Making Leveraged Exchange-Traded Funds Work for your Portfolio

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Abstract

We examine strategically incorporating broad stock market leveraged exchange-traded funds (LETFs) into infrequently rebalanced investment portfolios. We demonstrate that easily understandable and implementable strategies can enhance the risk-return profile of a portfolio containing LETFs. Our analysis shows that seemingly reasonable investment strategies may result in undesirable Omega ratios, with these effects compounding across rebalancing periods. By contrast, relatively simple dynamic strategies that systematically de-risk the portfolio once gains are observed can exploit this compounding effect, taking advantage of favorable Omega ratio dynamics. Our findings suggest that LETFs represent a valuable tool for investors employing dynamic strategies, while confirming their well-documented unsuitability for passive or static approaches.

Keywords: Asset allocation, leveraged ETFs, neural network
JEL codes: G11, G22
AMS codes: 91G, 65N06, 65N12, 35Q93

1 Introduction

Leveraged Exchange Traded Funds (LETFs) are exchange-traded funds (ETFs) replicating some multiple β of the daily returns of their underlying reference assets or indices before costs. In contrast with 'vanilla' ETFs (VETFs), which simply aim to replicate the returns of their underlying assets/indices before costs (i.e., $\beta = 1$), typical LETF multipliers are $\beta = 2$ or $\beta = 3$ of daily returns in the case of leveraged long exposure[1].

There are two diametrically opposed threads in the literature concerning LETFs. One set of authors (see e.g. Sullivan (2009); Pessina and Whaley (2021); Bednarek and Patel (2022)) suggest that LETFs are best avoided. This literature's point of view can perhaps be summarized by a quote from Pessina and Whaley (2021)

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[1]Inverse LETFs, with negative return multipliers, are not considered in this paper

> *"Levered and inverse products are not, and cannot be, effective investment management tools."*

On the other hand, there is a contrasting literature (see e.g. Bansal and Marshall (2015); Curcio and Dickerson (2017); Balter et al. (2025)) that comes to the completely opposite conclusion. For example, Bansal and Marshall (2015) note that

> *"We conclude that LETFs...have been wrongly maligned, and we believe that they should be considered for inclusion in aggressive portfolios."*

In actual practice, LETFs are popular with retail (Johnson, 2022) and institutional investors (DeVault et al., 2021). For a more complete review of the literature on LETFs, on both sides of this debate, we refer the reader to Van Staden et al. (2026).

In this work, our focus is limited to LETFs written on broad stock market indices—a critical qualification. LETFs based on broad, diversified indices like the S&P 500 reference an "underlying asset" (i.e. the market index itself) which has high levels of diversification, relatively lower volatility, and a tendency to exhibit positive long-term drift. For example, Figure 1.1(a) considers a simple buy-and-hold position for three ETFs referencing the S&P 500 as underlying asset: A standard VETF replicating the index (IVV), a LETF with daily returns multiplier $\beta = 2$ (SSO), and a LETF with daily returns multiplier $\beta = 3$ (UPRO).

In contrast, Figure 1.1(b) illustrates that in the case of the S&P Oil & Gas Exploration & Production Select Industry Index, which has about 50 constituents in a volatile sector, simple buy-and-hold positions in the VETF (XOP) and the corresponding LETF with multiplier $\beta = 2$ (GUSH) show how LETFs can indeed live up to their bad reputation.

However, we emphasize that we are *not* making the case for simple buy-and-hold strategies involving LETFs. Instead, we present relatively sophisticated yet easily understandable and implementable strategies that require only infrequent rebalancing. Figure 1.1 is only included as an illustration for why our focus remains on LETFs referencing broad stock market indices, rather than LETFs on niche sector indices.

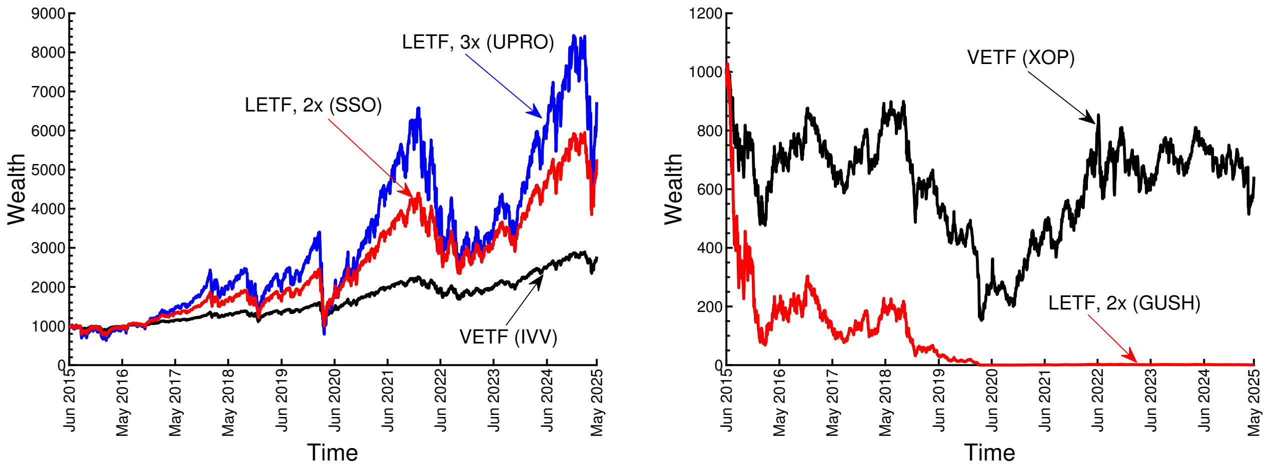
So how does one go about incorporating broad stock market LETFs into a portfolio, to take advantage of relative return behavior observed in Figure 1.1(a)? The contributions of this paper are as follows:

- We avoid the trap of using only available LETF market data, since LETFs were only introduced in 2006 (Bansal and Marshall (2015)) and therefore cannot provide a thorough historical perspective of LETF behaviour during different market and inflation regimes[2]. Figure 1.1(a) might reflect only the relatively benign market conditions (with notable exceptions) of the last decade, and we do not want to assume that broad stock market LETFs will always have the behavior relative to the VETF as observed in Figure 1.1(a).

  Instead, we construct a proxy returns time series for a VETF and LETF referencing a broad equity market index with data since 1926, to obtain a truly long-term, robust perspective on introducing LETFs to a portfolio. The synthetic time series appropriately incorporates ETF costs and interest, and is inflation-adjusted to enable realistic conclusions.

  As noted, we are focused on long term behavior, in this work, which necessitates construction of proxy returns. For a study comparing traded LETFs with a benchmark index, after 2006, we refer the reader to Curcio and Dickerson (2017).

[2] Many LETFs were only introduced much later. For example, the LETF GUSH illustrated in Figure 1.1(b) was only introduced in May 2015.



(a) *S&P 500: LETFs and VETF*

(b) *S&P Oil & Gas Exploration & Production Select Industry Index: VETF and LETF*

FIGURE 1.1: *LETFs written on broad stock market indices vs. LETFs on niche sector indices: Illustrative buy-and-hold portfolio values from investing all hypothetical initial wealth of $1000 in each VETF and LETF from June 2015 until May 2025. Please note that we are not making the case for simple buy-and-hold strategies in this paper - these figures are intended to illustrate the behavior of the LETF relative to its corresponding VETF over time. Returns have been adjusted for splits and dividends, data from nasdaq.com.*

- Intuitive explanations are provided as to why LETFs can enhance portfolio performance when used appropriately within the context of a dynamic strategy, which offers accessible insights for practitioners.
- The critical importance of the Omega ratio in evaluating medium-term LETF strategies is demonstrated, revealing how this metric captures performance dynamics that traditional risk measures may miss. Our analysis shows that while some seemingly reasonable LETF strategies exhibit Omega ratios below unity, others achieve ratios above unity, with compounding effects across multiple rebalancing periods. We show how systematic de-risking following gains allows investors to capitalize on favorable Omega ratio compounding. This provides the intuitive explanation for the superior path-dependent performance observed in the literature (Van Staden et al. (2026)).
- Finally, we construct optimal dynamic LETF investment strategies using a data driven neural network approach, which relies only on historical returns data, i.e. no parametric models are assumed for the return dynamics of underlying assets. The results confirm the benefits and risks, as well as significant improvements to Omega ratios, that can be realized by incorporating LETFs into portfolios.

While our findings strongly support the use of LETFs within dynamic, actively managed strategies, they simultaneously reinforce warnings against their use in passive or static investment contexts.

We would like to contrast this paper with our companion work on LETFs (Van Staden et al., 2026). In Van Staden et al. (2026) we focus on (i) closed form solutions of Hamilton-Jacobi-Bellman (HJB) equations and (ii) optimal strategies under cumulative information ratio criteria.

In contrast, in this work, we concentrate on the following:

- We build up intuition around the effectiveness of LETFs in *dynamic* strategies. As mentioned above, we conclude that a key explanatory factor is the Omega ratio. However, crucially, we base the Omega ratio on pathwise comparison performance criteria.

- We use an *across time aware* cumulative difference objective function (Van Staden et al., 2024a) for our data-driven neural network approach for solving the optimal stochastic control problem. This objective function attempts to avoid underperforming the benchmark during the entire investment horizon, not just at the terminal date.

The paper is structured as follows. Section 2 provides intuition under the assumption of parametric asset dynamics where the underlying stock index follows a jump diffusion process calibrated to historical stock market data, with corresponding illustrative investment results in Section 3. Building on these insights, Section 4 demonstrates the investment outcomes obtained by rebalancing to different fixed LETF weight allocations. Section 5 discusses our choice of the objective function in this paper for the neural network technique. Section 6 generalizes these results by using a data-driven neural network approach to determine the optimal dynamic allocation of wealth to the LETF at each rebalancing time, confirming the contrarian nature of strategies designed to take advantage of LETF behavior. Section 7 concludes the paper.

2 Intuition: jump diffusion

While the ultimate goal is to show LETF investment strategies without any parametric assumptions regarding the dynamics of the underlying assets (see Section 6), to gain the necessary intuition it is nevertheless instructive to start with simple parametric dynamics for the underlying assets.

We first consider the basic case where the underlying stock index follows a jump diffusion. For the readers convenience, we highlight key elements of the derivation in Avellaneda and Zhang (2010); Ahn et al. (2015); Van Staden et al. (2026).

