A Numerical Scheme for Guaranteed Minimum Withdrawal Benefits (GMWB)

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Motivation

Variable annuity products: sold by insurance companies to retail investors.

These products are guarantees on investments in pension plans.

From a paper we wrote in 2002 (segregated funds are a Canadian version of Variable Annuities)

> *"If one adopts the no-arbitrage perspective...in many cases these contracts appear to be significantly underpriced, in the sense that the current deferred fees being charged are insufficient to establish a dynamic hedge for providing the guarantee. This is particularly true for cases where the underlying asset has relatively high volatility. This finding might raise concerns at institutions writing such contracts."* Windcliff, Forsyth, LeRoux, Vetzal, North American Actuarial J., 6 (2002) 107-125

What Happened?

As described in a Globe and Mail article (Report on Business, December 2, 2008, "Manulife, in red, raises new equity,"), one of the large Canadian insurance companies, Manulife, posted a large mark-to-market writedown to account for losses associated with these segregated fund guarantees. From the Globe and Mail Streetwise Blog, November 7, 2008

> *"Concerns that the market selloff will translate into massive future losses at Canada's largest insurer sent Manulife shares reeling last month. Those concerns were a result of Manulife's strategy of not fully hedging products such as annuities and segregated funds, which promise investors income no matter what markets do."*

Why did this happen?

These products contain embedded options which allow the investors many opportunities to optimize the value of the guarantee.

Pricing of these products requires solution of an optimal stochastic control problem (an HJB PDE).

- This was beyond the technical abilities of most insurance companies
- Insurance companies used simplistic models which underestimated the risk involved.
- These models showed that there was no need to hedge these products, (Quote from actuary:) *"Over any ten year period, the market never goes down."*
- Insurance company executives were able to boast of large (apparent) profits, which then triggered rich bonus payments to traders and executives.

Retirement Risk Zone

Consider an investor with a retirement account, which is invested in the stock market
Over the long run (before retirement), it does not matter if

- the market first drops by 10% per year over several years and then goes up by 20% per year for several years; or
- the market first goes up by 20% per year and then drops by 10% per year

$$(.9)(.9)...(1.2)(1.2)... = (1.2)(1.2)...(.9)(.9)...$$

The Retirement Risk Zone II

This is not the case once the investor retires, and begins to make withdrawals from the retirement account

The outcomes will be very different in the cases:

- in the first few years after retirement, the market has losses, and the account is further depleted by withdrawals, followed by some years of good market returns; compared to
- a few years of good market returns, after retirement (including withdrawals), followed by some years of losses

Losses in the early years of retirement can be devastating in the long run! Early bad returns can cause complete depletion of the account.

A Typical GMWB Example

Investor pays \$100 to an insurance company, which is invested in a risky asset.

Denote amount in risky asset sub-account by $W = 100$.

The investor also has a virtual guarantee account $A = 100$.

Suppose that the contract runs for 10 years, and the guaranteed withdrawal rate is \$10 per year.

A Typical GMWB Example II

At the end of each year, the investor can choose to withdraw up to \$10 from the account. If $\$\gamma \in [0, 10]$ is withdrawn, then

$$\begin{aligned} W_{new} &= \max(W_{old} - \gamma, 0) \quad ; \quad \textit{Actual investment} \\ A_{new} &= A_{old} - \gamma \quad ; \quad \textit{Virtual account} \end{aligned}$$

This continues for 10 years. At the end of 10 years, the investor can withdraw anything left, i.e. $\max(W_{new}, A_{new})$.

Note: the investor can continue to withdraw cash as long as $A > 0$, *even if* $W = 0$ (recall that W is invested in a risky asset).

