $\mathcal{PNL}$ have polynomial growth.

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Therefore to show that groups in $\mathcal{PNL}$ are virtually abelian, it is enough to show that the Heisenberg group does not have a PNL word problem.

The Heisenberg group is the the group $H = \langle a, b, c | ac = ca, bc = cb, ab = cba \rangle$. If its word problem W were in PNL, so would $W \cap \{a^n b^n (a^{-1})^n (b^{-1})^n c^k | k, n \in \mathbb{N}\} = \{a^n b^n (a^{-1})^n (b^{-1})^n c^{n^2} | n \in \mathbb{N}\}$, meaning Petri nets could multiply.

Petri nets can't multiply

This is a consequence of the undecidability of Hilbert's 10th problem [Matiyasevich 1970]:

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This is a clear contradiction. As a corollary of this and the normal forms for elements of the word problem of H, we have:

Theorem

$H \notin \mathcal{PNL}$

and we have proved our result.

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- $PNL \not\subseteq coCF$ but $\mathcal{PNL} \subset co\mathcal{CF}$.

Future work

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- How many of these results transfer to Petri nets where λ-transitions are allowed?

Thank you!