Let the value of the stock index be denoted by $S(t)$ and the value of a risk free bond be denoted by $B(t)$. Assume that the stock index follows a jump diffusion process, which allows for non-normal returns. If a jump occurs $S(t) = \xi S(t^-)$, and

$$
\begin{aligned}
\frac{dS}{S(t^-)} &= (\mu - \lambda\kappa)\ dt + \sigma\ dZ + (\xi - 1)d\mathbb{Q} \\
& \quad d\mathbb{Q} = \begin{cases} 0 & ; \quad \text{probability } (1 - \lambda\ dt) \\ 1 & ; \quad \text{probability } \lambda\ dt \end{cases} \\
& \quad \kappa = E[\xi - 1] \\
& \quad \lambda = \text{ intensity of the Poisson process} \\
& \quad dZ = \text{ increment of a Wiener process .}
\end{aligned} \tag{2.1}
$$

Assume that $y = \log \xi$ follows a double exponential process(Kou, 2002), with density $g(y)$ given by

$$
g(y) \quad = \quad p_{up}\eta_1 e^{-\eta_1 y}\mathbf{1}_{y\geq 0} + (1 - p_{up})\eta_2 e^{\eta_2 y}\mathbf{1}_{y<0}. \tag{2.2}
$$

where p_{up} is the probability of an upward jump. Note as well that

$$
E[\xi] \quad = \tfrac{p_{up}\eta_1}{\eta_1 - 1} + \tfrac{(1-p_{up})\eta_2}{\eta_2+1} \ . \tag{2.3}
$$

Equation (2.1) implies that

$$
\frac{S(t)}{S(0)} \quad = \quad e^{(\mu-\lambda\kappa)t+\sigma(Z(t)-Z(0))} e^{-\sigma^2 t/2 + \sum_{i=0}^{\mathbb{N}(t)} \log \xi_i} \ , \tag{2.4}
$$

where $\mathbb{N}(t)$ counts the number of Poisson jumps with intensity λ in $(0,t)$, and ξ_i are drawn from the density (2.2), and $(Z(t) - Z(0)) \simeq \mathcal{N}(0,t)$ where $\mathcal{N}(0,t)$ is a draw from a normal distribution with mean zero and variance t.

The bond is considered to be risk-free and non-volatile

$$dB \;=\; rB\; dt \;. \tag{2.5}$$

In the absence of limited liability, the value of a leveraged ETF $V^\ell(t)$, which is continuously rebalanced to a weight of β in the stock index and $(1-\beta)$ in the bond index then follows the process (we assume $\beta > 1$)

$$\begin{aligned} \frac{dV^\ell}{V^\ell} &= \beta\left(\frac{dS}{S}\right) + (1-\beta)\left(\frac{dB}{B}\right) - c_\ell\; dt \\ &= \Big((1-\beta)r + \beta(\mu - \lambda\kappa) - c_\ell\Big)\; dt + \beta\sigma\; dZ + \beta(\xi - 1)\; d\mathbb{Q} \;, \end{aligned} \tag{2.6}$$

where c_ℓ is the leveraged ETF (denoted by LETF) expense ratio. Since we assume continuous rebalancing of the LETF, V^ℓ can only become negative due to jumps. Hence, to incorporate limited liability, i.e. the value of the LETF cannot become negative (Ahn et al., 2015; Van Staden et al., 2026), we can rewrite equations (2.6) to reflect this condition

$$\frac{dV^\ell}{V^\ell} \;=\; \Big((1-\beta)r + \beta(\mu - \lambda\kappa) - c_\ell\Big)\; dt + \beta\sigma\; dZ + \max(\beta(\xi - 1), -1)\; d\mathbb{Q} \;, \tag{2.7}$$

which implies that

$$\begin{aligned} \frac{V^\ell(t)}{V^\ell(0)} &= e^{-c_\ell t} e^{\big((1-\beta)r + \beta(\mu-\lambda\kappa) - \beta^2\sigma^2/2\big)t} e^{\beta\sigma(Z(t)-Z(0)) + \sum_{i=0}^{\mathbb{N}(t)} \log(\max(1+\beta(\xi_i - 1), 0))} \\ &= e^{-c_\ell t} e^{\big(1-\beta)r - \beta^2\sigma^2 2\big)t} \left(e^{(\mu-\lambda\kappa)t + \sigma(Z(t)-Z(0))}\right)^\beta e^{\sum_{i=0}^{\mathbb{N}(t)} \log(\max(1+\beta(\xi_i-1),0))} \;. \end{aligned} \tag{2.8}$$

We note that, in practice, LETFs are often replicated using index swaps (Guasoni and Mayerhofer, 2023). This is, of course, economically equivalent to equation (2.6).

Rewrite equation (2.4) as

$$\left(\frac{S(t)}{S(0)}\right)^\beta e^{\beta\sigma^2 t/2 - \beta\sum_{i=0}^{\mathbb{N}(t)} \log \xi_i} \;=\; \left(e^{(\mu-\lambda\kappa)t + \sigma(Z(t)-Z(0))}\right)^\beta \;. \tag{2.9}$$

Substitute equation (2.9) into equation (2.8) to obtain

$$\frac{V^\ell(t)}{V^\ell(0)} \;=\; e^{\{((1-\beta)r + \beta(1-\beta)\sigma^2/2 - c_\ell)t\}} \left(\frac{S(t)}{S(0)}\right)^\beta H(\beta,t) \;, \tag{2.10}$$

where

$$H(\beta,t) \;=\; \Pi_{i=0}^{\mathbb{N}(t)} \left(\frac{\max(1+\beta(\xi_i - 1), 0)}{\xi_i^\beta}\right) \;. \tag{2.11}$$

Now consider a portfolio containing an initial allocation of α^ℓ to the LETF and $(1-\alpha^\ell)$ to the risk free bond, with $0 \le \alpha^\ell \le 1$. Given an initial wealth $W(0)$, the total portfolio value at time t, denoted by $P^\ell(t)$, is then

$$\begin{aligned} \frac{P^\ell(t)}{W(0)} &= (1-\alpha^\ell)\left(\frac{B(t)}{B(0)}\right) + \alpha^\ell\left(\frac{V^\ell(t)}{V^\ell(0)}\right) \\ &= (1-\alpha^l)e^{rt} + \alpha^l e^{\left\{\left((1-\beta)r+\beta(1-\beta)\sigma^2/2-c_\ell\right)t\right\}}\left(\frac{S(t)}{S(0)}\right)^\beta H(\beta,t) \ . \end{aligned} \tag{2.12}$$

The value of a Vanilla ETF V^v is simply (with the same initial wealth $W(0)$)

$$\frac{V^v(t)}{W(0)} = e^{-c_v t}\left(\frac{S(t)}{S(0)}\right) \ , \tag{2.13}$$

where c_v is the expense ratio of the vanilla ETF (denoted by VETF). Consider a portfolio containing an initial allocation of α^v to the VETF and $(1-\alpha^v)$ to the risk free bond, with $0 \le \alpha^v \le 1$. The total portfolio value at time t is then

$$\frac{P^v(t)}{W(0)} = (1-\alpha^v)e^{rt} + \alpha^v e^{-c_v t}\left(\frac{S(t)}{S(0)}\right) \ , \tag{2.14}$$

assuming initial wealth $W(0)$. In the following, we will refer to $P^v(t)$ as the *benchmark* portfolio.

Remark 2.1 (Properties of equation (2.12)). *It is useful to note the following properties of equation (2.12)*

- $e^{\left\{\left((1-\beta)r+\beta(1-\beta)\sigma^2/2-c_\ell\right)t\right\}} < 1$ *; if* $\beta > 1, t > 0$,
- $H(\beta,t) < 1$ *; if* $\beta > 1, t > 0$, *(Van Staden et al., 2026),*

which implies that volatility, jumps and expenses act as a drag on the LETF. However, the power law term $(S(t)/S(0))^\beta$ *counteracts this drag if* $(S(t)/S(0)) > 1$ *(*$\beta > 1$*).*

2.1 GBM case

In order to gain some intuition, we first consider a case of geometric Brownian motion (GBM) dynamics. This can be formally obtained from the results in Section 2 by setting the jump intensity $\lambda = 0$.

More precisely, we obtain

$$\frac{P^\ell(t)}{W(0)} = (1-\alpha^l)e^{rt} + \alpha^l e^{\left\{\left((1-\beta)r+\beta(1-\beta)\sigma^2/2-c_\ell\right)t\right\}}\left(\frac{S(t)}{S(0)}\right)^\beta \ , \tag{2.15}$$

$$\frac{P^v(t)}{W(0)} = (1-\alpha^v)e^{rt} + \alpha^v e^{-c_v t}\left(\frac{S(t)}{S(0)}\right) \ , \tag{2.16}$$

$$\frac{S(t)}{S(0)} = e^{(\mu-\sigma^2/2)t}e^{\sigma(Z(t)-Z(0))} \ . \tag{2.17}$$

We use data from the Center for Research in Security Prices (CRSP) on a monthly basis over the 1926:1-2023:12 period.[3] Our base case tests use the CRSP US 30 day T-bill for the bond asset

[3] More specifically, results presented here were calculated based on data from Historical Indexes, ©2023 Center for Research in Security Prices (CRSP), The University of Chicago Booth School of Business. Wharton Research Data Services (WRDS) was used in preparing this article. This service and the data available thereon constitute valuable intellectual property and trade secrets of WRDS and/or its third-party suppliers.

and the CRSP value-weighted total return index for the stock asset. This latter index includes all distributions for all domestic stocks trading on major U.S. exchanges. All of these various indexes are in nominal terms, so we adjust them for inflation by using the U.S. CPI index, also supplied by CRSP. We use real indexes since investors should be focused on real (not nominal) wealth goals. The parameters in Table 2.1 are obtained using maximum likelihood.

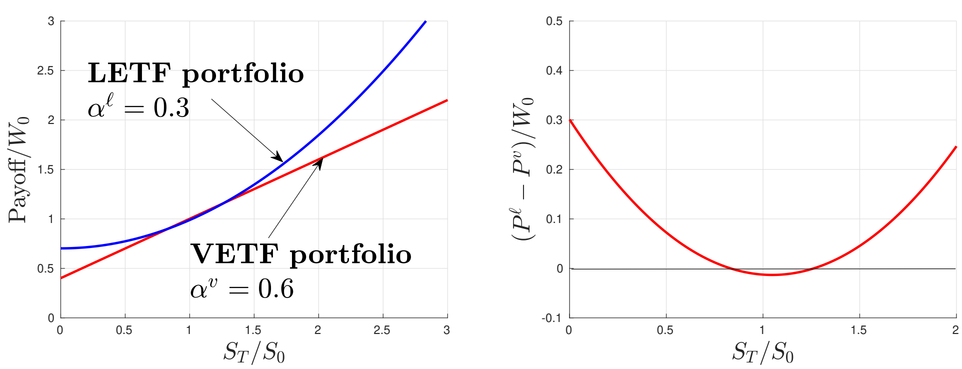
μ	0.0818
σ	.1849
LETF leverage	$\beta = 2$
30 day T-bill return r	0.0032
LETF expense ratio c_ℓ	.0089
VETF expense ratio c_v	0.0

TABLE 2.1: *Data for GBM example. Annualized parameters fit to CRSP monthly data, 1926:1-2023:12, inflation adjusted. The LETF expense ratio corresponds to that of the ProShares Ultra S&P 500 LETF with multiplier $\beta = 2$ (etfdb.com/etf/SSO, accessed 15 May 2025).*

We will assume that the leverage factor for the LETF is $\beta = 2$. After an initial allocation to the risk free bond and the LETF/VETF, there is no further rebalancing, with an initial investment horizon of $T = 1.0$ years.