Example: Order of Random Returns

Good Returns at Start

Time	Return (%)	Balance ($)	Withdrawal ($)
1	41.65	141.65	10
2	31.12	172.62	10
3	20.15	195.39	10
4	-30.25	129.31	10
5	18.05	140.85	10
6	16.82	152.86	10
7	20.12	171.60	10
8	7.44	173.62	10
9	-40.90	96.70	10
10	-7.5	80.20	10
Total Withdrawal Amount ($)			170.20
Ten year balance if no withdrawal ($)			151.37

Same Random Returns: Different Order

No GMWB: poor returns at start

Time	Return (%)	Balance ($)	Withdrawal ($)
1	-30.25	69.75	10
2	-40.90	35.31	10
3	16.82	29.57	10
4	7.44	21.03	10
5	41.65	15.62	10
6	20.12	6.75	6.75
7	31.12	0	0
8	18.05	0	0
9	20.15	0	0
10	-7.5	0	0
Total Withdrawal Amount ($)			56.75
Ten year balance if no withdrawal ($)			151.37

Unlucky Order of Returns: With GMWB

GMWB Protection

Time	Return (%)	Balance ($)	Withdrawal ($)
1	-30.25	69.75	10
2	-40.90	35.31	10
3	16.82	29.57	10
4	7.44	21.03	10
5	41.65	15.62	10
6	20.12	6.75	10
7	31.12	0	10
8	18.05	0	10
9	20.15	0	10
10	-7.5	0	10
Total Withdrawal Amount ($)			100
Ten year balance if no withdrawal ($)			151.37

Why is this useful?

The investor can participate in market gains, but still has a guaranteed cash flow, in the case of market losses.

This insulates pensioners from losses in the early years of retirement.

This protection is paid for by deducting a yearly fee α from the amount in the risky account W each year.

The simple form of GMWB described has many variants in practice: Guaranteed Lifetime Withdrawal Benefit (GLWB), ratchet increase of virtual account A if no withdrawals, etc.

We will keep things simple here, and look at the basic GMWB.

Most variable annuities sold in North America have some type of market guarantee.

Some More Details

The investor can choose to withdraw up to the specified contract *rate* G_r without penalty.

Usually, a penalty ($\kappa > 0$) is charged for withdrawals above G_r.

Let $\hat{\gamma}$ be the rate of withdrawal selected by the holder.

Then, the rate of actual cash received by the holder of the GMWB is

$$\hat{f}(\hat{\gamma}) = \begin{cases} \hat{\gamma} & \text{if } 0 \leq \hat{\gamma} \leq G_r, \\ \hat{\gamma} - \kappa(\hat{\gamma} - G_r) & \text{if } \hat{\gamma} > G_r. \end{cases}$$

Stochastic Process

Let S denote the value of the risky asset, we assume that the risk neutral process followed by S is

$$\begin{aligned} dS &= rSdt + \sigma SdZ \\ & \quad r = \text{ risk free rate}; \quad \sigma = \text{ volatility} \\ & \quad dZ = \phi\sqrt{dt} \ \ ; \ \ \phi \sim \mathcal{N}(0,1) \end{aligned}$$

The risk neutral process followed by W is then (including withdrawals dA).

$$\begin{aligned} dW &= (r-\alpha)Wdt + \sigma WdZ + dA, \quad \text{if } W > 0 \\ dW &= 0, \quad \text{if } W = 0 \\ & \quad \alpha = \text{ fee paid for guarantee } \ ; \ \ A = \text{ guarantee account} \end{aligned}$$

No-arbitrage Value

Let $V(W, A, \tau)$ ($\tau = T - t$, T is contract expiry) be the no-arbitrage value of the GMWB contract (i.e. the cost of hedging).

At contract expiry ($\tau = 0$) we have (payoff = withdrawal)

$$V(W, A, \tau = 0) \quad = \quad \max(W, A(1 - \kappa))$$

It turns out that it is optimal to withdraw at a rate $\hat{\gamma}$

- $\hat{\gamma} \in [0, G_r]$, or
- $\hat{\gamma} = \infty$ (instantaneously withdraw a finite amount)

Impulse Control

Let

$$\mathcal{L}V = \frac{1}{2}\sigma^2 W^2 V_{WW} + (r-\alpha) W V_W - rV.$$

Since we have the option of withdrawing at a finite rate at each point in (W, A, τ), Ito's Lemma and no-arbitrage arguments give

$$V_\tau - \mathcal{L}V - \max_{\hat{\gamma} \in [0, G_r]} \left(\hat{\gamma} - \hat{\gamma} V_W - \hat{\gamma} V_A \right) \geq 0$$

Note that $\hat{\gamma}$ is a finite withdrawal *rate*. Withdrawals only allowed if $A > 0$.

