Optimal Trade Execution: What is your broker doing?

Peter Forsyth[1] Stephen Tse[2] Yuying Li[1] H. Windcliff[2] S. Kennedy[2]

[1]Cheriton School of Computer Science
University of Waterloo

[2]Morgan Stanley
New York

Waterloo, August 29, 2013

The Basic Problem

Broker buys/sells large block of shares on behalf of client

- Large orders will incur costs, due to price impact (liquidity) effects
  - $\rightarrow$ e.g. rapidly selling a large block of shares will depress the price
- Slow trading minimizes price impact, but leaves exposure to stochastic price changes
- Fast trading will minimize risk due to random stock price movements, but price impact will be large
- What is the optimal strategy?

Industry Standard Approach

We will consider so-called *arrival price algorithms*

Optimality defined in terms of pre-trade price (the arrival price) compared to average *execution price*.

- Standard algorithm is the Almgren and Chriss (2001) technique

Objective of this talk

- The Almgren, Chriss strategy is based on an approximate solution to an *optimal stochastic control* problem
- We solve this problem using a fully numerical approach
  - ⇒ Industry standard method is significantly sub-optimal in many practical cases

Formulation

$$
\begin{aligned}
P &= \text{Trading portfolio} \\
&= B + AS \\
B &= \text{Bank account: keeps track of gains/losses} \\
S &= \text{Price of stock} \\
A &= \text{Number of units of the stock} \\
T &= \text{Trading horizon (e.g. one day)}
\end{aligned}
$$

For Simplicity: Sell Case Only

Sell

$$t = 0 \rightarrow B = 0, S = S_0, A = A_0$$
$$t = T \rightarrow B = B_T, S = S_T, A = 0$$

- B_T is the cash generated by trading in $[0, T]$
- Success is measured by B_T (proceeds from sale, relative to pre-trade market value $(A_0 S_0)$).
- Maximize $E[B_T]$, minimize $Var[B_T]$

$$\begin{aligned} E[\cdot] &= \text{Expectation} \\ Var[\cdot] &= \text{Variance (a measure of risk)} \end{aligned}$$

- Typically $T =$ one day

Basic Problem

Trading rate v[1] ($A =$ number of shares)

$$\frac{dA}{dt} = v \ .$$

Suppose that S follows geometric Brownian Motion (GBM) under the objective measure

$$dS = \mu S \ dt + \sigma S \ dZ$$

μ is the drift rate of S

σ is the volatility

$dZ = \phi\sqrt{dt}$

ϕ draw from a standard normal distribution

- Reasonable model for stock prices over periods $<$ one day

[1] This gives us the trading schedule over the day. For actually placing discrete trades, we need an order book model.

Temporary Price Impact: $S_{exec} = f(v)S$

The bank account B is assumed to follow

$$\begin{aligned}
\frac{dB}{dt} &= (-vS_{exec}) \\
&\quad S_{exec} \text{ is the execution price} \\
&\quad\quad = Sf(v) \\
&\quad f(v) \text{ is the temporary price impact} \\
f(v) &= \exp[\kappa_t v] \\
&\quad < 1 \;\text{ if selling: execution price} < \text{pre-trade price}
\end{aligned}$$

Pre-trade price is S (i.e. midpoint of bid-ask)

- We actually get $S_{exec} = f(v)S < S$ if selling[2]

[2] Trading rate $v < 0$ if selling.

Optimal Strategy

Define:

$$
\begin{aligned}
X &= (S(t), A(t), B(t)) = \text{State} \\
B_T &= \text{Proceeds from selling} \\
v(X, t) &= \text{trading rate} \\
E[\cdot] &= \text{expectation}
\end{aligned}
$$

Let

$$
\underbrace{E_{t,x}^{v(\cdot)}[\cdot]}_{Reward} = E[\cdot | X(t) = x] \text{ with } v(X(u), u), u \geq t
$$

$$
\text{being the strategy along path } X(u), u \geq t
$$

$$
\underbrace{\mathrm{Var}_{t,x}^{v(\cdot)}[\cdot]}_{Risk} = \mathrm{Var}[\cdot | X(t) = x] \text{ Variance under strategy } v(\cdot)
$$

Mean Variance: Standard Formulation

Our objective is to compute the strategies which generate the *efficient frontier*.

We construct the efficient frontier by finding the optimal control $v(\cdot)$ which solves (for fixed λ)

$$\sup_{v}\left\{\underbrace{E^v[B_T]}_{Reward} - \lambda \underbrace{Var^v[B_T]}_{Risk}\right\} \tag{1}$$

• Varying $\lambda \in [0, \infty)$ traces out the efficient frontier

• $\lambda = 0; \rightarrow$ we seek only maximize cash received, we don't care about risk.

• $\lambda = \infty \rightarrow$ we seek only to minimize risk, we don't care about the expected reward.

An efficient frontier is a curve in the $(Var^v[B_T], E^v[B_T])$plane.

- Each point on the efficient frontier is Pareto optimal
- For a given value of variance, no other strategy produces a higher expected gain.