Figure 2.1 shows the payoff diagrams from equations (2.15-2.17), assuming that the initial allocation fraction to the LETF for P^ℓ is $\alpha^\ell = 0.30$, compared the $\alpha^v = 0.60$, for P^v (the portfolio which uses a vanilla stock ETF). Since the leverage ratio for the LETF is $\beta = 2.0$, this allocation to the LETF in P^ℓ results in the same initial exposure to stock price changes as for the Vanilla ETF portfolio P^v.

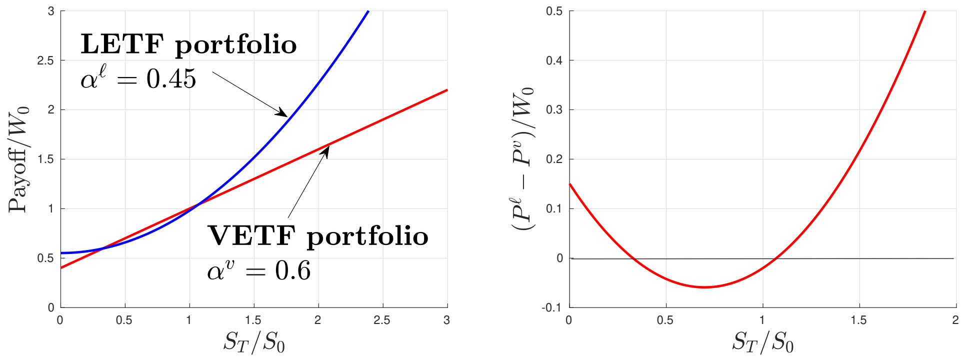
Figure 2.1 clearly shows the nonlinear payoff effect of using an LETF compared with using a VETF. Figure 2.1(b) focuses on the difference $(P^\ell - P^v)$. We can see that use of an LETF is a drag on performance near $S_T/S_0 \simeq 1$, but boosts performance for either large or small stock returns evaluated over the entire year. The enhanced payoff of the portfolio containing the LETF if $(S_T/S_0) < 1$ is due to the larger allocation to bonds, compared to the portfolio with the VETF.



(a) $P^\ell/W(0), P^v/W(0)$ (b) $(P^\ell - P^v)/W(0)$ *from Figure 2.1(a)*

FIGURE 2.1: *Payoff diagrams comparing $P^v/W(0), P^\ell/W(0)$. $T = 1.0$ years, from equations (2.15-2.17), GBM case. LETF leverage factor $\beta = 2.0$. Stock data fit of equation (2.17) to inflation adjusted CRSP index, 1926:1-2023:12. Data in Table 2.1. $\alpha^\ell = 0.3, \alpha^v = 0.6$.*

Figure 2.2 shows comparable results for $\alpha^\ell = 0.45$ with $\alpha^v = 0.60$. From Figure 2.2(b) we can see the underperformance of P^ℓ (the LETF portfolio) relative to the VETF portfolio P^v is now larger (compared to $\alpha^\ell = 0.30$), but has shifted to the zone $S_T/S_0 < 1$. Note as well that for $S_T/S_0 > 1$, the VETF portfolio has a very rapid increase in outperformance.



(a) $P^\ell/W(0), P^v/W(0)$ (b) $(P^\ell - P^v)/W(0)$ *from Figure 2.2(a)*

FIGURE 2.2: *Payoff diagrams comparing* $P^v/W(0), P^\ell/W(0)$. $T = 1.0$ *years, from equations (2.15-2.17), GBM case. LETF leverage factor* $\beta = 2.0$. *Stock data fit of equation (2.17) to inflation adjusted CRSP index, 1926:1-2023:12. Interest rate* r *from T-bills, inflation adjusted, 1926:1-2023:12. Data in Table 2.1.* $\alpha^\ell = 0.45, \alpha^v = 0.6$.

We can see from Figures 2.1(b) and 2.2(b), that the area of underperformance for $\alpha^\ell = 0.45$ is larger compared to $\alpha^\ell = 0.30$. However, at this point, we cannot say anything more about the performance of these choices for α^ℓ unless we know the probability distribution of S_T/S_0.

2.2 Deficiencies of Sharpe Ratio Criteria

A standard approach for evaluating investment strategies is based on comparing Sharpe ratios. However, Sharpe ratios are inappropriate for portfolios containing LETFs.

Recall that the SDEs for the LETF (V^ℓ) and the VETF (V^v) are (GBM case)

$$\begin{aligned} \frac{dV^\ell}{V^\ell} &= \Big((\mu - r)\beta + r - c_\ell \Big) \, dt + \beta\sigma \, dZ \\ \frac{dV^v}{V^v} &= \Big(\mu - c_v \Big) \, dt + \sigma \, dZ \; . \end{aligned} \tag{2.18}$$

Given the existence of a risk free bond, equation (2.5), then, the continuously compounded Sharpe ratio $\mathbb{S}$ for an ETF $V(T)$ over an investment horizon $[0,T]$ is (Goetzmann et al., 2002; Bernard and Vanduffel, 2014)

$$\mathbb{S} = \frac{E[V(T)] - V(0)e^{rT}}{std[V(T)]}, \tag{2.19}$$

where $E[\cdot]$ and $std[\cdot]$ respectively denote expectation and standard deviation. From the properties of GBM (2.17), we have

$$\mathbb{S}^\ell = \frac{1 - e^{-((\mu - r)\beta - c_\ell)T}}{\big(e^{(\beta\sigma)^2 T} - 1\big)^{1/2}} \quad ; \quad \mathbb{S}^v = \frac{1 - e^{-(\mu - r - c_v)T}}{\big(e^{(\sigma)^2 T} - 1\big)^{1/2}} \; . \tag{2.20}$$

Taking the limit as $T \to 0$ in equation (2.20),

$$\lim_{T\to 0} \frac{\mathbb{S}^\ell}{\sqrt{T}} = \frac{\left(\mu - r - \frac{c_\ell}{\beta}\right)}{\sigma} \quad ; \quad \lim_{T\to 0} \frac{\mathbb{S}^v}{\sqrt{T}} = \left(\frac{\mu - r - c_v}{\sigma}\right) , \tag{2.21}$$

and, if we set the fees $c_\ell = c_v = 0$, then the limits in equation (2.21) are identical. In the following, we refer to $\mathbb{S}^\ell/\sqrt{T}$, $\mathbb{S}^v/\sqrt{T}$ as the annualized Sharpe ratios. Using the data from Table 2.1, but setting $c_\ell = c_v = 0$, Table 2.2 shows the numerical values for the annualized Sharpe ratios for V^ℓ and V^v. The Sharpe ratios for both ETFs are identical in the limit $T \to 0$, and very close for $T = 1.0$ years.

Based on Sharpe ratios alone, it would appear that a portfolio containing an LETF and risk free bonds should have performance comparable to a portfolio with a VETF and bonds (with a suitable allocation in bonds to account for the LETF leverage). However, Figures 2.1 and 2.2 show that this is clearly not the case.

This demonstrates that Sharpe ratios are not really useful in analyzing the performance of a portfolio containing an LETF compared with the use of a VETF. The deficiency of the Sharpe ratio in this case can be attributed to the nonlinear option-like payoff of the LETF, which is not captured by the Sharpe ratio (Dybvig and Ingersoll, 1982; Lhabitant, 2000; Goetzmann et al., 2002). In the following, we will examine detailed statistics for the entire distribution of a pathwise comparison performance measure, in order to remedy the deficiencies of Sharpe ratios.

	$T \to 0$	$T = 1.0$ years
LETF $\frac{\mathbb{S}^\ell}{\sqrt{T}}$	0.425	0.380
VETF $\frac{\mathbb{S}^v}{\sqrt{T}}$	0.425	0.405

TABLE 2.2: *Compound annualized Sharpe ratios $\mathbb{S}^\ell/\sqrt{T}, \mathbb{S}^v/\sqrt{T}$, equation (2.20). GBM example. Parameters in Table 2.1. Fees set to zero, $c_\ell = c_v = 0$.*

2.3 Pathwise Performance Comparison and the Omega Ratio

If an investor is considering replacing a VETF by an LETF in a portfolio, she is undoubtedly concerned with pathwise performance comparison. For example, along each stochastic path, what is the probability that the LETF portfolio will outperform the VETF portfolio? This is particularly relevant in the institutional context. For example, pension plan and endowment investment management is often evaluated by comparing to a constant weight benchmark based on index ETFs (Canadian Pension Plan, 2021; Norges Bank, 2021; Ennis, 2021). In our case, the benchmark portfolio is P^v.

Consequently, we will examine the distribution of $R(T) = P^\ell(T)/P^v(T)$. This will give us information about the performance of the LETF portfolio compared to the benchmark VETF portfolio, along each realized path. If $R(T) > 1$, this indicates that the LETF portfolio outperforms the VETF portfolio. In order to generate statistics for $R(T) = P^\ell(T)/P^v(T)$ we will use Monte Carlo simulation.

Let $F(R_T)$ be the CDF of R_T. The Omega ratio (Keating and Shadwick, 2002) at level L is defined as

$$\Omega(L) = \frac{\int_L^\infty (1 - F(R_T))\ dR_T}{\int_{-\infty}^L F(R(T)\ dR_T} = \frac{E[\max(R_T - L, 0)]}{E[\max(L - R_T, 0)]}$$
$$R_T = \frac{P_T^\ell}{P_T^v}\ . \tag{2.22}$$

The Omega ratio is a measure of upside versus downside, with respect to the level L. A well known problem with the Omega ratio is the choice of the level L in equation (2.22). However, in the case of an Omega ratio based on R_T, the choice of L is clear.

Remark 2.2 (Choice of level L for the Omega ratio (2.22))**.** *We are interested in pathwise outperformance of the LETF portfolio compared to the VETF portfolio. Since outperformance is indicated if $R_T > 1$, we will examine $\Omega(L)$, with $L = 1$. This is a direct measure of outperformance of the LETF portfolio, compared to the VETF portfolio.*

2.4 Pathwise Statistics

Table 2.3 shows the summary statistics for $R_T = P^\ell(T)/P^v(T)$ for these simulations. ES(5%) is the expected shortfall at the five per cent level, i.e. the mean of the worst 5% of the outcomes.

