# Unambiguity in automata theory

Thomas Colcombet
DCFS 2015

Waterloo, June 26, 2015

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[Jurdzinski 98]
ParityGames in UP∩CoUP

[Bourke&Tewari&Vinodchandran 07]
Planar reachability in UL (unambiguous logspace)

[Allender&Reinhardt 97]
UL and NL coincide in the non-uniform setting (open in the uniform one)

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Word automata

Transducers

Infinite word automata

Infinite tree automata

Tropical automata

Register automata

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**Remark:** in general it is easy to decide if an automaton is unambiguous.
Proof: *Take the product of the automaton with itself + 1 bit, such that it accepts an input iff there exist two distinct runs of the original automaton. Test for emptiness.*

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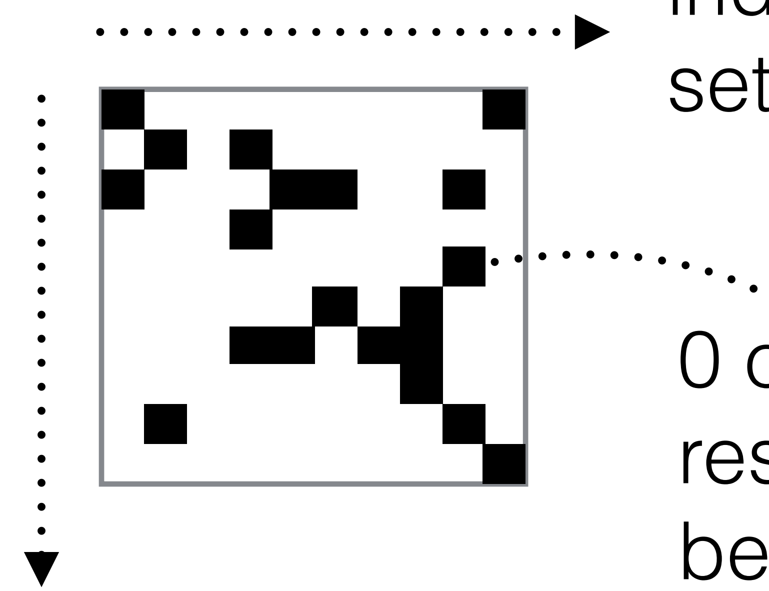
One sees the problem as the (input) matrix:

The communication complexity depends on the shape of the matrix.

indexed by the set of A-inputs



0 or 1 if the resulting input belongs to the problem.

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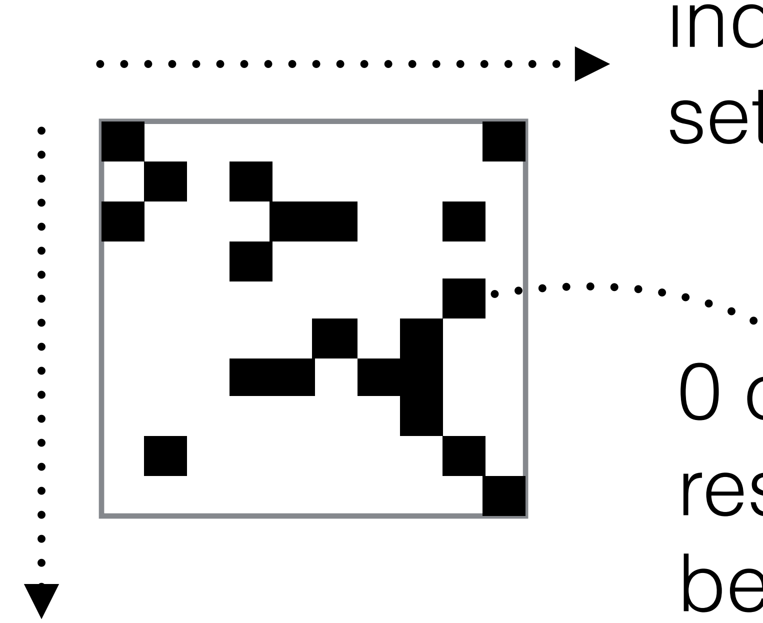


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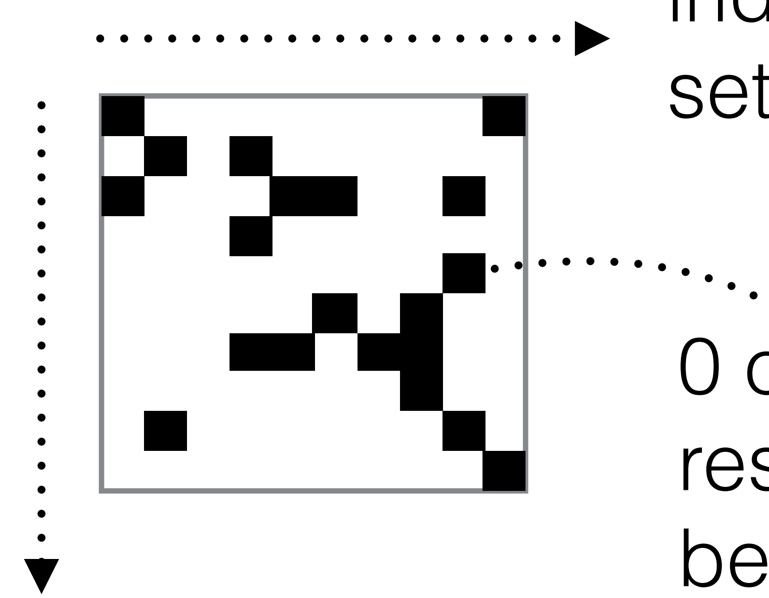


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Usually the story continues with randomized protocols (processes can flip coins)… (yields O(log(n)) bits in the above example).

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If $u_1 v_1 \in L, \ldots, u_n v_n \in L$
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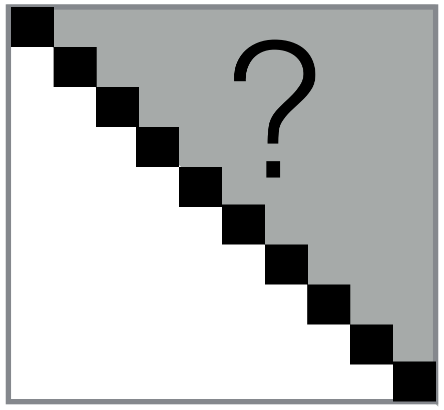

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Since rank(rectangle)=1 and rank is subadditive,

rank(M)≤unamb-comp(M)

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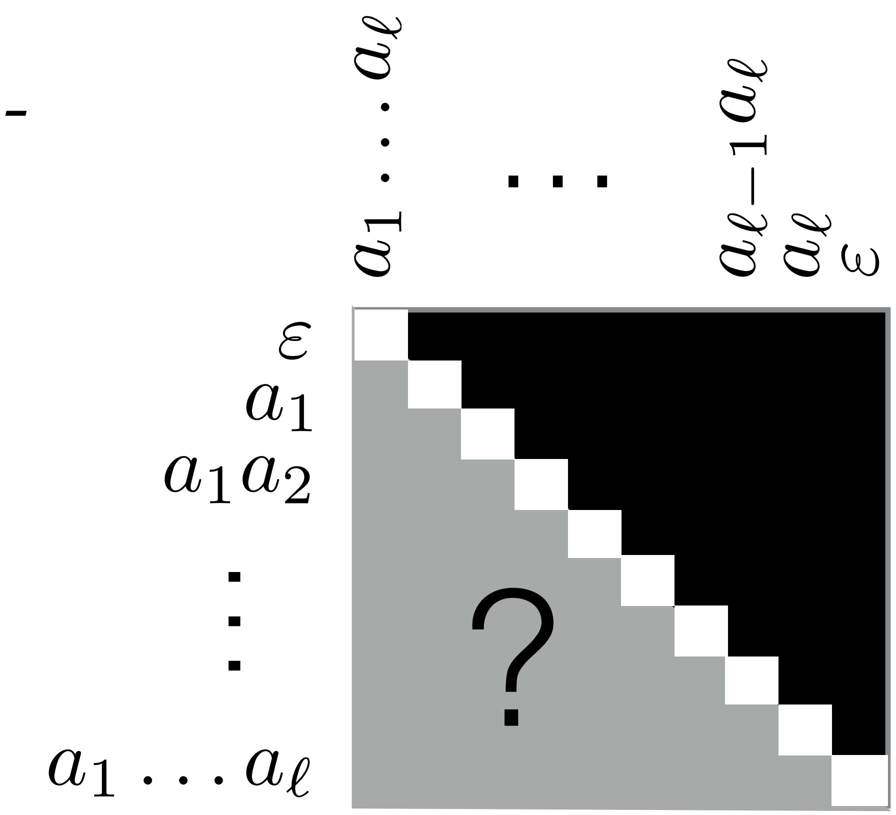

