A fixed point theorem for non-monotonic functions

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The Knaster-Tarski fixed point theorem

Theorem (Knaster–Tarski) Suppose that L is a complete lattice and $f : L \to L$ is monotonic. Then f has extremal fixed points. Moreover, the set of all fixed points of f is a complete lattice by itself.

Applications in CS

- automata and languages
- programming logics and verification
- semantics
- algorithms, etc

Fixed points of non-monotonic functions

Fixed points of non-monotonic functions in CS

- Boolean automata and language equations (Brzozowski-Leiss 1980)
- logic programming (Przymusiński 1989, Rondogiannis-Wadge 2005)
- semantics of parallelism (Chen 2003)

Logic programming

- Przymusiński's well founded semantics
  - abstract fixed point theory by Denecker, Fitting, Gelder, Truszczyński...
- Rondogiannis-Wadge semantics
  - abstract fixed point theory: this work

Stratified complete lattices

κ a fixed limit ordinal

Definition Suppose that $L = (L, \leq)$ is a complete lattice We call L a **stratified complete lattice** if it is equipped with preorderings $\sqsubseteq_\alpha$, $\alpha < \kappa$ such that:

- $\forall x, y\ \forall \alpha, \beta\ (x \sqsubseteq_\alpha y \wedge \beta < \alpha\ \Rightarrow\ x =_\beta y)$
- $\forall x, y\ ((\forall \alpha\ x =_\alpha y)\ \Rightarrow\ x = y)$

Example

$L = (V^Z, \leq)$: $V : F_0 < F_1 < \ldots < F_\alpha < \ldots < 0 < \ldots < T_\alpha < \ldots < T_1 < T_0\ (\alpha < \kappa)$

$f \sqsubseteq_\alpha g$ iff

1. $\forall z \forall \beta < \alpha\ ((f(z) = F_\beta \Leftrightarrow g(z) = F_\beta) \wedge (f(z) = T_\beta \Leftrightarrow g(z) = T_\beta))$
2. $\forall z\ ((f(z) = T_\alpha \Rightarrow g(z) = T_\alpha) \wedge (g(z) = F_\alpha \Rightarrow f(z) = F_\alpha))$

Axioms

A1. $\forall x \forall \alpha \exists y\ (x =_\alpha y \wedge \forall z (x \sqsubseteq_\alpha z \Rightarrow y \leq z))$

Fact: y is uniquely determined by x and α. Notation $x|_\alpha$

A2. $\forall (x_i)_{i \in I(\neq \emptyset)} \forall y \forall \alpha\ (x_i =_\alpha y, i \in I \Rightarrow \bigvee_{i \in I} x_i =_\alpha y)$

A3. $\forall x, y \forall \alpha < \kappa\ (x \leq y \Rightarrow x|_\alpha \leq y|_\alpha)$

A4. $\forall x, y \forall \alpha < \kappa\ ((x \leq y \wedge (\forall \beta < \alpha\ x =_\beta y))\ \Rightarrow\ x \sqsubseteq_\alpha y)$

Example When $f \in V^Z$, $\alpha < \kappa$:

$$f|_\alpha(z) = \begin{cases} f(z) & \text{if } f(z) \in \{F_\beta, T_\beta : \beta < \alpha\} \\ F_{\alpha+1} & \text{otherwise} \end{cases}$$

Stratified complete lattices satisfying A1–A4 will be called **models**.

Lattice theorem

Definition Let L be a model and $x, y \in L$. We define $x \sqsubseteq y$ iff $x = y$ or $\exists\alpha\ x \sqsubset_\alpha y$.

Theorem (with P. Rondogiannis) If L is a model, then $(L, \sqsubseteq)$ is a complete lattice with the same extremal elements.

Weakly monotonic functions

Definition Suppose that L is a model and $f : L \to L$. We call f α-**monotonic** for some $\alpha < \kappa$ if

$$\forall x, y(x \sqsubseteq_\alpha y \ \Rightarrow f(x) \sqsubseteq_\alpha f(y))$$

Moreover, we call f **weakly monotonic** if it is α-monotonic for all $\alpha < \kappa$.

Example Let $\kappa = \Omega$ and define $\wedge : V^2 \to V$ as the binary minimum (infimum) operation and $\neg : V \to V$ by

$$\neg(x) = \begin{cases} T_{\alpha+1} & \text{if } x = F_\alpha \\ F_{\alpha+1} & \text{if } x = T_\alpha \\ 0 & \text{if } x = 0 \end{cases}$$

P is a propositional logic program over Z i.e., P is a countable set of instructions

$$z \leftarrow p_1; \cdots ; p_m; \neg q_1; \cdots ; \neg q_n$$

where $z, p_i, q_j \in Z$. Let $\Phi_P : V^Z \to V^Z$ be defined by:

$$\Phi(f)(z) = \bigvee_{z \leftarrow p_1; \cdots ; p_m; \neg q_1; \cdots ; \neg q_n \in P} f(p_1) \wedge \cdots \wedge f(p_m) \wedge \neg f(q_1) \wedge \cdots \wedge f(q_n)$$

Then Φ_P is weakly monotonic.

Fixed point theorem

Theorem (with P. Rondogiannis) Suppose that L is a model and $f : L \to L$ is weakly monotonic. Then f has a least fixed point and a greatest fixed point w.r.t. the ordering $\sqsubseteq$.

Theorem (ZE) Suppose that L is a model and $f : L \to L$ is weakly monotonic. Then the fixed points of f form a complete lattice w.r.t. the ordering $\sqsubseteq$.

Fixed point theorem

Example

$$P: \quad p \leftarrow \; ; \quad s \leftarrow \neg q \; ; \quad t \leftarrow \neg t$$

$$\begin{aligned} p &= T_0 \\ s &= \neg q \\ t &= \neg t \end{aligned}$$

Minimal model: $(p, T_0), (q, F_0), (s, T_1), (t, 0)$

p is 'more true' than s since there is a rule that says p is true, while s is true only because q is false by default

Another model: $(p, T_0), (q, T_0), (s, F_1), (t, 0)$

Well-founded model: $(p, T), (q, F), (s, T), (t, 0)$

Inverse limit models

Inverse system of **projections**

$$h^\alpha_\beta : L_\alpha \to L_\beta, \quad \beta < \alpha < \kappa$$
$$h^\beta_\gamma \circ h^\alpha_\beta = h^\alpha_\gamma, \quad \gamma < \beta < \alpha$$

Inverse limit complete lattice:

$$L_\infty = \{(x_\alpha)_{\alpha<\kappa} \in \prod_{\alpha<\kappa} L_\alpha : \forall \beta < \alpha \ h^\alpha_\beta(x_\alpha) = x_\beta\}$$

equipped with the pointwise ordering

Definition Let $x = (x_\gamma)_{\gamma<\kappa}$ and $y = (y_\gamma)_{\gamma<\kappa}$ in L_∞ and $\alpha < \kappa$. We define $x \sqsubseteq_\alpha y$ iff $x_\alpha \leq y_\alpha$ and $x_\beta = y_\beta$ for all $\beta < \alpha$.

Proposition (ZE) L_∞ is a stratified complete lattice. It is a model iff each h^α_β is locally completely additive:

$$\forall X \subseteq L_\alpha \forall y \in L_\beta \ (h^\alpha_\beta(X) = \{y\} \ \Rightarrow h^\alpha_\beta(\bigvee X) = y)$$

Representation theorem

Theorem (ZE) Every model is isomorphic to the model determined by the limit of an inverse system over complete lattices with locally completely additive projections.

Future work

Comparison with well-founded fixed points

Applications

Bibliography

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