LQ Embedding (Zhou and Li (2000), Li and Ng (2000))

Equivalent formulation: for fixed λ, if $v^*(\cdot)$ maximizes

$$\sup_{v(\cdot)\in\mathbb{Z}} \Big\{ \underbrace{E_{t,x}^{v}[B_T]}_{Reward} - \lambda \underbrace{Var_{t,x}^{v}[B_T]}_{Risk} \Big\} ,$$

$$\mathbb{Z} \text{ is the set of admissible controls} \tag{2}$$

then there exists a $\gamma = \gamma(t, x, E[B_T])$ such that $v^*(\cdot)$ minimizes [3]

$$\inf_{v(\cdot)\in\mathbb{Z}} E_{t,x}^{v(\cdot)} \left[\left(B_T - \frac{\gamma}{2} \right)^2 \right] . \tag{3}$$

$\hookrightarrow$ Equation (3) can be solved using dynamic programming.

[3] Strictly speaking, since some values of γ may not represent points on the original frontier, we need to use the algorithm in Tse, Forsyth, Li (2012) to remove these points.

Hamilton Jacobi Bellman (HJB) Equation

Let

$$
\begin{aligned}
V(s, \alpha, b, \tau) = & \text{ Value Function} \\
= & \inf_{v(\cdot) \in \mathbb{Z}} \left\{ E_{t,x}^{v(\cdot)} \left[(B_T - \frac{\gamma}{2})^2 \;\middle|\; S(t) = s, A(t) = \alpha, B(t) = b \right\} \\
& x = (s, \alpha, b) \\
& s = \text{ stock price} \\
& \alpha = \text{ number of units of stock} \\
& b = \text{ cash obtained so far} \\
& T = \text{ Trading horizon} \\
& \tau = T - t = \text{ time running backwards} \\
& \mathbb{Z} = [v_{\min}, 0] \qquad \text{(Only selling permitted)}
\end{aligned}
$$

HJB Equation for Optimal Control $v^*(\cdot)$

We can use dynamic programming to solve for

$$V(s, \alpha, b, \tau) \quad = \quad \inf_{v(\cdot) \in \mathbb{Z}} E_{t,x}^{v(\cdot)} \left[\left(B_T - \frac{\gamma}{2} \right)^2 \right] . \qquad (4)$$

Then, using some stochastic calculus, $V(s, \alpha, b, \tau)$ is determined by

$$\frac{\partial V}{\partial \tau} \quad = \quad (\frac{\sigma^2 s^2}{2}) \frac{\partial^2 V}{\partial s^2} + \mu s \frac{\partial V}{\partial s} + \inf_{v \in \mathbb{Z}} \left[-vsf(v) \frac{\partial V}{\partial b} + v \frac{\partial V}{\partial \alpha} \right]$$

$$\mathbb{Z} = [v_{min}, 0]$$

with initial condition $V(s, \alpha, b, \tau = 0) = (b - \gamma/2)^2$.

Numerical Method: Mean Variance Efficient Frontier

Recall that

$$V(s, \alpha, b, \tau = 0) \;\; = \;\; (b - \gamma/2)^2$$

Numerical Algorithm

- Pick a value for $\gamma \in [0, \infty)$
  - Solve HJB equation numerically[4] (i.e. on a grid) for optimal control $v = v(s, \alpha, b, \tau)$
  - Store control at all grid points
  - Simulate trading strategy using a Monte Carlo method (use stored optimal controls)
  - Compute mean, standard deviation
  - This gives a single point on the efficient frontier
- Repeat

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[4]We need to be sure that our numerical algorithm converges to the correct solution, the *viscosity solution* (Forsyth (2011) Applied Numerical Mathematics)

But solving the HJB equation requires some work

- But this is considered too complex in industry
- So, the original (Almgren and Chriss) paper made several approximations (e.g. $v(\cdot)$ independent of $S(t)$).
- In fact, a careful read of this paper, shows that the objective function (after the approximations) is not actually Mean Variance, but is Mean Quadratic variation
  - $\rightarrow$ Risk measure is *Quadratic Variation* not *Variance*

Formally, the quadratic variation risk measure is defined as

$$\text{Risk}_t = E\left[\int_t^T \left(dP(t')\right)^2 \right]$$

$$P = \text{ Trading Portfolio } = AS + B$$

This is the quadratic variation of the portfolio value process.

Mean Quadratic Variation

This measures risk in terms of the average variability of the portfolio along the entire trading path.
Find optimal strategy $v(\cdot)$ which maximizes (for fixed λ)

$$\sup_{v(\cdot)\in\mathbb{Z}} \left\{ \underbrace{E_{x,t}^{v(\cdot)}\left[B_T\right]}_{Reward} - \lambda \underbrace{E_{x,t}^{v(\cdot)}\left[\int_t^T \left(dP(t')\right)^2\right]}_{Risk} \right\} \tag{5}$$

One can easily derive the HJB equation for the optimal control $v^*(\cdot)$ for Mean Quadratic variation optimal strategies

Varying λ will trace out a curve in the expected value, standard deviation plane

This problem is much simpler to solve than the Mean Variance problem (see: Almgren and Chriss (2001))