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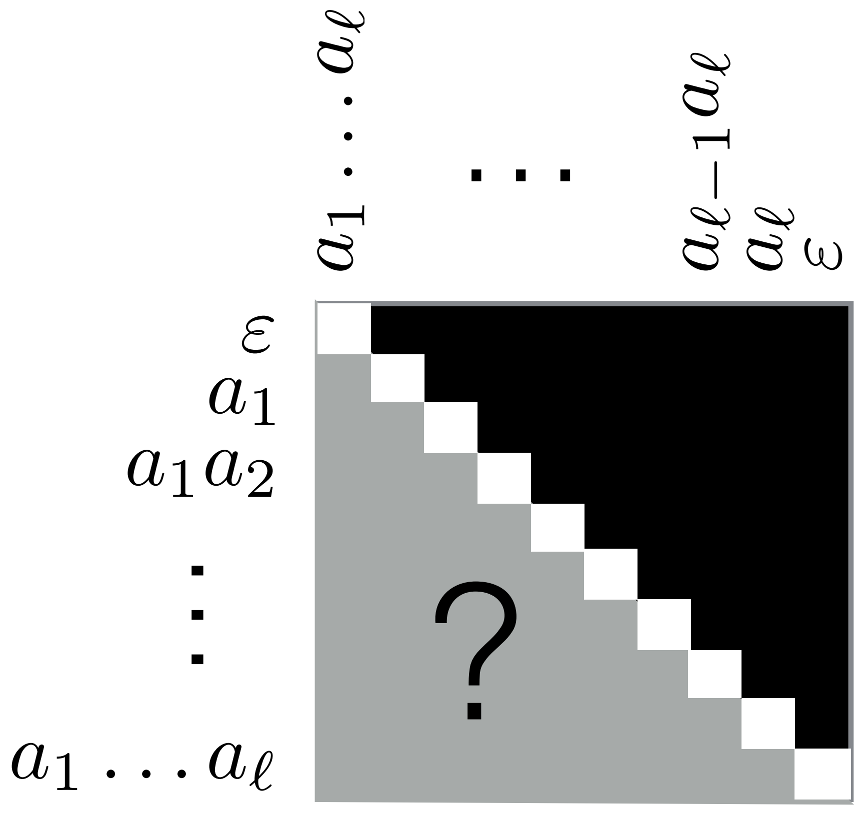

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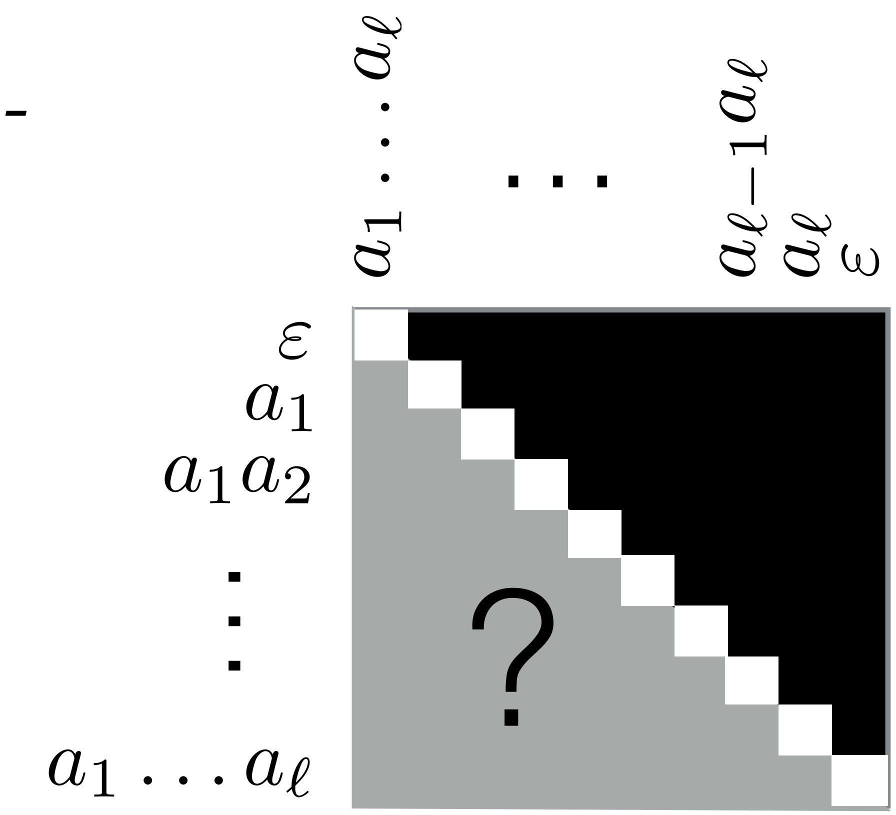


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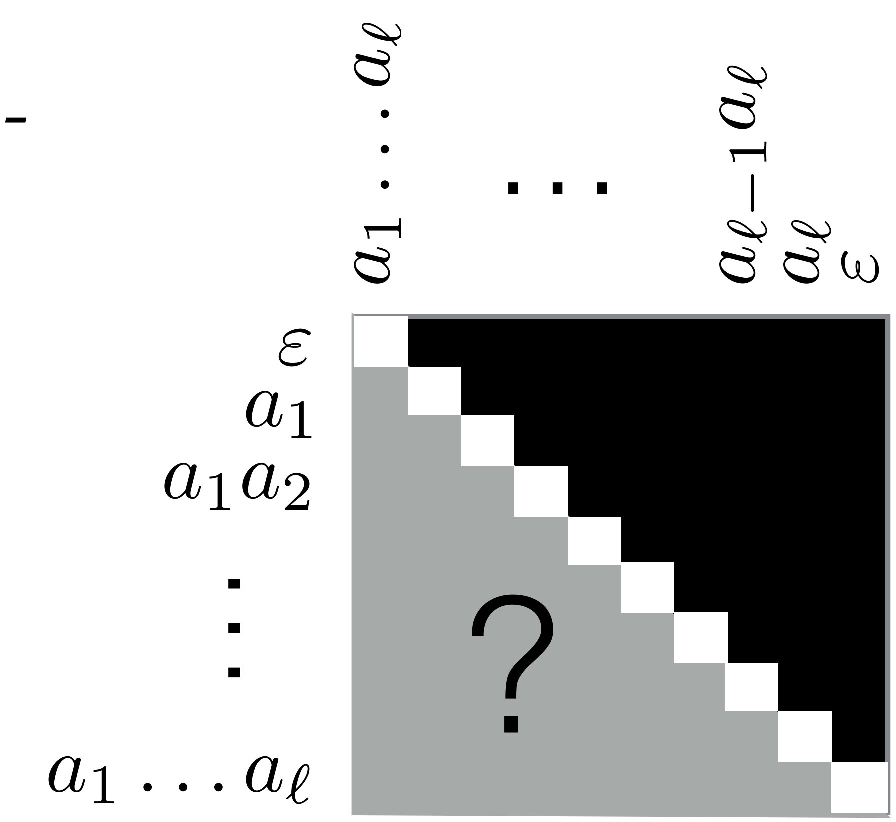


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**Corrolary:** The universality of unambiguous automata is in CoNP.

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*Hence, the number of accepted words of length at most n is computable in time polynomial in n and Universality is in P.*

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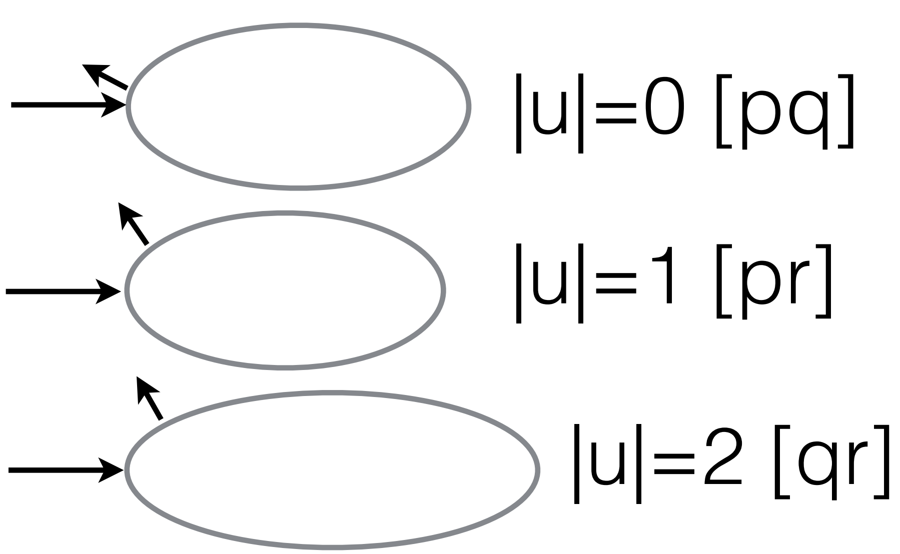

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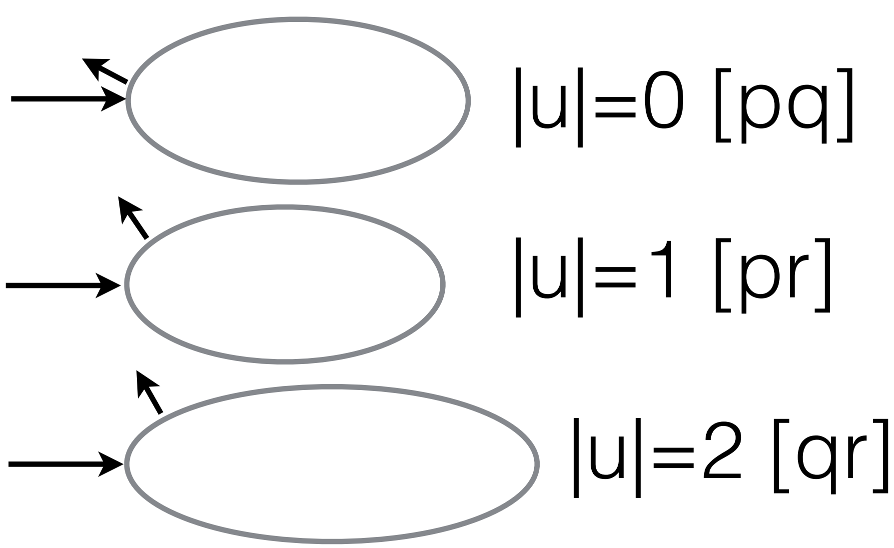