Table 2.3 shows that the LETF $\alpha^\ell = .45$ portfolio has $E[R_T] > 1$ and $Median[R_T] \simeq 1$, in contrast to the underperformance of the mean and median (relative to the VETF portfolio) for the $\alpha^\ell = .30$ portfolio. Of course, this comes at a cost, since the expected shortfall $ES(5\%)$ and the $5'th$ percentiles are worse for $\alpha^\ell = .45$ compared to $\alpha^v = .30$.

However, $\Omega(1) = 1.82$ for $\alpha^\ell = 0.45$ compared to $\alpha^\ell = 0.30$, which has $\Omega(1) = 0.71$. More intuitively, $\alpha^\ell = .45$ has a significant upside compared to the downside (in terms of outperformance). In contrast, $\alpha^\ell = 0.30$ has more downside compared to upside.

	E[R_T]	Median[R_T]	$R_T : 5^{th}$ percentile	$R_T : 95^{th}$ percentile	ES(5%)	$\Omega(1)$	$Prob[R_T > 1]$
$\alpha^\ell = 0.3$	0.9981 (.0001)	.9919	.9871	1.0294	.9871	0.7130	.2739
$\alpha^\ell = 0.45$	1.0145 (.0004)	.9995	.9367	1.1426	.9315	1.8247	.4978

TABLE 2.3: *Statistics for $R(T) = P^\ell(T)/P^v(T)$. $\alpha^v = 0.6$. $T = 1.0$ years, from equations (2.15-2.17), GBM case. LETF leverage factor $\beta = 2.0$. Numbers in brackets are the standard error estimate at the 95% confidence level. Stock data fit of equation (2.17) to inflation adjusted CRSP index, 1926:1-2023:12. Interest rate r from T-bills, inflation adjusted, 1926:1-2023:12. Data in Table 2.1. 10^5 MC simulations. ES(5%) is the mean of the worst 5% of the outcomes. $\Omega(1)$ defined in equation (2.22).*

Table 2.4 examines a more long term strategy. We consider an investment horizon of ten years. Initially, and annually thereafter, the VETF and LETF strategies are rebalanced with α^v and α^ℓ fractions in the stock ETF. For the $\alpha^\ell = .30$ case, E[R_T] and Median[R_T] are less than one, with $\Omega(1) \simeq 0.40$. For this value of α^ℓ, the LETF portfolio has does not seem to be worthwhile: there is a consistent drag on performance relative to the VETF portfolio, and the Omega ratio indicates pronounced downside. Note that the Omega ratio is reduced compared to the one year case (Table 2.3).

In contrast, for the LETF portfolio (ten year case, Table 2.4) with $\alpha^\ell = .45$, both E[R_T] Median[R_T] are greater than one (indicating outperformance) and with $\Omega(1) \simeq 6.4$, indicating significant upside. Note that Omega for the ten year strategy is much larger than for the one year case, indicating that the Omega ratio is compounded by repeated rebalancing. As well, the 95^{th} percentile for $\alpha^\ell = .45$ indicates that the final distribution of R_T has a large right skew. Of course, there is no free lunch here, the 5th percentile and ES(5%) is worse for $\alpha^\ell = .45$ compared to $\alpha^\ell = 0.30$.

	E[R_T]	Median[R_T]	$R_T : 5^{th}$ percentile	$R_T : 95^{th}$ percentile	ES(5%)	$\Omega(1)$	$Prob[R_T > 1]$
$\alpha^\ell = 0.3$	.9808 (.0003)	.9716	.9158	1.0766	.9077	0.39891	.2910
$\alpha^\ell = 0.45$	1.152 (.001)	1.1142	.8324	1.5992	.7822	6.3933	.7136

TABLE 2.4: *Statistics for $R(T) = P^\ell(T)/P^v(T)$. $\alpha^v = 0.6$. $T = 10.0$ years, rebalanced annually, from equations (2.15-2.17), GBM case. LETF leverage factor $\beta = 2.0$. Numbers in brackets are the standard error estimate at the 95% confidence level. Stock data fit of equation (2.17) to inflation adjusted CRSP index, 1926:1-2023:12. Interest rate r from T-bills, inflation adjusted, 1926:1-2023:12. Data in Table 2.1. 10^5 MC simulations. ES(5%) is the mean of the worst 5% of the outcomes. $\Omega(1)$ defined in equation (2.22).*

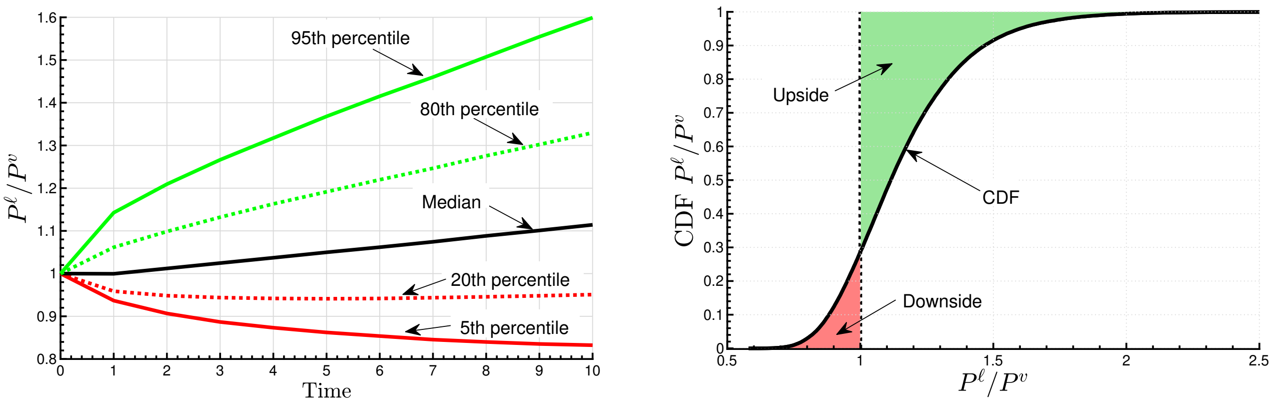
As discussed previously, portfolios which use LETFs are fundamentally nonlinear. Hence traditional performance metrics are inappropriate (Goetzmann et al., 2002). To this end, we consider the percentiles of our pathwise comparison performance metric over time. This ensures that we have a good measure of downside risk and upside gain, along each stochastic path. Further insight regarding the behaviour of the ten year rebalanced LETF strategy with $\alpha^\ell = 0.45$ can be seen in Figure 2.3. The median, 80th and 95th percentiles of P^ℓ/P^v are monotonically increasing over time, and greater than one. The 20th percentile of $R_t = P^\ell/P^v$ stabilizes at about 0.95, while the 5th percentile is $\simeq 0.83$ at the ten year mark. We remind the reader that these are pathwise measures, hence this may be an acceptable level of risk for enhanced upside. Note that the LETF portfolio is rebalanced to 0.55 in bonds, while the benchmark strategy has only 0.40 in bonds at each rebalancing date. Figure 2.3(b) shows the CDF of $R_T = P^\ell(T)/P^v(T)$, which illustrates the favourable Omega ratio with the $\alpha^\ell = 0.45$ strategy.

3 Jump Diffusion

We will now assume that the underlying stock index follows a jump diffusion process (2.1), with double exponential jump size distribution (2.2). The final expression for the value of the LETF portfolio is given in equation (2.12), and the final value of the VETF portfolio is given in equation (2.14). Note that the payoff of the LETF portfolio, as a function of the return of the underlying stock index $(S(T)/S(0))$ is no longer deterministic, in contrast to the GBM case. The SDE parameters are fit to the CRSP data 1926:1-2023:12, see Appendix B.

Table 3.1 shows the results with $\alpha^\ell = 0.45$ using various rebalancing frequencies, along with the case of $\alpha^\ell = 0.30$ (rebalancing annually). In all cases, the VETF portfolio is rebalanced to a weight of $\alpha^v = 0.60$ in stocks.

As for the GBM case, the result using $\alpha^\ell = 0.30$ is unimpressive. In particular, the mean and median of $R_T = P^\ell(T)/P^v(T)$ are less than one, and $\Omega(1) = 0.47$ indicating more downside than upside.



(a) *Percentiles P^ℓ/P^v.* (b) *CDF P^ℓ/P^v. Outperformance if $P^\ell/P^v > 1$.*

FIGURE 2.3: *Comparing P^ℓ/P^v. $\alpha^\ell = 0.45, \alpha^v = 0.60$. 10^5 MC simulations. $T = 10.0$ years, from equations (2.15-2.17), GBM case. Annual rebalancing. LETF leverage factor $\beta = 2.0$. Stock data fit of equation (2.17) to inflation adjusted CRSP index, 1926:1-2023:12. Interest rate r from T-bills, inflation adjusted, 1926:1-2023:12. Data in Table 2.1. $T = 1$ yr. $\alpha^\ell = 0.45, \alpha^v = 0.6$. $\Omega(1)$ (equation (2.22)) is the ratio of the upside area to the downside area.*

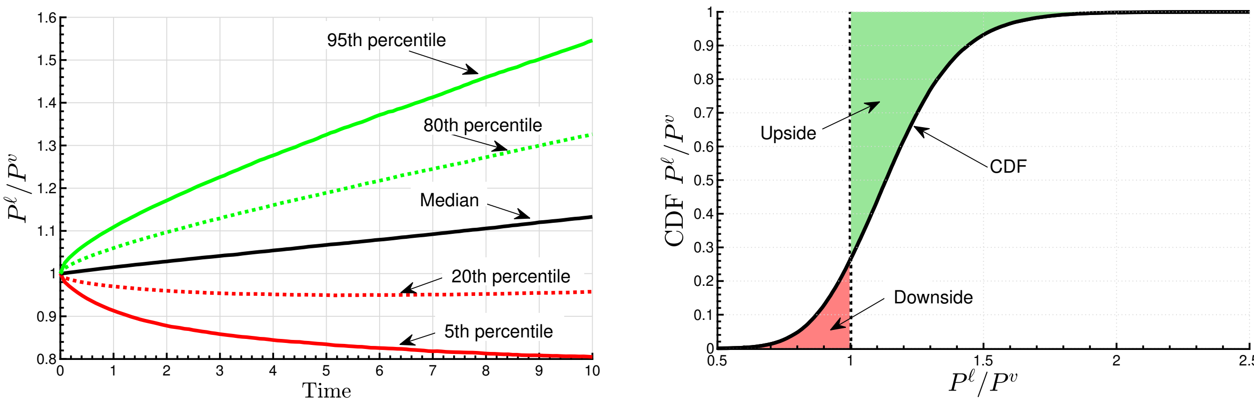
In contrast, the results for $\alpha^\ell = 0.45$ are more interesting. For all rebalancing frequencies, the mean and median of R_T are larger than one, and $\Omega(1) > 5.0$. Of course, this comes at the cost of more left tail risk, as seen in the ES(5%) and the 5th percentile. As the rebalancing interval decreases, the 95th percentile of R_T decreases slightly, balanced by an increase in the 5th percentile. $\Omega(1)$ also increases as the rebalancing interval decreases.