Numerical Examples: Mean Variance vs. Mean Quadratic Variation

Simple case: GBM, zero drift [5]

$$dS = \sigma S \; dZ$$

Recall temporary Price Impact ($S_{exec} = f(v)S$) :

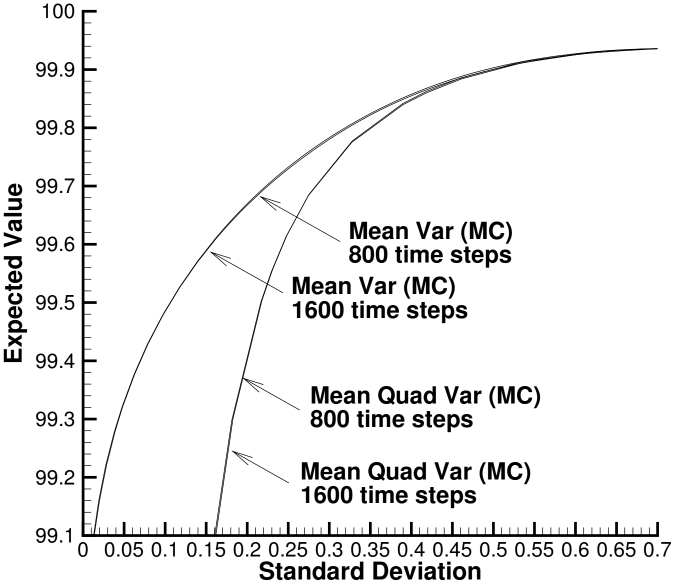
$$f(v) = exp(\kappa_t v)$$

T	s_{init}	α_{init}	Action
1/250 (One Day)	100	1.0	Sell

Case	σ	κ_t	Percentage of Daily Volume
1	0.2	2.4×10^{-6}	20.0%
2	1.0	2×10^{-6}	16.7%

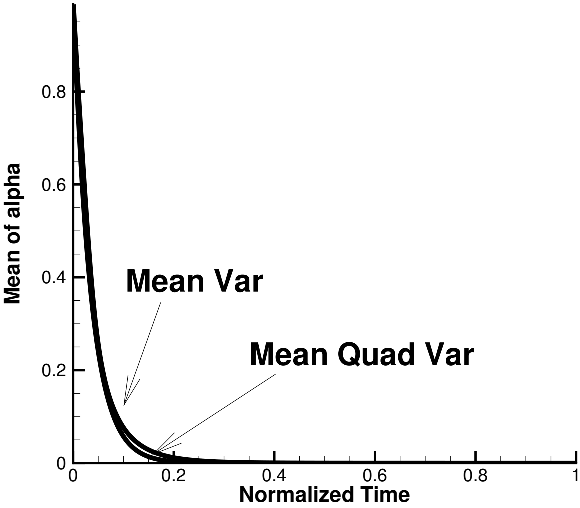
[5] Over one day, drift is negligible

$\sigma = .2$, 20% daily volume, $S_{init} = 100$



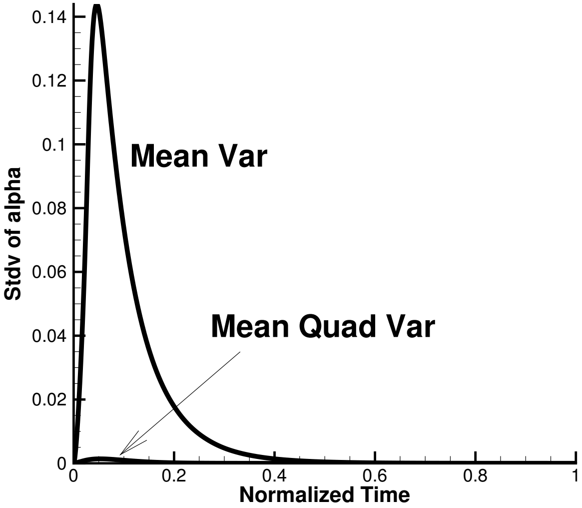
Mean Share Position (α) vs. Time

- $\sigma = 1.0$, 16.7% daily volume
- Mean: 99.29.
- Std(Mean Variance) $= 0.68$
- Std(Mean Quadratic Variation) $= 0.93$



Standard Deviation of Share Position (α) vs. Time

- $\sigma = 1.0$, 16.7% daily volume
- Mean: 99.29.
- Std(Mean Variance) $= 0.68$
- Std(Mean Quadratic Variation) $= 0.93$



Conclusions: Optimal for Who?

- From a client point of view
  - Client is only concerned with the mean and variance of final cash position
    - $\rightarrow$ Mean Variance is the optimal strategy
- From the bank point of view
  - Banks care about risk during the trading day
  - Mean Quadratic variation controls this risk
    - $\rightarrow$ Easier to compute than Mean Variance optimal strategies
    - $\rightarrow$ Easy to manage trading (share position has small standard deviation)
  - But
    - $\rightarrow$ This is sub-optimal for the client
- So, banks are doing what is best for them, not for their clients

  $\Rightarrow$ Are you surprised?