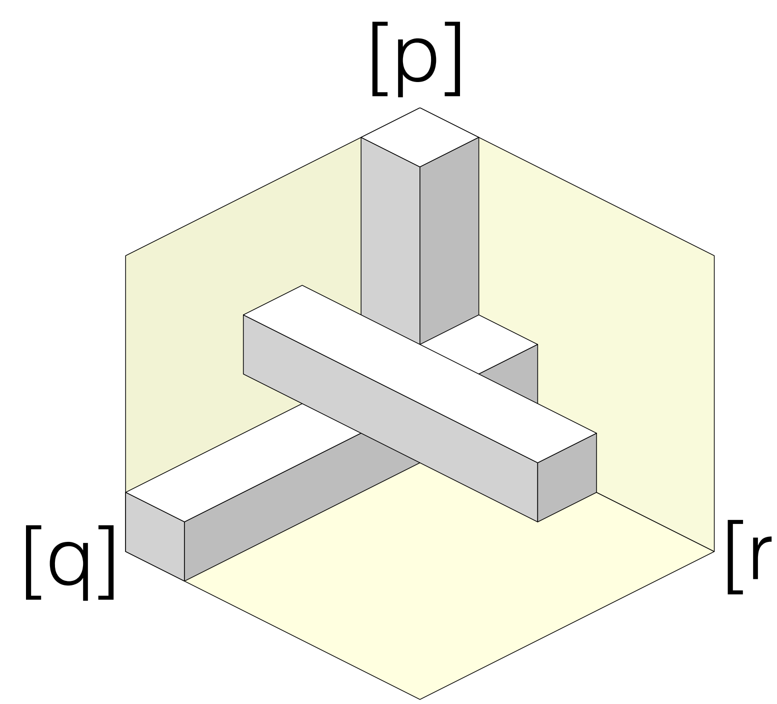

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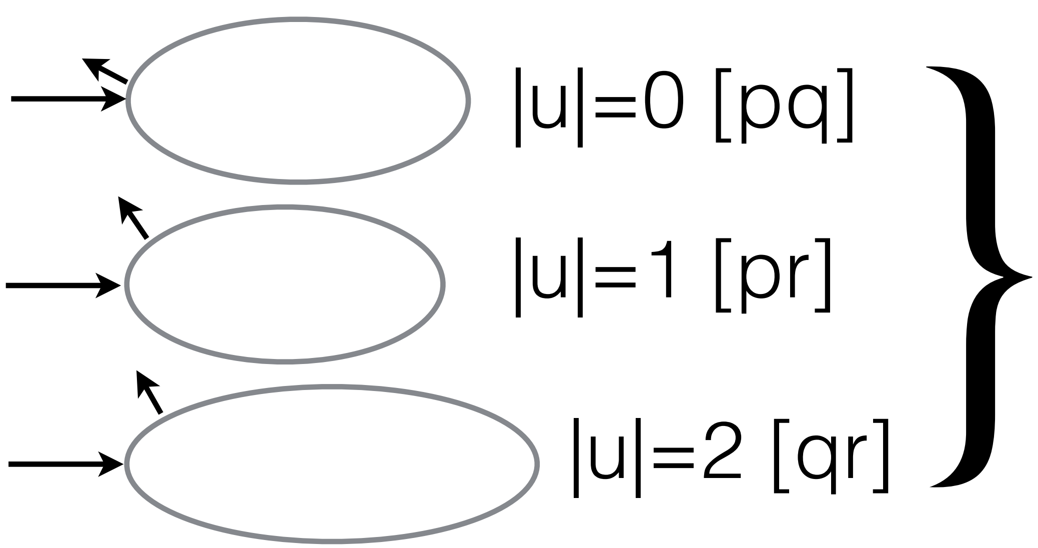
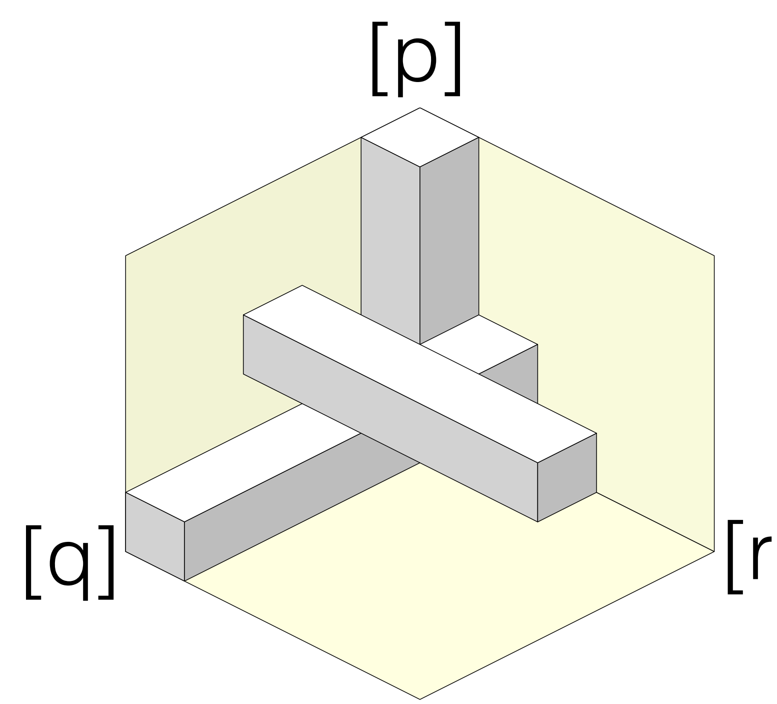



An automaton
for the complement
has to contain a
cycle of length pqr.



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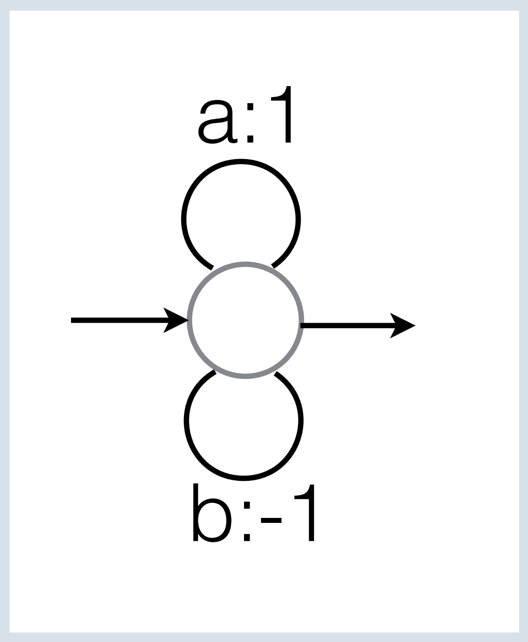


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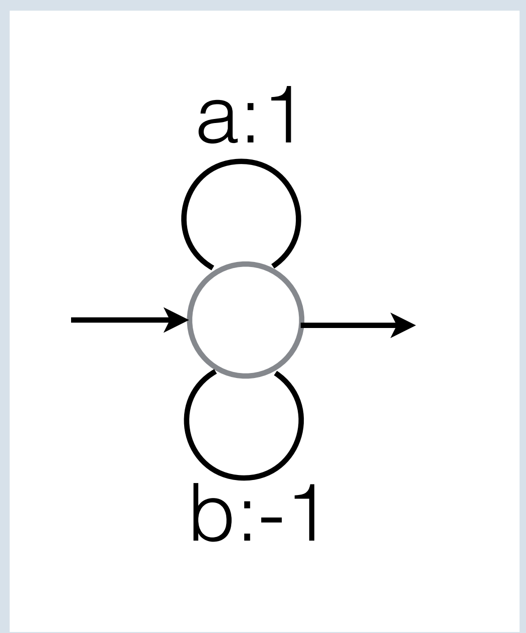