Remark 3.1 (Effect of rebalancing frequency.)**.** *Table 3.1 shows that for a holding period of ten years, the effect of rebalancing more frequently than annually is quite small.*

Figure 3.1 gives more details for the simulation with $\alpha^\ell = 0.45$ and monthly rebalancing. The median, 80th and 95th percentiles of $R(t) = P^\ell(t)/P^v(t)$ increase monotonically from one, and the 20th percentile stabilizes to end up with a value of $[R_T]_{20th} = 0.96$. The risk does show up at the 5th percentile (0.8 at ten years). However, this level of risk may be quite acceptable, given the Omega(1) ratio, as seen in Figure 3.1(b).

Rebalancing interval	E[R_T]	Median[R_T]	$R_T : 5^{th}$ percentile	$R_T : 95^{th}$ percentile	ES(5%)	$\Omega(1)$	$Prob[R_T > 1]$
			$\alpha^\ell = 0.30$				
Yearly	0.9831 (.0004)	0.9769	0.9026	1.0838	0.8671	0.4690	0.3139
			$\alpha^\ell = 0.45$				
Yearly	1.1561 (.002)	1.1228	0.7712	1.6502	0.6952	5.0141	0.7012
Quarterly	1.1504 (.001)	1.1313	0.7983	1.5646	0.7239	5.6502	0.7299
Monthly	1.1494 (.001)	1.1330	0.8033	1.5487	0.7308	5.8412	0.7367

TABLE 3.1: *Statistics for $R(T) = P^\ell(T)/P^v(T)$, 10^5 MC simulations. $\alpha^v = 0.6$. $T = 10.0$ years, rebalancing intervals shown. Jump diffusion case, see equations (2.12) and (2.14). LETF leverage factor $\beta = 2.0$. Numbers in brackets are the standard error estimate at the 95% confidence level. SDE parameters fit to inflation adjusted CRSP index, 1926:1-2023:12. See Appendix B for the parameters. ES(5%) is the mean of the worst 5% of the outcomes.*



(a) *Percentiles* P^ℓ/P^v. *20th percentile at* $t = 10$ *is* 0.96. *80th percentile (t=10):* 1.33.

(b) CDF P^ℓ/P^v. *Outperformance if* $P^\ell/P^v > 1$.

FIGURE 3.1: *Comparing* P^ℓ/P^v. $\alpha^\ell = 0.45, \alpha^v = 0.60$. 10^5 *MC simulations.* $T = 10.0$ *years. Jump diffusion case, see equations (2.12) and (2.14). Monthly rebalancing. LETF leverage factor* $\beta = 2.0$. *SDE parameters fit to inflation adjusted CRSP index, 1926:1-2023:12. See Appendix B for the parameters.* $\alpha^\ell = 0.45, \alpha^v = 0.6$. $\Omega(1)$ *(equation (2.22)) is the ratio of the upside area to the downside area.*

4 Summary of Fixed Weight Simulations

The overall trends observed for the results obtained by rebalancing to fixed weights are consistent for both GBM and jump diffusion models.

At first sight, a reasonable strategy might appear to choose the LETF weight as $\alpha^\ell = 0.3$, with $\alpha^v = 0.6$. This would give both portfolios similar exposure to the stock index,[4] with the borrowing cost of the LETF offset by the larger bond component of P^ℓ. The idea here would be to gain exposure to the nonlinear left and right tails of the payoff (see Figure 2.1). However, for reasonable market parameters, this advantage is outweighed by the LETF fee and the volatility/jump drag, see Remark 2.1.

Increasing the exposure of the LETF with a weight $\alpha^\ell = 0.45$, with $\alpha^v = 0.60$, results in a much more interesting strategy. In this case, there is a gain in median, 80th and 95th percentile's of $R_T = P^\ell(T)/P^v(T)$, which is offset by an increased left tail risk. However, some investors may find this tradeoff appealing.

The key feature which contrasts the $\alpha^\ell = 0.3$ with the case $\alpha^\ell = 0.45$ appears to be the effect the Omega ratio. For one year investment horizons, the Omega ratio for $\alpha^\ell = 0.3$ is less than one, while the Omega ratio for $\alpha^\ell = 0.45$ is larger than one. This effect seems to compound for more rebalancing times over longer intervals, in the sense that the Omega ratio for $\alpha^\ell = 0.3$ decreases for longer time horizons, while the Omega ratio for $\alpha^\ell = 0.45$ increases for longer time horizons.

Of course, we have used a very simple strategy, involving constant weight rebalancing. Different choices of α^ℓ will result in different tradeoffs between risk and reward. However, these results indicate that LETFs can be advantageous even with fairly naive strategies.

Clearly, dynamic allocation of the amount invested in the LETF should result in even better investment policies. This requires use of more sophisticated algorithms for asset allocation. In addition, use of parametric SDE models of asset price dynamics is prone to misspecification.

In Section 6, we will explore the use of a Neural Network asset allocation strategy, coupled with a data-driven approach to simulating portfolios containing LETFs and VETFs.

[4] Recall that $\beta = 2.0$.

5 Choice of Objective Function for the Dynamic Strategy

Recall our definition of the terminal wealth ratio $R_T = P^\ell(T)/P^v(T)$, which can be regarded as a reward, e.g., $R_T > 1$ implies outperformance. The tracking error, or risk, can be measured by $(R_T - 1)^2$. We now allow a dynamic strategy with $\alpha^\ell = \alpha^\ell(t, P^\ell, P^v)$. In other words, α^ℓ is a function of time and the feature variables (P^ℓ, P^v). To avoid notational clutter, in the following, we will occasionally refer to the control as $\alpha^\ell(t)$, with the dependence on (P^ℓ, P^v) understood.

Consider the following optimal control problem: find α^ℓ which solves

$$\sup_{\alpha^\ell} E\left[R_T - \lambda(R_T - 1)^2\right] \ , \tag{5.1}$$

where $E[\cdot]$ is the expectation operator, and $\lambda > 0$ is a scalarization parameter (which can also be interpreted as a risk aversion). The terminal wealth $P^\ell(T)$ is generated by the control α^ℓ, the fraction of wealth in the LETF. Equation (5.1) expresses the tradeoff between reward (i.e. portfolio outperformance) and risk.

Varying λ in equation (5.1) traces out an efficient frontier in the $(E[(R_T - 1)^2], E[R_T])$ plane. If we examine this curve for a fixed value of $E[(R_T - 1)^2])$, then the control $\alpha^\ell(t)$ corresponding to this value of λ generates the largest possible value of $E[R_T]$, with this fixed value of tracking error $E[(R_T - 1)^2]$. In other words, we are maximizing outperformance $E[R_T]$ for a given risk budget, which is specified by the permissible tracking error $E[(R_T - 1)^2]$.

It is not obvious how to specify an appropriate value of λ in (5.1), without constructing the entire efficient frontier. To motivate an appealing choice for λ, we can rewrite the weighted risk and reward as below:

$$\begin{aligned} \sup_{\alpha^\ell} E\left[R_T - \lambda(R_T - 1)^2\right] &= \inf_{\alpha^\ell} E\left[\left(R_T - \left(1 + \frac{1}{2\lambda}\right)\right)^2\right] \\ &= \inf_{\alpha^\ell} E\left[\left(e^{\delta T} - R_T\right)^2\right], \qquad e^{\delta T} = 1 + \frac{1}{2\lambda} \ . \end{aligned} \tag{5.2}$$

The parameter δ now has the convenient interpretation of an annualized outperformance target.

A possible objection to the weighted risk and reward (5.2) is that this is independent of the actual size (value) of the benchmark P^v. It appears to be particularly egregious to investors when the benchmark P^v does well, and the active portfolio P^ℓ does poorly.[5]

To overweight periods of strong performance for the benchmark portfolio, we rescale the argument of $E[\cdot]$ in (5.2) by multiplying by $(P^v(T))^2$ and consider instead

$$(P^v(T))^2 \left(e^{\delta T} - R_T\right)^2 = \left[\left(P^v(T)e^{\delta T} - P^\ell(T)\right)^2\right] \ .$$

This gives us a new control problem

$$\inf_{\alpha^\ell} E\left[\left(P^v(T)e^{\delta T} - P^\ell(T)\right)^2\right] \ . \tag{5.3}$$

However, as noted in Forsyth et al. (2023); Van Staden et al. (2024a), a problem with objective function (5.3) is that the risk and reward are only measured at the end of the investment horizon T.

[5]See Financial Times, February 5, 2026, Mary McDougall, *"Has USS's investment strategy worked?"*

In order to provide control over the risk of underperformance during the entire investment horizon, we change problem (5.3) to

$$\inf_{\alpha^\ell} E\left[\int_0^T \left(P^v(t)e^{\delta t} - P^\ell(t)\right)^2 \; dt\right] \; . \tag{5.4}$$

Objective function (5.4) is referred to as the cumulative tracking difference (CD) in Van Staden et al. (2024a).

6 Data-driven Machine Learning Approach

Instead of using fixed portfolio allocations to the LETF as in Section 4, we now use a data-driven neural network approach to determine the optimal *dynamic* allocation of wealth to the LETF at each rebalancing time. The proportion to invest in the LETF is given by a neural network, and we assume that there is no short-selling or leveraging of the investment in the LETF.

The neural network is trained on nearly a century of bootstrapped historical data, spanning 1926:01 to 2023:12, where LETF and VETF returns are synthetically constructed. Learning the optimal investment strategy using such a long data period ensures that periods of exceptional market volatility and high inflation are also included, rather than focusing on the most recent decades since the inception of LETFs. See Appendix A for more information on the construction of proxy returns time series for the LETF and VETF. We remind the reader that all returns are inflation adjusted.