Impulse Control II

We also have the option of withdrawing a finite amount instantaneously (withdrawing at an infinite rate) at each point in (W, A, τ)

$$V(W, A, \tau) - \sup_{\gamma \in (0, A]} \left[V(\max(W - \gamma, 0), A - \gamma, \tau) + (1 - \kappa)\gamma - c \right] \geq 0 \ .$$

where γ is a finite withdrawal *amount*.

$c > 0$ is a fixed cost (which can be very small). This is required to make the Impulse Control problem well-posed.

Note that this equation specifies that any amount in the remaining guarantee account can be withdrawn instantaneously (i.e. $\gamma \in (0, A]$) with a penalty.

HJB Variational Inequality

Since it must be optimal to either withdraw at a finite rate or withdraw a finite amount at each point, then this can all be written compactly as a Hamilton Jacobi Bellman Variational Inequality

$$\min\Big\{ V_\tau - \mathcal{L}V - \max_{\hat{\gamma}\in[0,G_r]} \big(\hat{\gamma} - \hat{\gamma}V_W - \hat{\gamma}V_A\big),$$

$$V - \sup_{\gamma\in(0,A]} \big[V(\max(W-\gamma,0), A-\gamma, \tau) + (1-\kappa)\gamma - c \big] \Big\}$$

$$= 0$$

Previous Work

- Milevsky, Salisbury, (2006, Insurance: Mathematics and Economics), pose GMWB pricing problem as a singular control.
- Dai, Kwok, Zong, (Mathematical Finance, 2008), solve singular control formulation using a penalty method.
- Zakamouline (Mathematical Methods Operations Research, 2005) argues that in general one can pose singular control problems as impulse control with negligible difference (infinitesimal fixed cost)
  - Claims that impulse control is more general
- Chen, Forsyth (Numerische Mathematik, 2008), solve impulse control formulation (this lecture)

Alternative Approach: Discrete Withdrawal Times

Rather than attempt to solve the HJB Impulse Control problem directly, let's replace this problem by a *discrete withdrawal* problem

- Assume that the holder can only withdraw at discrete *withdrawal* times $\tau_1, ..., \tau_N$, with $\tau_{i+1} - \tau_i = \Delta t_w$
- Use dynamic programming idea, work backwards from $t = T(\tau = 0)$, so that $V(W, A, 0) = \max(W, A(1-\kappa))$
- During the interval from $\tau = 0$ to $\tau = \tau_1$ (the first withdrawal time going backwards) we solve

$$V_\tau - \mathcal{L}V = 0 \;\; ; \;\; \mathcal{L}V = \frac{1}{2}\sigma^2 W^2 V_{WW} + (r - \alpha) W V_W - rV .$$

Optimum Strategy: Discrete Withdrawals

At τ_1, we assume that the holder withdraws the optimum *amount* γ

$$V(W, A, \tau_1^+) = \max_{\gamma \in [0,A]} \left[V\big(\max(W-\gamma, 0), A-\gamma, \tau_1\big) + f(\gamma) \right],$$

where now the cash flow term is

$$f(\gamma) = \begin{cases} \gamma & \text{if } 0 \leq \gamma \leq G, \\ \gamma - \kappa(\gamma - G) - c & \text{if } \gamma > G. \end{cases}$$

$$G = G_r \Delta t_w$$

Discrete Withdrawals

Then, from τ_1^+ to τ_2, we solve

$$V_\tau - \mathcal{L}V = 0 \quad ; \quad \text{No A dependence in } \mathcal{L}V$$

Then, we determine the optimum withdrawal at τ_2^+, and so on, back down to $\tau = T(t = 0)$ today.

This would appear to be a reasonable approximation to reality.

In fact, most real contracts allow only discrete withdrawals.

Discrete Withdrawal: A Numerical Scheme

Define nodes in the W direction $[W_0, W_1, \ldots, W_{i_{\max}}]$, and in the A direction $[A_0, A_1, \ldots, A_{j_{\max}}]$.