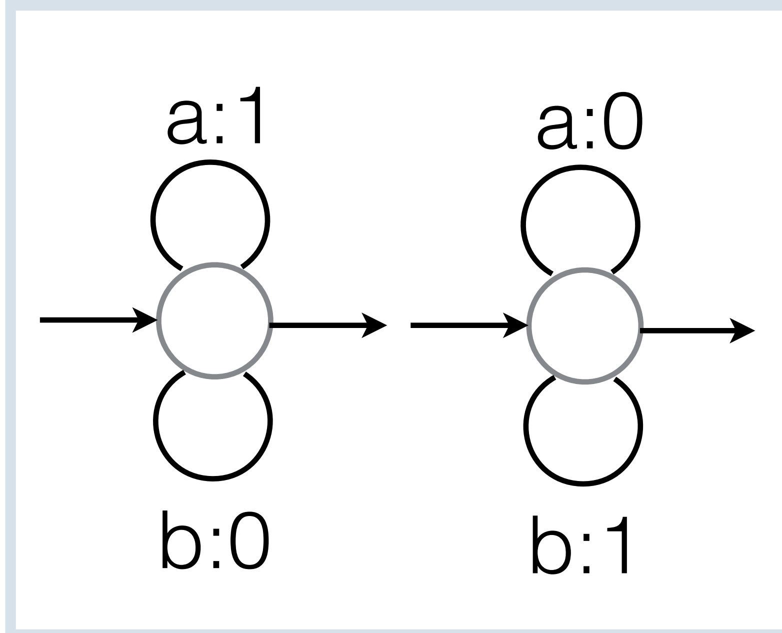


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# Tropical automata

Hashiguchi
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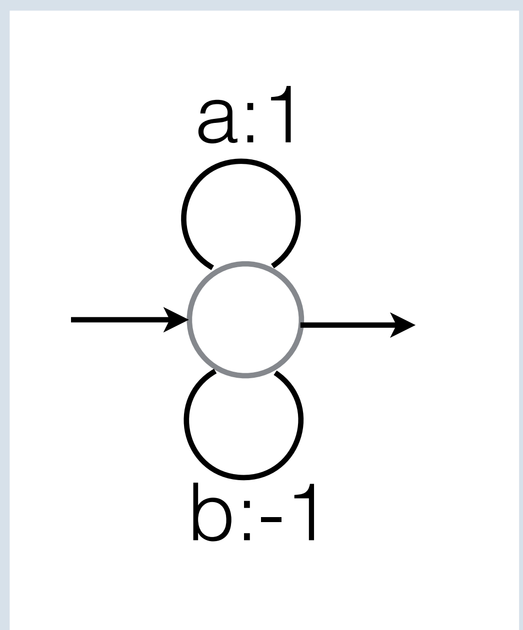


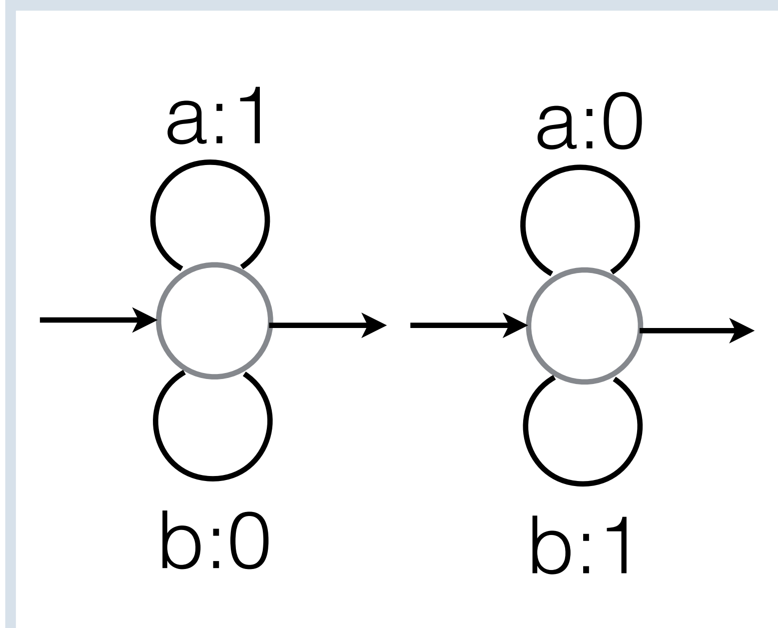


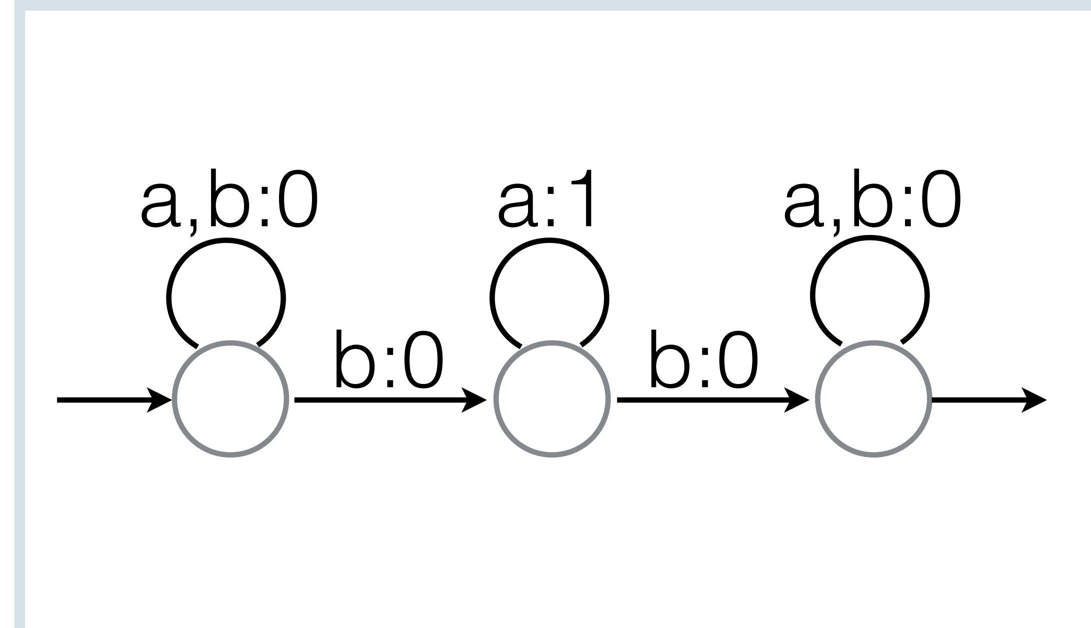


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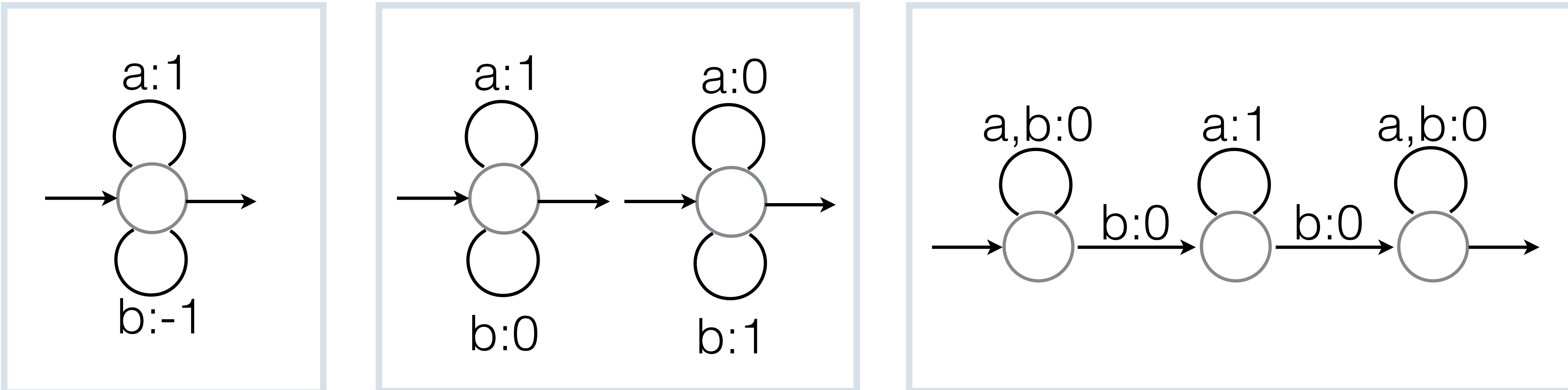


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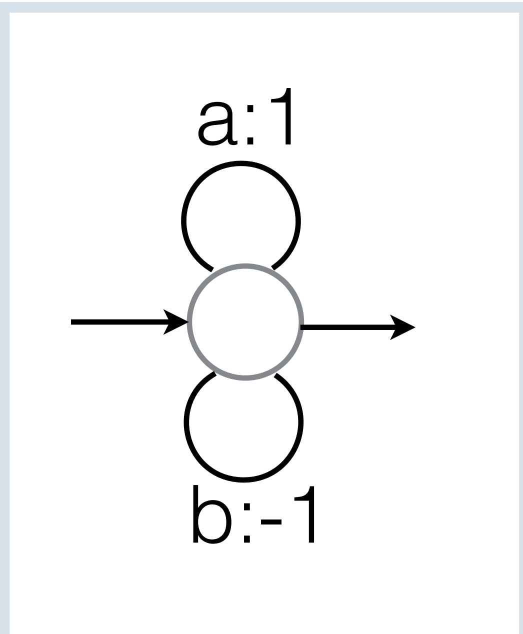