We emphasize the use of stationary block bootstrap resampling rather than fixed block bootstrap resampling to generate training/testing data for the neural network. In summary, the stationary block bootstrap method (Politis and Romano (1994)) randomly varies block lengths according to a geometric distribution, which better preserves the temporal dependence structure of financial returns while introducing greater variability in the resampled sequences. This approach addresses two key limitations of fixed block bootstrap; first, it reduces the repetitiveness that can arise from using uniform block lengths, and second, it maintains adequate variance in the resampled return series by avoiding overly rigid partitioning of the original time series. Furthermore, the stationary block bootstrap method is popular both in in academic settings (Anarkulova et al. (2022)) and practical applications (Cogneau and Zakalmouline (2013); Dichtl et al. (2016); Scott and Cavaglia (2017); Cavaglia et al. (2022); Simonian and Martirosyan (2022)).

We assume that both the LETF and VETF portfolios are rebalanced quarterly, i.e. at $N_{rb} = 40$ discrete rebalancing times during the investment time horizon $[t_0 = 0, T = 10 \text{ years}]$,

$$\mathcal{T} \;\; = \;\; \{t_n = n\Delta t |\, n = 0,...,39\}\,, \qquad \Delta t = 0.25. \tag{6.1}$$

In contrast to the previous sections, we now do not need to specify any parametric dynamics for the underlying assets. Instead, the neural network will simply learn the optimal investment in the LETF using bootstrapped historical returns, which incorporates all empirical return characteristics, including for example fat tails, volatility clustering, serial correlation, asymmetric return distributions, stochastic volatility, inflation regime changes, and other stylized facts of financial markets that are difficult to capture in parametric models. Suppose the bootstrapped returns over the time interval $[t_{n-1}, t_n]$ for the LETF, VETF and 30-day T-bill are given by $\mathcal{R}_\ell(t_n), \mathcal{R}_v(t_n)$ and $\mathcal{R}_B(t_n)$ respectively. Assuming zero contributions, the LETF portfolio P^ℓ and VETF portfolio P^v have the following dynamics respectively,[6]

$$P^{\ell}(t_{n+1}) = P^{\ell}(t_n) \cdot \left[\alpha^{\ell}(t_{n-1}) \cdot (1 + \mathcal{R}_{\ell}(t_n)) + \left(1 - \alpha^{\ell}(t_{n-1})\right) \cdot (1 + \mathcal{R}_B(t_n))\right], \quad (6.2)$$

$$P^{v}(t_{n+1}) = W(t_n) \cdot [\alpha^{v} \cdot (1 + \mathcal{R}_v(t_n)) + (1 - \alpha^{v}) \cdot (1 + \mathcal{R}_B(t_n))]. \quad (6.3)$$

We continue assuming a proportion in the VETF of $\alpha^v = 0.60$ as in previous sections, but now we wish to determine the dynamic proportion to invest in the LETF at each rebalancing time, $\alpha^{\ell}(t_n)$, $t_n \in \mathcal{T}$. What is more, we want to determine an *optimal* proportion in the LETF, $\alpha^{\ell*}(t_n)$, $t_n \in \mathcal{T}$, which requires the specification of the an investment objective.

For this purpose, we choose the cumulative tracking difference or CD objective (see Forsyth et al. (2023); Van Staden et al. (2024a)) as given in equation (5.4) aimed at targeting a favourable tracking difference of the LETF portfolio relative to the VETF portfolio over the investment time horizon. We can interpret P^v as a benchmark portfolio, consisting of bonds and an equity index, rebalanced to a constant weight $\alpha^v = 0.60$ at each rebalancing time. We remind the reader that use of a constant weight benchmark is commonplace for institutional investors (Canadian Pension Plan, 2021; Norges Bank, 2021; Ennis, 2021).

With annual outperformance target parameter δ, we consider a discretized form of the CD objective (5.4) over a time horizon of $T = 10$ years with quarterly rebalancing,

$$(CD(\delta)): \qquad \inf_{\alpha^{\ell}(t_n), t_n \in \mathcal{T}} \left[\sum_{n=0}^{39} \left(P^{\ell}(t_n) - e^{\delta \cdot t_n} \cdot P^{v}(t_n)\right)^2\right]. \quad (6.4)$$

How can (6.4) be solved without difficulty? We use the neural network approach of Li and Forsyth (2019); Van Staden et al. (2024b)), which can be classified as a "global-in-time" machine learning approach (see Hu and Laurière (2024)) to stochastic control problems like (6.4). Only a single optimization problem is solved which determines the parameters of a single neural network, with time and total portfolio values as feature variables. In addition, since the neural network uses an architecture that enforces the investment constraints of no short-selling and no leveraged positions automatically, the optimization in (6.4) is unconstrained. For more information and solution implementation details, see Li and Forsyth (2019); Van Staden et al. (2024b).

Once the problem (6.4) is solved, we can obtain the optimal dynamic allocation to the LETF $\alpha^{\ell*}(t_n)$ at any rebalancing time $t_n \in \mathcal{T}$, as the output of the neural network for inputs features time t_n, LETF portfolio value $P^{\ell}(t_n)$ and VETF portfolio value $P^{v}(t_n)$.

Considering two values of the annual outperformance target parameter δ in the CD objective (6.4), namely $\delta = 0.02$ and $\delta = 0.04$, we consider the summary statistics in Table 6.1 of the terminal ratio $R(T) = P^{\ell}(T)/P^{v}(T)$ obtained using the corresponding optimal LETF allocations $\alpha^{\ell*}$ and VETF allocation of $\alpha^v = 0.6$. Comparing Table 6.1 with the results of previous sections, for example Table 3.1, we observe a significant improvement in the Omega ratio $\Omega(1)$ and $Prob[R_T > 1]$ from using the optimal dynamic strategies $\alpha^{\ell*}$ rather than a fixed weight allocation α^{ℓ} to the LETF. However, we also note that there is no free lunch, in the sense that the significant improvement in performance as per the Omega ratio also accompanied by an increase in downside risk, as observed by a reduction in the mean of the worst 5% of outcomes (ES(5%)) of $R(T)$.

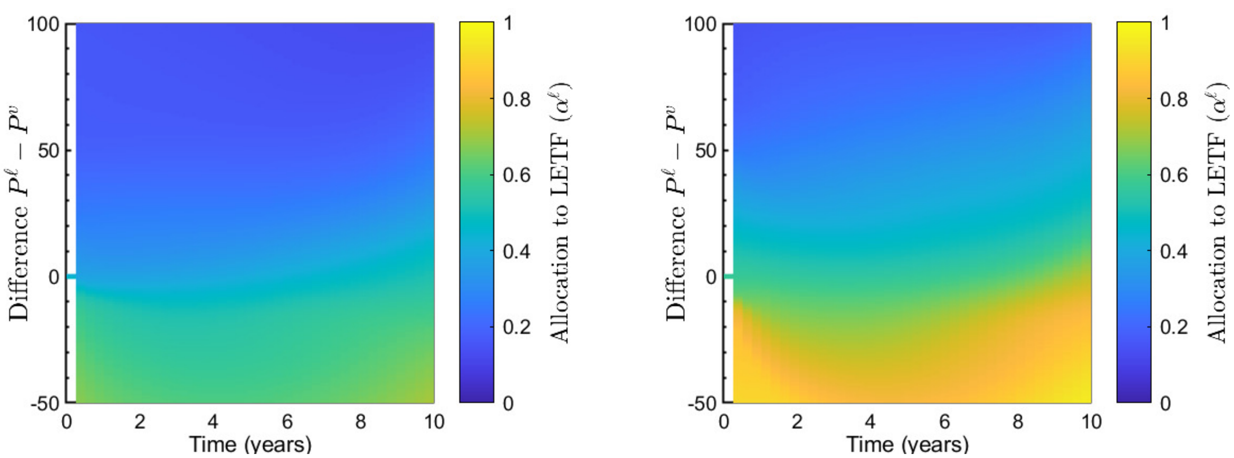
To compare the optimal dynamic strategies $\alpha^{\ell*}$ for $CD(\delta = 0.02)$ and $CD(\delta = 0.04)$ to the fixed allocations α^{ℓ} to the LETF considered in the preceding sections, Figure 6.1 illustrates $\alpha^{\ell*}(t_n)$ as a function of the difference $P^{\ell}(t_n) - P^{v}(t_n)$ and time t_n. As expected given the outperformance

[6]We remind the reader that our notation $\alpha^{\ell}(t)$ is understood to be short hand for $\alpha^{\ell}(t, P^{\ell}(t), P^{v}(t))$.

Rebalancing interval	$\mathrm{E}[R_T]$	Median$[R_T]$	$R_T: 5^{th}$ percentile	$R_T: 95^{th}$ percentile	ES(5%)	$\Omega(1)$	$Prob[R_T > 1]$
			Using optimal $\alpha^{\ell *}$ for $CD\,(\delta = 0.02)$				
Quarterly	1.1153 (<0.001)	1.1423	0.8857	1.2407	0.7134	7.8316	0.8956
			Using optimal $\alpha^{\ell *}$ for $CD\,(\delta = 0.04)$				
Quarterly	1.2419 (<0.001)	1.3073	0.6995	1.4576	0.4919	8.0831	0.8814

TABLE 6.1: *Statistics for $R(T) = P^{\ell}(T)/P^{v}(T)$, 5×10^5 bootstrapped historical data paths. Optimal LETF allocation $\alpha^{\ell *}$ determined using CD objective function with targets δ as shown, with VETF allocation of $\alpha^v = 0.6$. $T = 10.0$ years, quarterly rebalancing. LETF leverage factor $\beta = 2.0$. Numbers in brackets are the standard error estimate of the mean at the 95% confidence level. ES(5%) is the mean of the worst 5% of the outcomes.*

targets, the optimal allocation $\alpha^{\ell *}$ for $CD\,(\delta = 0.02)$ illustrated in Figure 6.1(a) is generally smaller (or less aggressive) than the corresponding allocation $\alpha^{\ell *}$ for $CD\,(\delta = 0.04)$ illustrated in Figure 6.1(a). However, both Figure 6.1(a) and Figure 6.1(a) illustrates how the optimal LETF strategy $\alpha^{\ell *}$ is contrarian, in the sense that smaller (resp. larger) values of the difference $P^{\ell}(t_n) - P^{v}(t_n)$ results in larger (resp. smaller) allocations to the LETF. In other words, following strong (resp. weak) LETF returns, the investment in the LETF is decreased (resp. increased). This enables the investor to "lock-in" periods of strong returns while simultaneously reducing risk.