Let $V_{i,j}^n \simeq V(W_i, A_j, \tau^n)$. $[V^n]_{i,j} = V_{i,j}^n$.

Let $(\mathcal{L}_h V)_{i,j}^n$ be a discrete form of the operator $\mathcal{L}V$.

Away from withdrawal times, we solve

$$\frac{V_{i,j}^{n+1} - V_{i,j}^n}{\Delta \tau} = (\mathcal{L}_h V)_{i,j}^{n+1}$$

A Numerical Scheme II

At withdrawal time τ_n, we then solve the local optimization problem at each node

$$V_{i,j}^{n+} = \max_{\gamma_{i,j}^n \in [0, A_j]} \left[\mathcal{I}_{i,j}(\gamma_{i,j}^n) V^n + f\left(\gamma_{i,j}^n\right) \right],$$

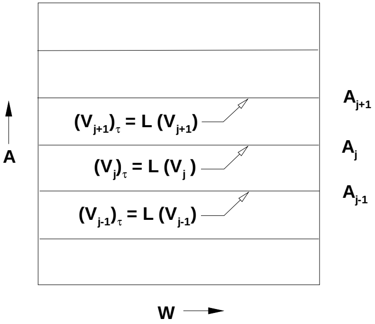
where $\mathcal{I}$ is an interpolation operator

$$\begin{aligned} \mathcal{I}_{i,j}(\gamma) V^n &= V^n(\max(W_i - \gamma, 0), A_j - \gamma) \\ &\quad + \text{ interpolation error} \end{aligned}$$

We use a linear interpolant of $V_{i,j}^n$ to determine the optimum withdrawal at each node $\gamma_{i,j}^n$.

Numerical Scheme III

- Away from withdrawal times, we solve a decoupled set of 1-d PDEs.
- At withdrawal times, we solve a set of decoupled optimization problems.



Vast majority of CPU time spent solving the local optimization problem at each node:

$$V_{i,j}^{n+} = \max_{\gamma_{i,j}^n \in [0,A_j]} \left[\mathcal{I}_{i,j}(\gamma_{i,j}^n) V^n + f\left(\gamma_{i,j}^n\right) \right] ,$$

- This is *embarrassingly parallel*, but requires access to global data.

Obvious Question

If we let $\Delta\tau_w \to 0$, does this discrete withdrawal approximation converge to the solution of the Impulse Control HJB equation?

If we allow discrete withdrawals every timestep, then our numerical method is

$$V_{i,j}^{n+1} - \max_{\gamma_{i,j}^n \in [0,A_j]} \left[\mathcal{I}_{i,j}(\gamma_{i,j}^n) V^n + f\left(\gamma_{i,j}^n\right) \right] - \Delta\tau \left(\mathcal{L}_h V \right)_{i,j}^{n+1} = 0 \ .$$

where the cash flow term $f\left(\gamma_{i,j}^n\right)$ is

$$\begin{aligned} f(\gamma) \qquad &= \begin{cases} \gamma & \text{if } 0 \leq \gamma \leq G, \\ \gamma - \kappa(\gamma - G) - c & \text{if } \gamma > G. \end{cases} \\ G &= G_r \Delta\tau \end{aligned}$$

and $\mathcal{I}$ is a linear interpolation operator

Viscosity Solution

The HJB equation does not necessarily have differentiable solutions.
We seek the viscosity solution of the HJB impulse control problem.

- Briefly, a viscosity solution is defined in terms of smooth test functions.
- These smooth test functions touch the *viscosity* solution at a single point, and are always *above* or *below* the solution elsewhere.
- The viscosity solution is *squeezed* between these nearby test functions.
- The viscosity solution does not necessarily satisfy the PDE in any conventional sense.

Basic Convergence Result

Theorem (Convergence to the Viscosity Solution (Barles, Souganidis (1993)))

Any numerical scheme which is consistent, l_∞ stable, and monotone, converges to the viscosity solution.