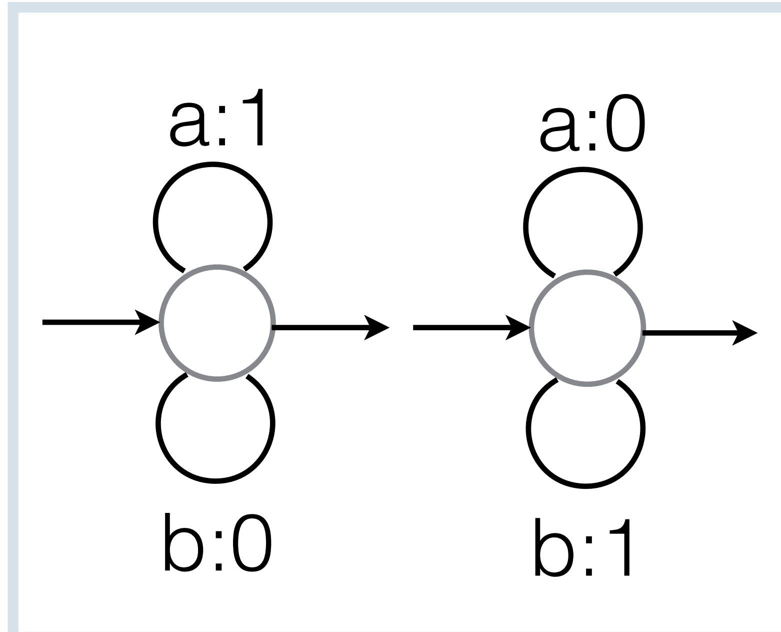


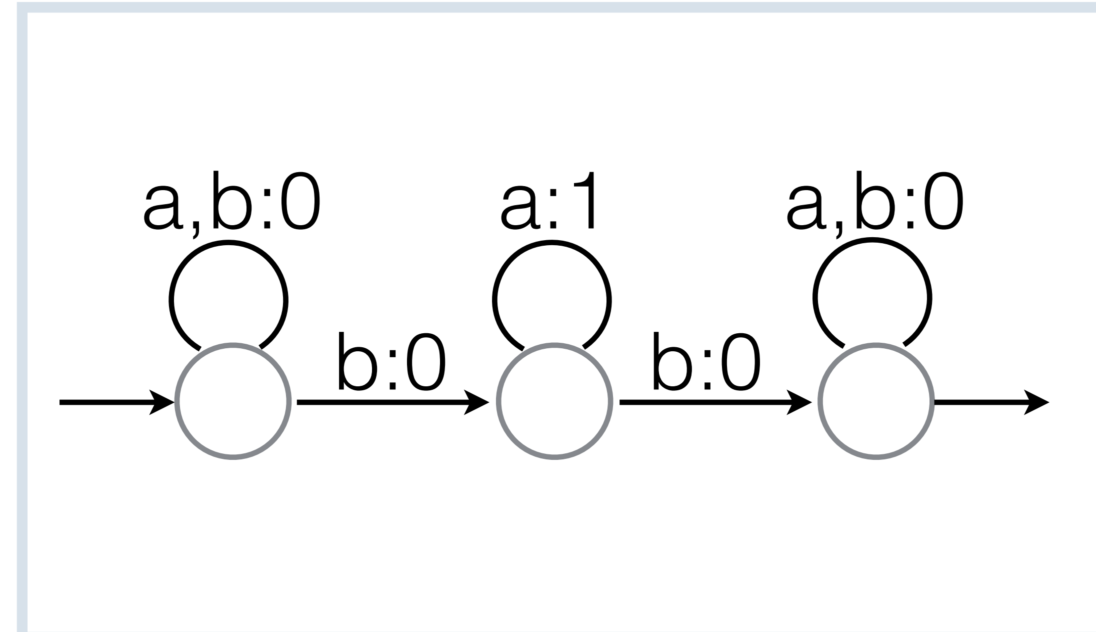


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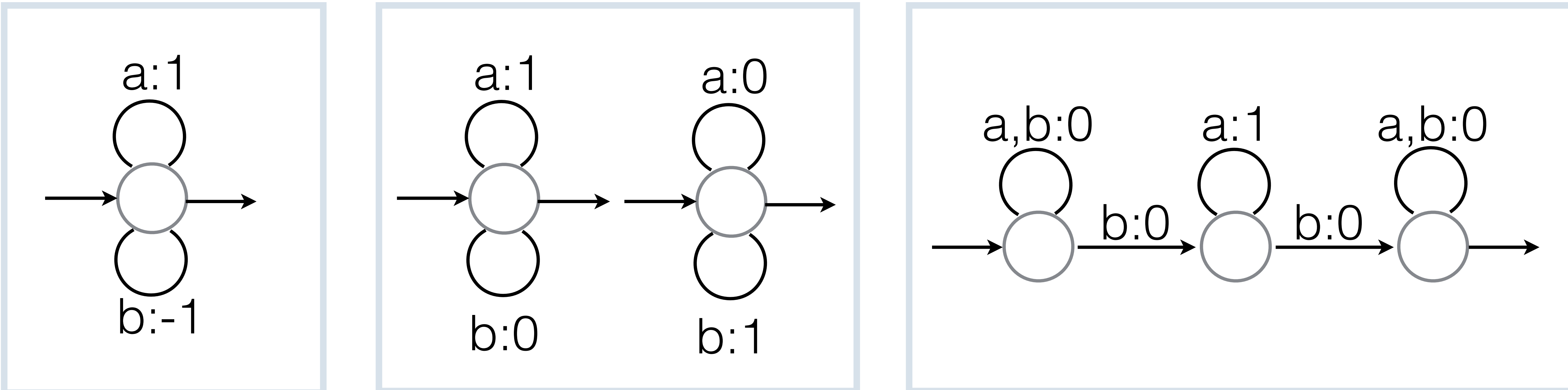


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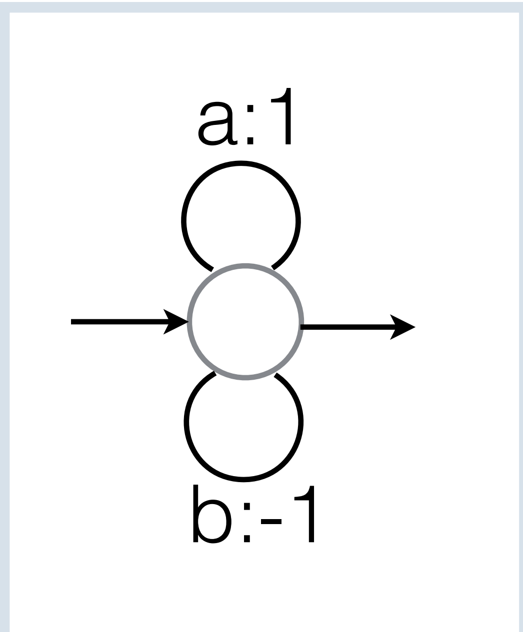


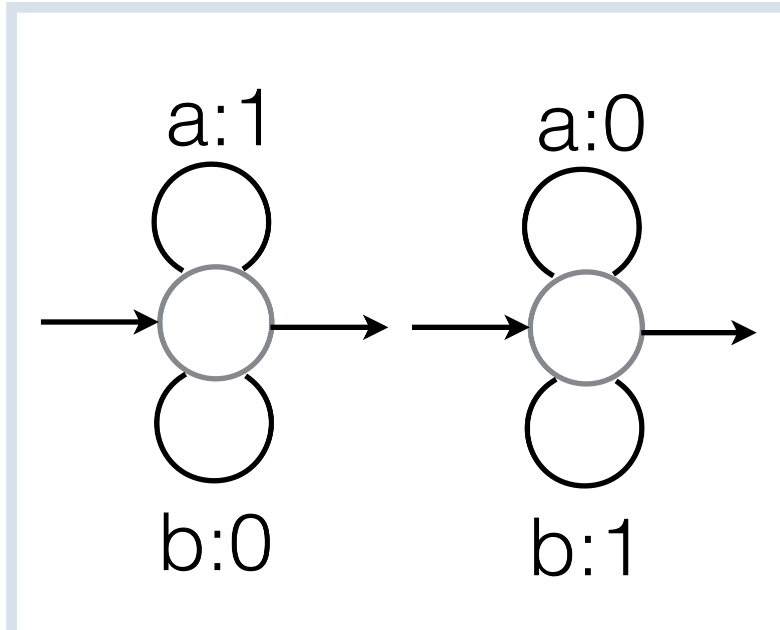


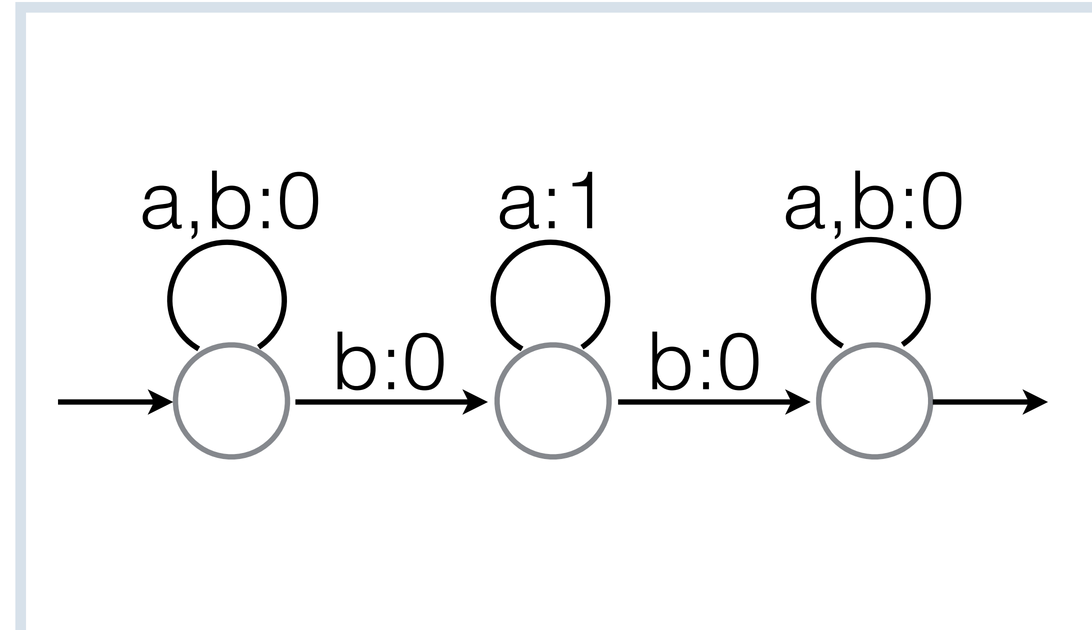


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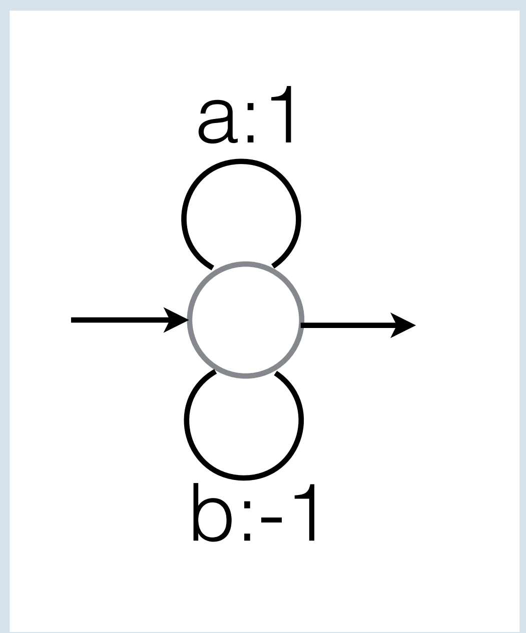