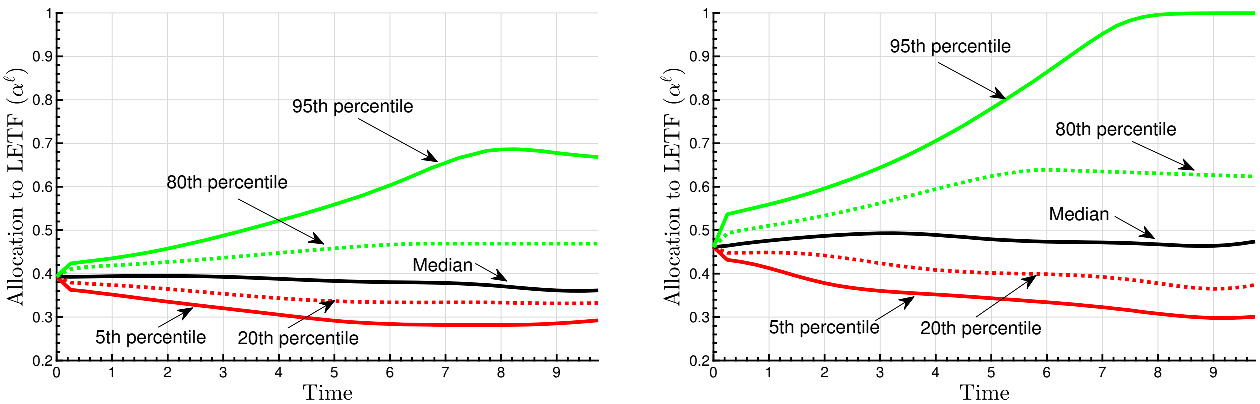


(a) *Optimal $\alpha^{\ell *}$ for $CD\,(\delta = 0.02)$.* (b) *Optimal $\alpha^{\ell *}$ for $CD\,(\delta = 0.04)$.*

FIGURE 6.1: *Optimal dynamic LETF allocation $\alpha^{\ell *}(t_n)$ illustrated as a function of the difference $P^{\ell}(t_n) - P^{v}(t_n)$ and time t_n (years), using the CD objective function with for two different values of the annual outperformance target parameter δ. At time $t = 0$, the difference is $P^{\ell}(t_0) - P^{v}(t_0) = 0$ by definition. The LETF leverage factor is $\beta = 2.0$.*

Of course, not all values of $\alpha^{\ell *}$ illustrated in Figure 6.1 are equally likely to be implemented. Figure 6.2 illustrates selected percentiles of the optimal allocation $\alpha^{\ell *}$ over time on the bootstrapped historical data used to train the neural network. We observe that in the case of $\alpha^{\ell *}$ for the $CD(\delta = 0.02)$ objective (Figure 6.2(a)), it is typical for $\alpha^{\ell *}$ to take on values in the interval [0.30,0.45], with median values of around 0.35 to 0.4. In the case of $\alpha^{\ell *}$ for the $CD(\delta = 0.04)$ objective (Figure

6.2(b)), while $\alpha^{\ell *}$ can take on more extreme values due to the more aggressive outperformance target, the median value is around 0.5. These results are perhaps expected given the the simulations of the preceding sections: to achieve the high Omega ratio results illustrated in Table 6.1 over the longer time horizon of 10 years, the optimal dynamic allocation strategy $\alpha^{\ell *}$ reduces exposure to the LETF to below 0.3 slowly (see 5th percentiles in Figure 6.2), and then only when the LETF has performed well compared to the VETF in prior periods (see Figure 6.1).



(a) *Percentiles of optimal $\alpha^{\ell *}$ for $CD(\delta = 0.02)$.*

(b) *Percentiles of optimal $\alpha^{\ell *}$ for $CD(\delta = 0.04)$.*

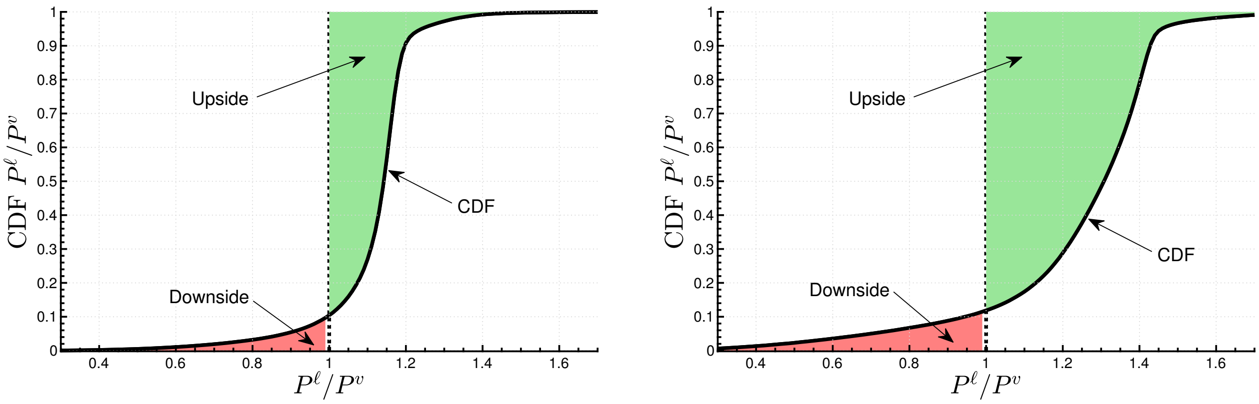
FIGURE 6.2: *Percentiles of the optimal dynamic LETF allocation $\alpha^{\ell *}(t_n)$ over time using the CD objective function with two different values of the annual outperformance target parameter δ. Results shown using 5×10^5 bootstrapped historical data paths. LETF leverage factor $\beta = 2.0$. The same scale on the y-axis is used to facilitate comparison.*

Recalling from Table 6.1 that we obtain $\Omega(1) = 7.83$ when using $\alpha^{\ell *}$ for $CD(\delta = 0.02)$, whereas the Omega ratio increases to $\Omega(1) = 8.08$ when using $\alpha^{\ell *}$ for $CD(\delta = 0.04)$. In other words, the more aggressive outperformance target of $\delta = 0.04$ in the CD objective also increased the Omega ratio of the resulting strategy. However, Figure 6.3 illustrates what can already be deduced from Table 6.1, namely that as the target increases from $\delta = 0.02$ to $\delta = 0.04$, both the upside and the the downside outcomes increase in terms of frequency and severity. However, with the higher target $\delta = 0.04$ the upside increases significantly *more* relative to the increase in the downside, resulting in the higher observed Omega ratio in Table 6.1.

As noted previously, traditional volatility based risk measures (such as Sharpe ratio) are inappropriate for portfolios which contain nonlinear instruments, such as LETFs. It is more useful to examine the percentiles of $R_t = P^{\ell}(t)/P^{v}(t)$. Specifically, the fifth percentile of R_t is a measure of downside risk across the entire investment period.

Figure 6.4 considers the percentiles of the portfolio value ratio P^{ℓ}/P^{v} over time when using the optimal dynamic LETF allocation $\alpha^{\ell *}$. Note that the same scale on the y-axis is used to facilitate comparison. We observe that the the optimal $\alpha^{\ell *}$ for $CD(\delta = 0.04)$ outperforms the optimal $\alpha^{\ell *}$ for $CD(\delta = 0.02)$ in terms of all percentiles shown for the ratio P^{ℓ}/P^{v} except for the 5th percentile, which is significantly worse using the more aggressive target. However, as noted in Section 4, this might be an appealing tradeoff for some investors.

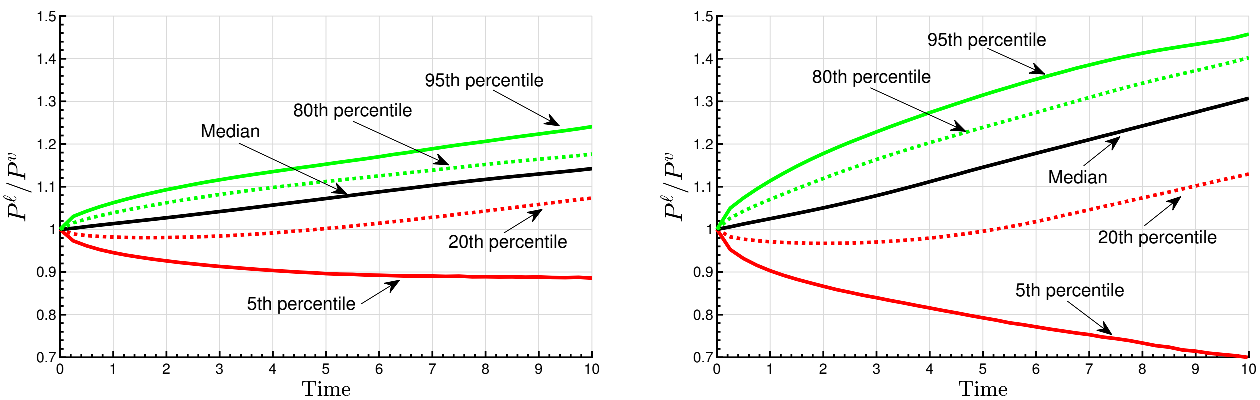
Since future asset returns will never replicate historical return paths precisely, we consider the preceding illustrative investment results using bootstrapped historical data (i.e. using stationary block bootstrap resampling) to be significantly more informative than single historical return path subsets when illustrating performance.



(a) *CDF P^ℓ/P^v using optimal $\alpha^{\ell*}$ for CD(δ = 0.02).*

(b) *CDF P^ℓ/P^v using optimal $\alpha^{\ell*}$ for CD(δ = 0.04).*

FIGURE 6.3: *Comparing P^ℓ/P^v at time $T = 10.0$ years, outperformance if $P^\ell/P^v > 1$. Optimal dynamic $\alpha^{\ell*}$ is used, obtained via the CD objective function with two different values of the annual outperformance target parameter δ, $\alpha^v = 0.60$. Results shown using 5×10^5 bootstrapped historical data paths. LETF leverage factor $\beta = 2.0$. $\Omega(1)$ (equation (2.22)) is the ratio of the upside area to the downside area.*



(a) *Percentiles of P^ℓ/P^v using optimal $\alpha^{\ell*}$ for CD($\delta = 0.02$).*

(b) *Percentiles of P^ℓ/P^v using optimal $\alpha^{\ell*}$ for CD($\delta = 0.04$).*

FIGURE 6.4: *Percentiles of P^ℓ/P^v over time, outperformance if $P^\ell/P^v > 1$. Optimal dynamic $\alpha^{\ell*}$ is used, obtained via the CD objective function with two different values of the annual outperformance target parameter δ, $\alpha^v = 0.60$. Results shown using 5×10^5 bootstrapped historical data paths. LETF leverage factor $\beta = 2.0$. The same scale on the y-axis is used to facilitate comparison.*