Consistent Discrete scheme applied to smooth test function satisfies a $\limsup$, $\liminf$ condition, as mesh, timestep $\rightarrow 0$ (smooth test functions squeeze the solution)

Stability (l_∞) Solution bounded in l_∞ as mesh, timestep $\rightarrow 0$.

Monotonicity: What does it mean?

Monotonicity

Let $V^n_{i,j}$, $Q^n_{i,j}$ be two discrete solutions to the same HJB equation.

Lemma (Discrete Arbitrage Inequality)

If $V^n_{i,j}, Q^n_{i,j}$ are generated using a monotone scheme, and $\forall i, j,\ Q^0_{i,j} \geq V^0_{i,j}$, then

$$Q^n_{i,j} \geq V^n_{i,j} \quad ; \quad \forall i, j; \forall n$$

In other words, if the payoff $Q(W, A, 0) \geq V(W, A, 0)$, then this inequality must hold at all earlier times, for the discrete solution, regardless of the timestep or meshsize.

This is a discrete version of the financial no-arbitrage condition. (i.e. financial equivalent of mass conservation)

Does it Converge?

We want to show that this scheme converges as $\Delta\tau$, ΔA, ΔW $\rightarrow 0$ to the viscosity solution of

$$\min\Big\{ V_\tau - \mathcal{L}V - \max_{\hat{\gamma}\in[0,G_r]} \big(\hat{\gamma} - \hat{\gamma}V_W - \hat{\gamma}V_A\big),$$
$$V - \sup_{\gamma\in(0,A]} \big[V(\max(W-\gamma,0), A-\gamma, \tau) + (1-\kappa)\gamma - c\big]\Big\}$$
$$= 0$$

This seems intuitively obvious, but there are some subtle points.

Monotonicity, Stability and Consistency

Lemma (Monotonicity and Stability)

Provided $(\mathcal{L}_h V^{n+1})$ is discretized using a positive coefficient method and linear interpolation is used when solving the local optimization problem at each node, then the scheme is unconditionally l_∞ stable and monotone.

Proof.

Straightforward □

Lemma (Consistency)

Provided the discrete operator $(\mathcal{L}_h V^{n+1})$ is consistent in the classical sense, and linear interpolation is used to solve the local optimization problem at each node, then the numerical scheme is consistent as defined in (Barles, Souganidis (1991)).

Proof.

Not so straightforward ($\liminf, \limsup$ form needed for boundary conditions) □

Convergence

Theorem (Strong Comparison Result)

The GMWB Impulse Control problem satisfies the Strong Comparison Result, i.e. there is a unique, continuous viscosity solution to the Impulse Control Problem. (Seydel, 2008)

Theorem (Convergence to the Viscosity Solution)

The discrete withdrawal numerical method, with withdrawal interval $\Delta t_w \rightarrow 0$ *converges to the unique viscosity solution of the Impulse Control problem.*

Proof.

This scheme is consistent, stable, and monotone, hence converges to the viscosity solution (Barles, Souganidis (1991)). □

One scheme for all problems

So, we now have a single scheme which

- Can be used to price GMWB contracts with finite withdrawal intervals (the usual case in real contracts, i.e. withdrawals only allowed once or twice a year)
- We can also price GMWB contracts in the limit as the withdrawal interval $\rightarrow 0$
- Convergence to the Impulse Control problem guaranteed
- No need for different method for these two cases!
- Scheme is simple and intuitive to implement $\rightarrow$ might be actually used by practitioners.

Examples

Recall that the investor pays no extra up-front fee for the guarantee (only the initial premium w_0).

The insurance company deducts an annual fee α from the balance in the sub-account W.

Problem: let $V(\alpha, W, A, \tau)$ be the value of the GMWB contract, for given yearly guarantee fee α.