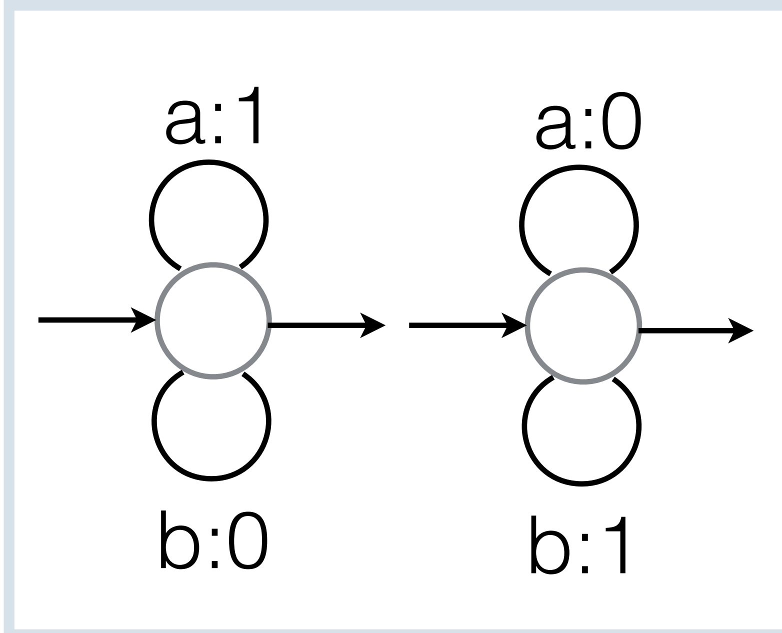


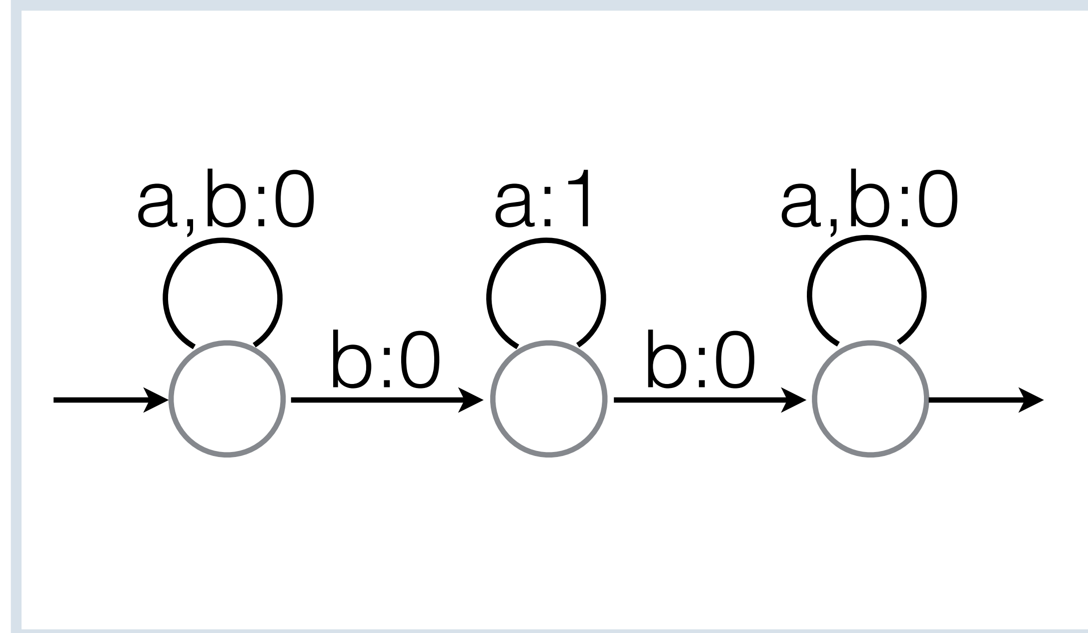


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Note that min-+ and max-+ semantics coincide over unambiguous automata. This yields unambiguous tropical automata.

# Being both min and max

**Theorem** [Lombardy&Mairesse06]: A function from words to integers that is both min-+ and max-+ rational is (effectively) recognized by an unambiguous tropical automaton.

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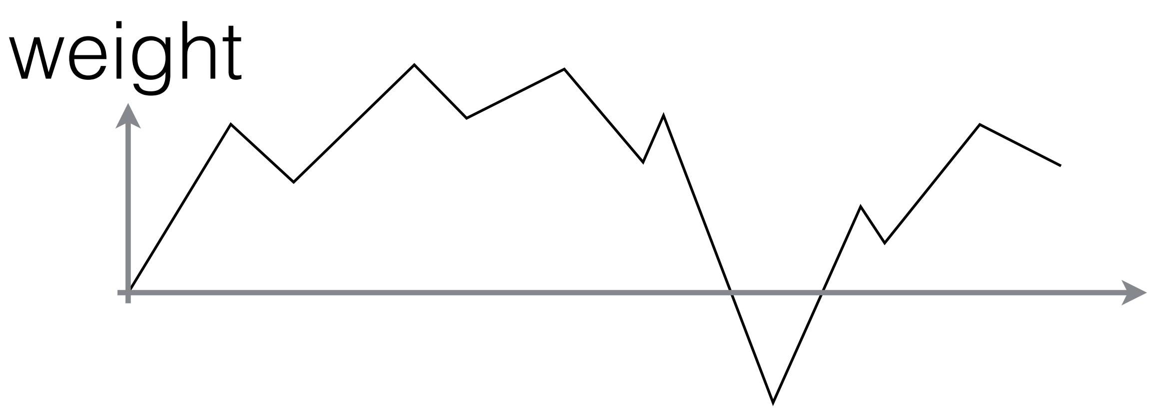

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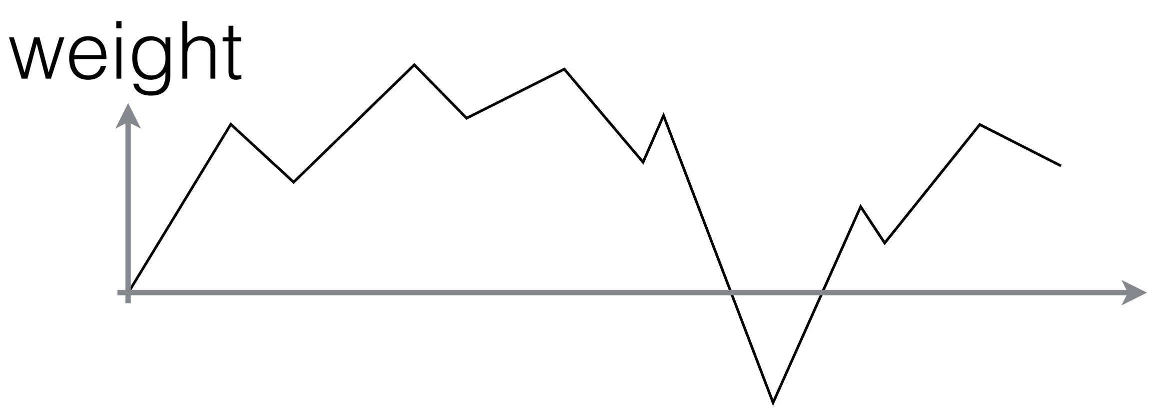


Claim 1: *There exists k such that the run never goes below -k.*

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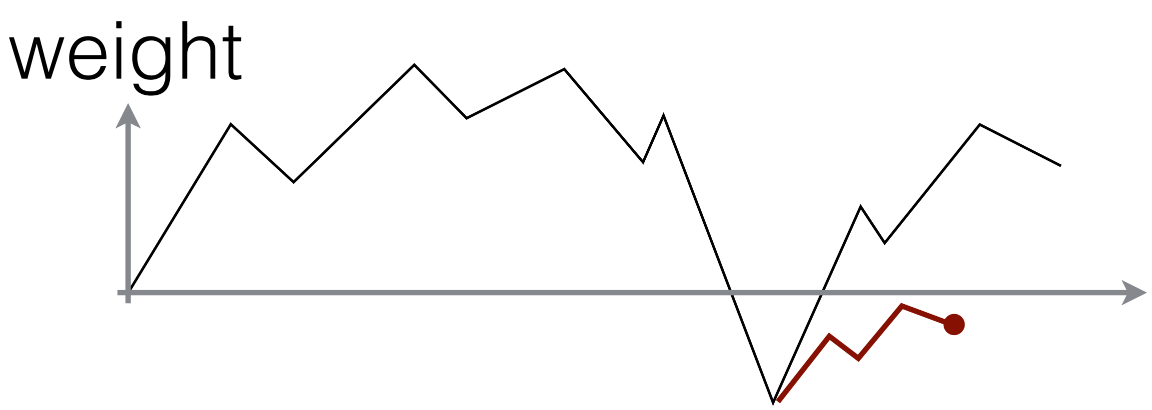