However, some practitioners may find bootstrapped results abstract or difficult to interpret intuitively. To provide more relatable and concrete illustrations, we examine how the optimal dynamic $\alpha^{\ell*}$ LETF portfolios and benchmark VETF portfolio would have performed during distinct 10-year historical periods. Table 6.2 demonstrates the portfolio performance starting with \$100 initial wealth invested over $T = 10$ years, beginning at the indicated month. Note that these results necessarily also rely on synthetically constructed historical returns for LETFs (see Appendix A). Table 6.2 confirms the advantages of strategically incorporating LETFs into investment portfolios for long-term investors, even if only using infrequent (quarterly) rebalancing. Note that these single historical paths can be regarded as out of sample compared to the bootstrap paths. This is because the probability of seeing an actual historical path in the bootstrap paths is vanishingly small (Ni et al., 2022).[7]

Starting month of investment:	Final month ($T = 10$ years)	Terminal wealth for each portfolio, initial investment \$100, quarterly rebalancing, zero contributions		
		VETF portfolio $\alpha^v = 0.60$	LETF portfolio Dynamic $\alpha^{\ell *}$ $CD(\delta = 0.02)$	LETF portfolio Dynamic $\alpha^{\ell *}$ $CD(\delta = 0.04)$
Jan 1970	Dec 1979	93	102	108
Jan 1975	Dec 1984	176	200	224
Jan 1980	Dec 1989	225	259	303
Jan 1985	Dec 1994	197	222	252
Jan 1990	Dec 1999	250	297	359
Jan 1995	Dec 2004	187	212	236
Jan 2000	Dec 2009	88	73	57
Jan 2005	Dec 2014	141	161	186
Jan 2010	Dec 2019	188	218	260
Jan 2013	Dec 2022	164	189	219
Jan 2014	Dec 2023	158	182	216

TABLE 6.2: *Historical path results: Investing initial wealth of \$100 over $T = 10.0$ years, starting at the beginning of the month as indicated, and following the optimal LETF allocation $\alpha^{\ell *}$ determined using CD objective function with targets δ as shown, as well as the VETF allocation of $\alpha^v = 0.6$. Quarterly rebalancing, LETF leverage factor $\beta = 2.0$, zero contributions are made to the portfolio.*

A key exception to the excellent performance of the LETF portfolios relative to the VETF portfolio is the period January 2000 to December 2009, where the VETF portfolio comfortably outperforms both LETF portfolios. However, this period is special, not only in the sense that it contains two major stock market events (the dot-com crash and the Global Financial Crisis), but that the GFC occurs near the end of the investment time horizon, which disproportionately affects the LETF portfolios.

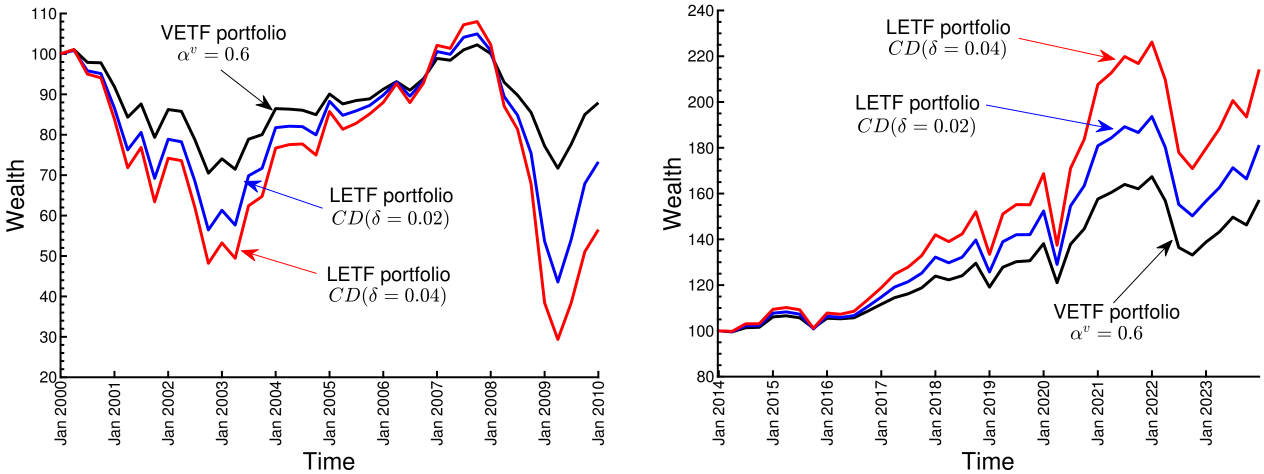
To illustrate this observation, Figure 6.5 compares the evolution of portfolio wealth over time for the different strategies, starting in January 2000 and January 2014. Note that the hypothetical investor starting in January 2014 will experience, over the time horizon of $T = 10$ years, strong early post-GFC growth, followed by the Covid-19 disruptions, subsequent recovery, as well as the 2022 bear market and a period of rising interest rates and inflation.

For both Figure 6.5(a) and Figure 6.5(b), we observe that regardless of the stock market crash in question, the LETF portfolios experience larger peak-to-trough declines but also faster post-crash recovery. The 10-year period starting in January 2000 therefore terminates in December 2009 before the LETF investor can take advantage of the post-crash recovery observed in Figure 6.5(b).

7 Conclusion

This paper explores how broad stock market LETFs can enhance portfolio performance when incorporated within dynamic, actively managed strategies, despite their well-documented unsuitability for passive investment strategies.

[7] Using typical parameters, Ni et al. (2022) show that the probability of finding an actual historical path amongst the bootstrap paths is less than 10^{-29}.



(a) *Start investing in Jan 2000.* (b) *Start investing in Jan 2014.*

FIGURE 6.5: *Historical path results over time, selected months: Investing initial wealth of \$100 over $T = 10.0$ years, starting at the beginning of the month as indicated, and following the optimal LETF allocation $\alpha^{\ell *}$ determined using CD objective function with targets δ as shown, as well as the VETF allocation of $\alpha^v = 0.6$. Quarterly rebalancing, LETF leverage factor $\beta = 2.0$, zero contributions are made to the portfolio.*

The results, which are based on synthetic LETF returns constructed with nearly 100 years of market data, reveals that the key to successful incorporation of a broad stock market LETF into a portfolio requires a contrarian investment strategy, i.e. reducing exposure to the LETF systematically following a period of gains. We observed that the Omega ratio is a critical metric for evaluating LETF strategies, with results showing that while naive approaches often yield Omega ratios below unity, dynamic strategies can achieve substantially higher Omega ratios that benefits from (infrequent) portfolio rebalancing. Using a data-driven neural network approach, we discuss the construction of optimal LETF allocation strategies without restrictive parametric assumptions, learning the strategy directly from historical returns to capture market complexity while maintaining practical implementability through infrequent rebalancing.

Our results demonstrate that accessible dynamic approaches to broad stock market LETF investment, requiring only quarterly rebalancing, can deliver potentially desirable risk-return benefits for the active investor.

8 Declaration

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Appendices

A Constructing synthetic LETF and VETF returns since 1926

Since LETFs were only introduced in 2006 (Bansal and Marshall (2015)), our goal of deriving a more realistic historical perspective of LETF behaviour during different market and inflation regimes requires the use of proxy data. Note that a similar construction of proxy returns time series for LETFs has also been done in Bansal and Marshall (2015), but with details regarding the inflation adjustment and choice of underlying index differ. As in for example Bansal and Marshall (2015) and Leung and Sircar (2015), we assume that the ETF managers achieve a negligible tracking error with respect to the underlying index.

The steps followed in constructing proxy time series of historical returns of a VETF and LETF referencing a broad US equity market index, we proceed as follows:

1. We obtain daily returns for the VWD index and 30-day Treasury bills from the CRSP[8]. The CRSP VWD index, capitalization-weighted index consisting of all domestic stocks trading on major US exchanges, is assumed to be the broad stock market index referenced by the VETF and LETF. We use data for the historical time period 1926:01 to 2023:12, to ensure that periods of exceptional market volatility and high inflation are also included.

2. Multiply each daily return by the returns multiplier β, where we used $\beta = 2$ for the LETF and $\beta = 1$ for the VETF, and construct a time series of monthly returns. This follows from the results in Section 2, with $\left(\delta V^{\ell}/V^{\ell}\right)$ and $(\delta V^{v}/V^{v})$ representing the daily returns of the LETF and VETF, respectively:

$$\frac{\delta V^{\ell}}{V^{\ell}} \simeq \beta \cdot \left(\frac{\delta S}{S}\right) + (1-\beta) \cdot \left(\frac{\delta B}{B}\right) - c_{\ell} \cdot \delta t, \tag{A.1}$$

$$\frac{\delta V^{v}}{V^{v}} \simeq \frac{\delta S}{S} - c_{v} \cdot \delta t. \tag{A.2}$$

   In this approximation, $\delta S/S$ denotes the daily returns of the equity market index, $\delta B/B$ the daily returns of the 30-day T-bills, and $\delta t = 1/252$. To reflect values similar to those observed in the market (see Section 2), we assume expense ratios of $c_{\ell} = .0089$ and $c_{v} = 0.00$, and a LETF leverage factor of $\beta = 2$.

3. Finally, all time series were inflation-adjusted using inflation data from the US Bureau of Labor Statistics[9].

Note that the proxy time series for VETF and LETF returns are *only* used when bootstrapping data sets providing the training/testing data for the data-driven neural network approach followed in Section 6. In contrast, the parametric dynamics for the results of Section 2 , Section 3 and Section 4 do not require historical proxy returns, since the LETF dynamics can be obtained by calibrating equations (2.1) and (2.2) to historical data as described in Appendix B.

B Parameters for jump diffusion model, fit to CRSP data 1926:1-2023:12.

The parameters for equations (2.1) and (2.2) are fit to the CRSP data, inflation adjusted, with results in Table B.1. We use the filtering technique described in (Mancini, 2009; Cont and Mancini, 2011; Dang and Forsyth, 2016) to estimate the parameters.

We also assume that the expense ratios are $c_{\ell} = .0089$ and $c_{v} = 0.0$, and the LETF leverage factor is $\beta = 2$.

[8] Calculations were based on data from the Historical Indexes 2024©, Center for Research in Security Prices (CRSP), The University of Chicago Booth School of Business. Wharton Research Data Services was used in preparing this article. This service and the data available thereon constitute valuable intellectual property and trade secrets of WRDS and/or its third party suppliers.

[9] The annual average CPI-U index, which is based on inflation data for urban consumers, were used - see http://www.bls.gov.cpi

	μ	σ	λ	p_{up}	η_1	η_2
CRSP Index (real)	0.08732	0.1477	0.3163	0.2258	4.3591	5.5337

TABLE B.1: *Estimated annualized parameters for double exponential jump diffusion model. Value-weighted CRSP index deflated by the CPI. Sample period 1926:1 to 2023:12. The average real return of a 30 day T-bill in the same period was* $r = 0.0032$.

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