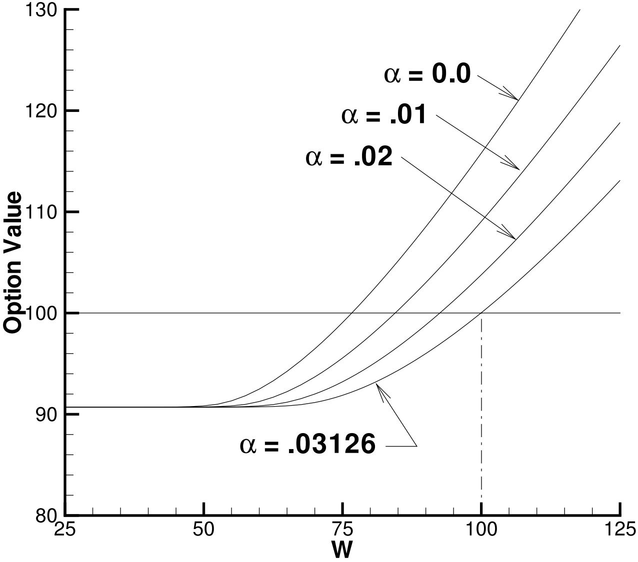
Assume that the investor pays an initial premium w_0 at $t = 0$ $(\tau = T)$.

Find the no-arbitrage fee α such that $V(\alpha, w_0, w_0, T) = w_0$ (we do this by a Newton iteration).

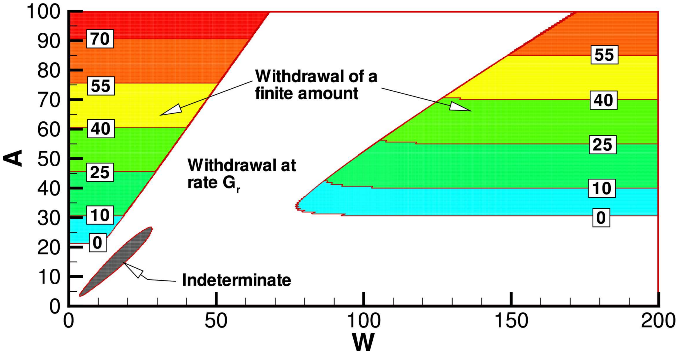
Data

Parameter	Value
Expiry time T	10.0 years
Interest rate r	.05
Maximum withdrawal rate G_r	10/year
Withdrawal penalty κ	.10
Volatility σ	.30
Initial Lump-sum premium w_0	100
Initial guarantee account balance	100
Initial sub-account value	100
Continuous Withdrawal	Yes

The No-arbitrage Fee ($t = 0$, $A = 100$)



Optimal Withdrawal Strategy



$t = 0$, fair fee charged for $w_0 = 100$.

• Indeterminate region: any withdrawal in $[0, G_r]$ is optimal (value is unique, control is not unique).

No-arbitrage Fee

- $\sigma = .15 \rightarrow \alpha = .007$ (70 bps)
- $\sigma = .20 \rightarrow \alpha = .014$ (140 bps)
- $\sigma = .30 \rightarrow \alpha = .031$ (310 bps)
- Current volatility of $S\&P \simeq .25$
- Typical fees charged: $\alpha = .005$ (50 bps) too low for current market conditions.
- Insurance companies seem to be charging fees based on marketing considerations, not hedging costs.
- Fee should be even higher if other (typical) contract options considered

Other Issues

Can easily use the same method if we assume underlying process is a jump diffusion (Chen, Forsyth, SIAM J. Scientific Computing (2007)).

Effect of discrete withdrawals, volatility, non-optimal withdrawals, etc. (Chen, Vetzal, Forsyth, Insurance: Mathematics and Economics (2008)).

A penalty method for singular control formulation of a GMWB (Dai et al, Mathematical Finance (2008)), (Huang, Forsyth, Working paper (2009)).

Impulse control for a Guaranteed Minimum Death Benefit (Belanger, Forsyth, Labahn, Applied Mathematical Finance (2009)).

Summary

- We have developed a single scheme which can be used to price GMWB contracts with finite withdrawal intervals, and the limiting case of infinitesimal withdrawal intervals
- In the case of infinitesimal withdrawal intervals, we have proven convergence to the viscosity solution of the Impulse Control problem
- For an infinitesimal fixed cost, solutions agree with a singular control formulation
- Insurance companies seem to be charging fees which are too low to cover hedging costs. Another subprime problem?