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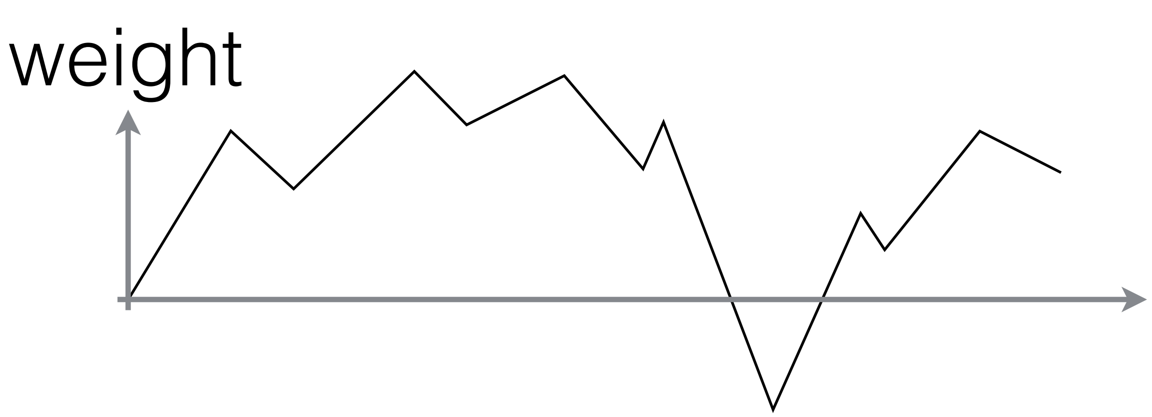


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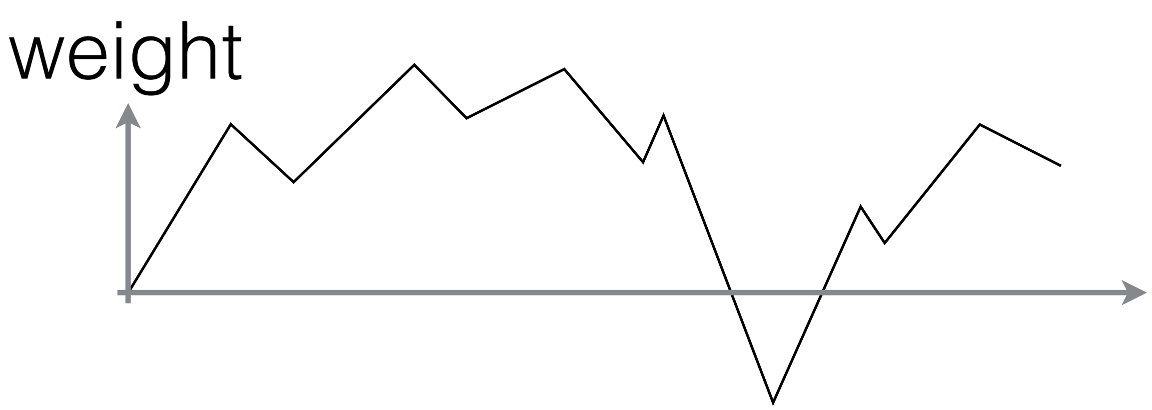


Claim 2: *There is k' such that the partial weight cannot decrease of more than k'.*

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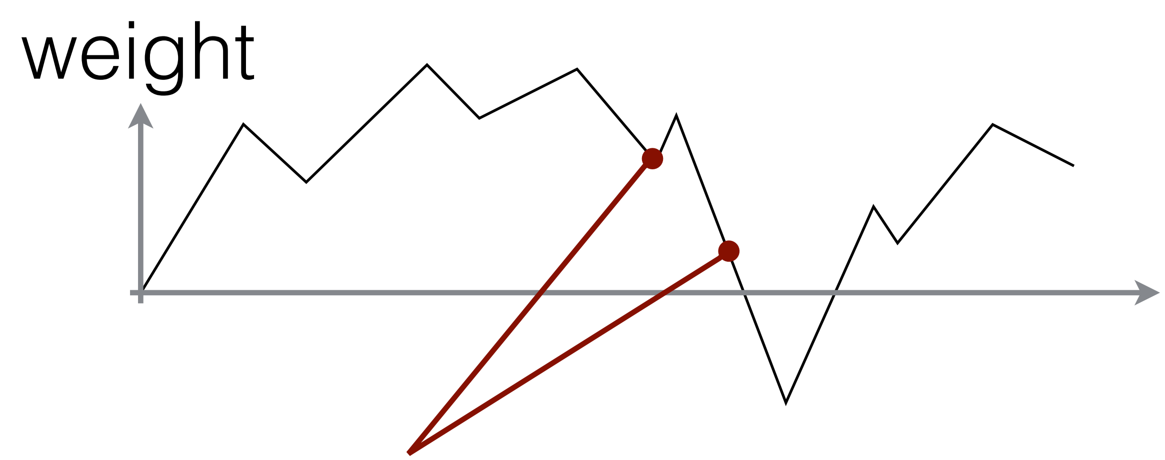


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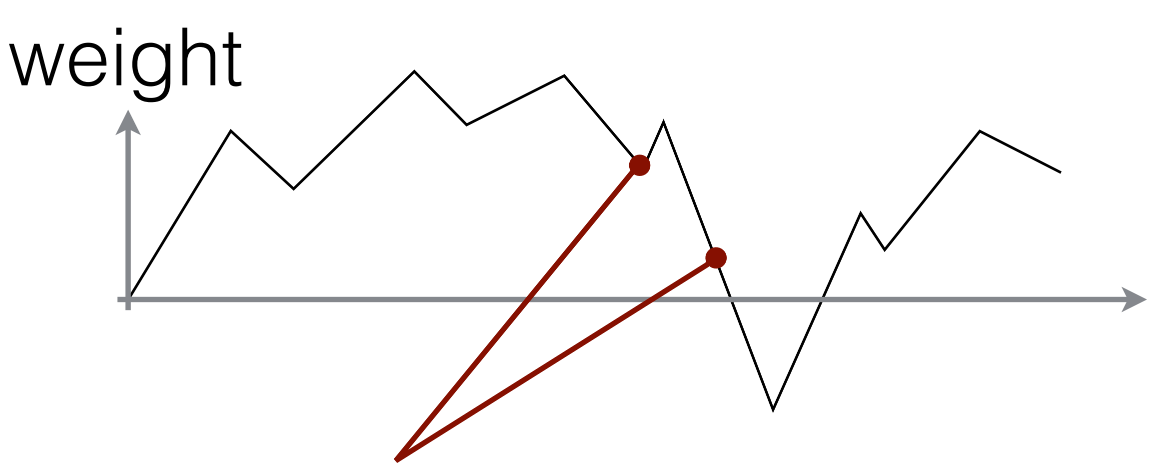


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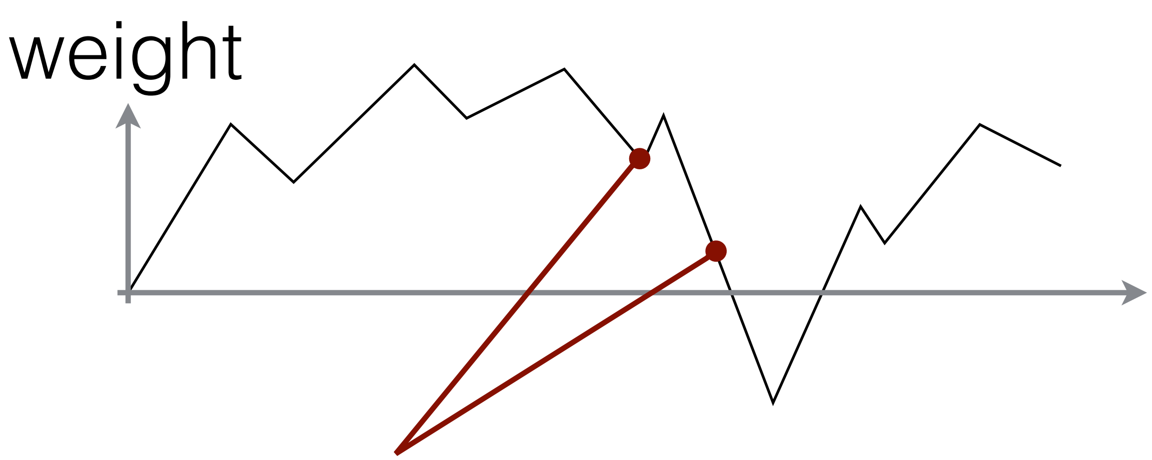


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*Thus, all runs of weight 0 have all their intermediate values the interval [-k,k'].*

*Hence, an automaton keeping weights in this interval can recognize runs of weight 0.*

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( Along the same ideas:

**Proposition** [Krob94] **(Fatou property):**
If a min-+ rational function f is such that f≥0
Then it is recognized by a min-+ automaton with only non-negative weights. )

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*It can be made unambiguous by keeping the lexicographic least run.*

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**Conjecture (separation):** Given a max-+ regular function f and a min-+ regular function g such that $f \leq g$, then there exists an unambiguous regular function h such that

$$f \leq h \leq g \,.$$

# Unambiguity for other forms of automata

# Other forms of automata

Transducers

Infinite word automata

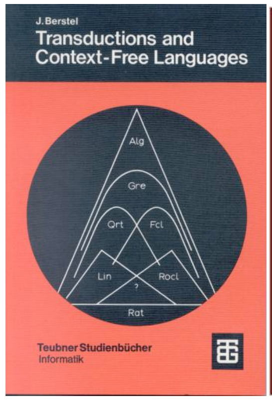
Register automata

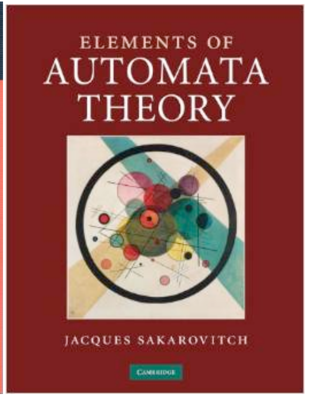
Infinite tree automata

# Other forms of automata

## Transducers

A transducer that recognizes a relation that happens to be a function is equivalent to an unambiguous transducer.

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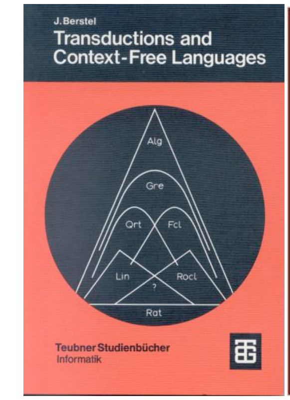


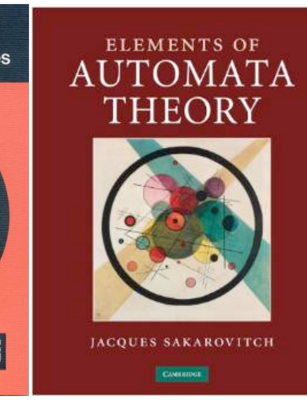


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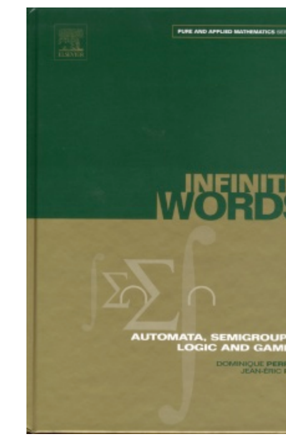


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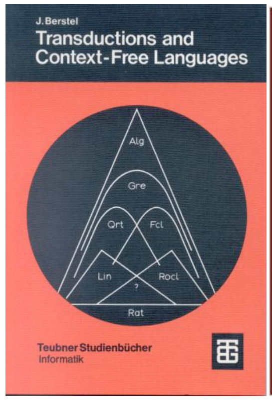


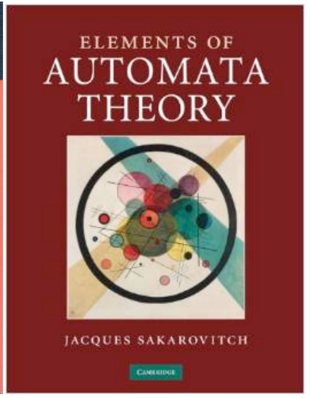


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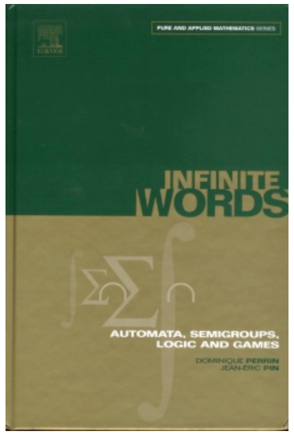


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Register automata are not closed under complemented and universality is undecidable.

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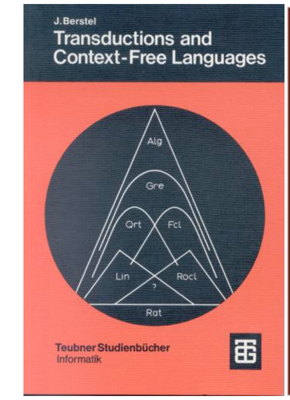


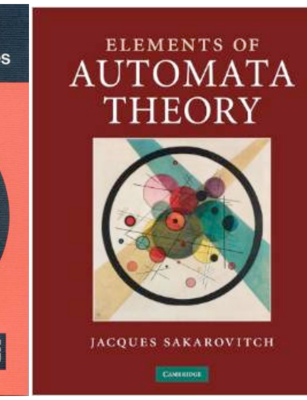


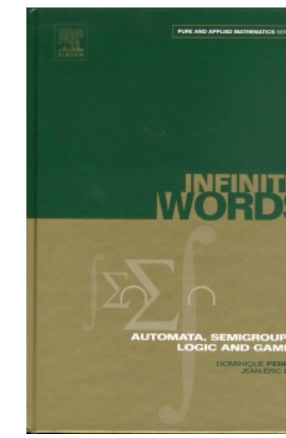


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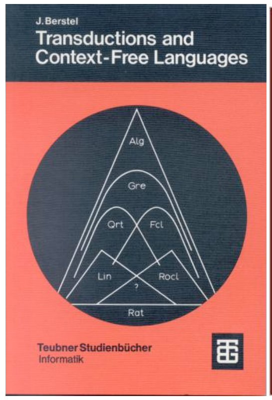
Non-deterministic Büchi automata are closed under Boolean operations and projection, but can't be determinized!
However, these can be made unambiguous [Carton&Michel00] (prophetic).

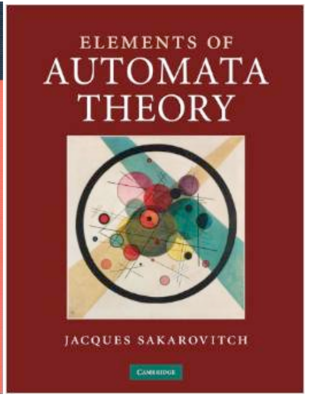
## Register automata

Register automata are not closed under complemented and universality is undecidable.
Conjecture: Unambiguous automata are closed under complement, and universality is decidable.

## Infinite tree automata

# Other forms of automata



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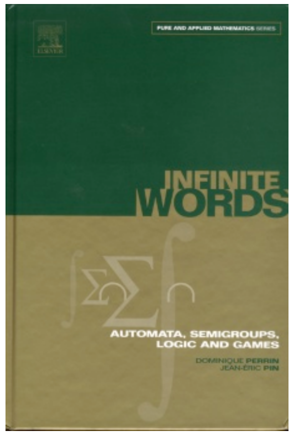


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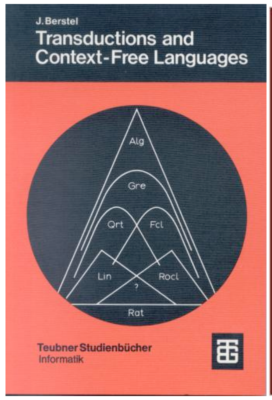


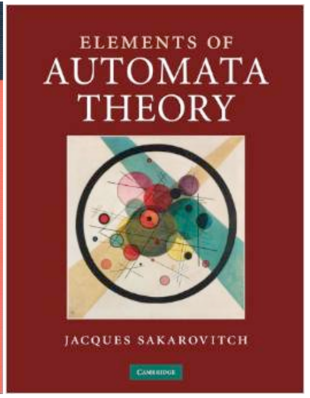


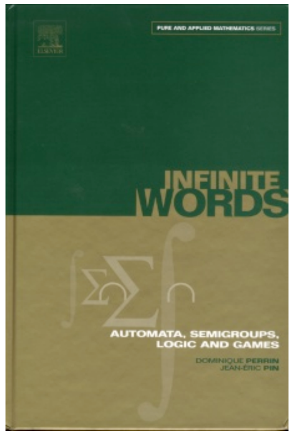


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Problem: Decide if a language is accepted by an unambiguous automaton.

# Conclusion

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Unambiguity arises naturally in automata theory in many situations:

- When non-deterministic automata are too wild (bad complexity or even undecidability of universality/equivalence, etc),
  but deterministic are too weak (because not closed under mirror…)
- When regular lookahead is added to deterministic automata.
- When this corresponds to a characterization result
  (e.g. min-plus and max-plus, transducers).

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Things are not yet well understood, and many questions remain open.

# Some open problems

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Is it polynomial to complement unambiguous word automata?

Is it possible to separate disjoint non-deterministic automata by unambiguous automata of polynomial size?

Is it possible to separate min-+ and max-+ automata by unambiguous tropical automata?

Can we decide if a min-+ automaton is equivalent to an unambiguous one?

Can we complement unambiguous register automata, and decide universality?

Is it possible to separate register automata by unambiguous ones?

Is it possible to decide if a language of infinite trees is recognized by some unambiguous automaton?

Are unambiguous automata over tame trees as expressive as general automata?