

# Smart leverage? Rethinking the role of Leveraged Exchange Traded Funds in constructing portfolios to beat a benchmark

Pieter M. van Staden\* Peter A. Forsyth† Yuying Li‡

March 22, 2025

## Abstract

Leveraged Exchange Traded Funds (LETFs), while extremely controversial in the literature, remain stubbornly popular with both institutional and retail investors in practice. While the criticisms of LETFs are certainly valid, we argue that their potential has been underestimated in the literature due to the use of very simple investment strategies involving LETFs. In this paper, we systematically investigate the potential of including a broad stock market index LETF in long-term, dynamically-optimal investment strategies designed to maximize the outperformance over standard investment benchmarks in the sense of the information ratio (IR). Our results exploit the observation that positions in a LETF deliver call-like payoffs, so that the addition of a LETF to a portfolio can be a convenient way to add inexpensive leverage while providing downside protection. Under stylized assumptions, we present and analyze closed-form IR-optimal investment strategies using either a LETF or standard/vanilla ETF (VETF) on the same equity index, which provides the necessary intuition for the potential and benefits of LETFs. In more realistic settings of infrequent trading, leverage restrictions and additional constraints, we use a neural network-based approach to determine the IR-optimal strategies, trained on bootstrapped historical data. We find that IR-optimal strategies with a broad stock market LETF are not only more likely to outperform the benchmark than IR-optimal strategies derived using the corresponding VETF, but are able to achieve partial stochastic dominance over the benchmark and VETF-based strategies in terms of terminal wealth, even if investment in the VETF can be leveraged. Our results help to explain the empirical appeal of LETFs to investors, and encourage the reconsideration in academic research of the role of broad stock market LETFs within the context of more sophisticated investment strategies.

**Keywords:** Asset allocation, leveraged investing, portfolio optimization, neural network

**JEL classification:** G11, C61

## 1 Introduction

Leveraged Exchange Traded Funds (LETFs) are exchange traded funds (ETFs) with the stated objective of replicating some multiple  $\beta$  of the daily returns of their underlying reference assets/indices before costs, where values of  $\beta$  of  $+2$ ,  $+3$ ,  $-2$  and  $-3$  are commonly used. In contrast, standard/vanilla ETFs (VETFs) aim simply to replicate the returns of their underlying assets/indices before costs (i.e.  $\beta = 1$ ).

A review of the academic literature suggests that incorporating LETFs in investment strategies are commonly regarded with at least some suspicion, if not outright distrust. “*Just say no to leveraged ETFs*”, the title of a recent article (Bednarek and Patel (2022)), provides perhaps the most succinct summary of the broadly negative view of LETFs that permeates the literature. There are certainly good reasons for these negative perceptions of LETFs. A common criticism in the literature focuses on the “compounding” effect of LETF returns, which arises since a LETF returning  $\beta$  times the *daily* returns of the underlying index obviously does not imply that the LETF also returns  $\beta$  times the *quarterly* returns of the underlying index. This observation, along with the time decay and volatility decay of LETF positions, results in the potential wealth-destroying effects of LETF investments which increase with the holding time horizon (Mackintosh (2008), Carver (2009), Sullivan

---

\*National Australia Bank, Melbourne, Victoria, Australia 3000. The research results and opinions expressed in this paper are solely those of the authors, are not investment recommendations, and do not reflect the views or policies of the NAB Group. [pieter.vanstaden@gmail.com](mailto:pieter.vanstaden@gmail.com)

†Cheriton School of Computer Science, University of Waterloo, Waterloo ON, Canada, N2L 3G1, [paforsyt@uwaterloo.ca](mailto:paforsyt@uwaterloo.ca)

‡Cheriton School of Computer Science, University of Waterloo, Waterloo ON, Canada, N2L 3G1, [yuying@uwaterloo.ca](mailto:yuying@uwaterloo.ca)

(2009), Charupat and Miu (2011)). The poor investment outcomes using LETFs reported in the literature should therefore not come as a surprise if an investor uses a basic buy-and-hold investment strategy (Charupat and Miu (2011), Bednarek and Patel (2022), Leung and Santoli (2012)), or very simple (if not outright naive) portfolio rebalancing rules typically considered in conjunction with unrealistically frequent rebalancing<sup>1</sup> from the perspective of long-term investors (Cheng and Madhavan (2009), Avellaneda and Zhang (2010), Bansal and Marshall (2015), DeVault et al. (2021)). While there are a few studies observing that LETFs could have a role within diversified portfolios, especially where the investor might have relatively aggressive performance targets (Bansal and Marshall (2015), Hill and Foster (2009)), wish to circumvent onerous leverage restrictions or large margin rates on borrowing (DeVault et al. (2021)), or wish to outperform broad market indices using very simple strategies with frequent rebalancing (Knapp (2023); Trainor et al. (2018)), these perspectives remain the exception to the mainstream academic view and tend to leave questions regarding the formulation of practical investment strategies unanswered.

However, the contrast between the general perceptions in the academic literature and investment practice could not be more profound. LETFs have consistently remained incredibly popular financial products since their introduction in 2006, as emphasized by recent headlines such as “*Investors pump record sums into leveraged ETFs*” (Financial Times, November 2022<sup>2</sup>) and “*Retail investors snap up triple-leveraged US equity ETFs*” (Financial Times, May 2024<sup>3</sup>). LETFs consistently dominate the top 10 most popular ETFs listed on US exchanges<sup>4</sup>.

Perhaps more significantly, LETFs also enjoy substantial popularity among institutional investors. Analyzing the quarterly reports by institutional investment managers with at least US\$100 million in assets under management that were filed with the SEC from September 2006 to December 2016, DeVault et al. (2021) finds that more than 20% of the reports reference at least one LETF in the end-of-quarter portfolio allocation.

Leaving speculative trading aside, what could explain the appeal of LETFs for institutional investors? Suppose an investor wants to leverage returns in a cost-effective way which also offers some downside protection. Since the requirement of downside protection rules out simple leverage, such an institutional investor has broadly speaking at least two options, namely (a) engage with a hedge fund or fund manager to use for example managed futures strategies, or (b) follow a dynamic trading strategy using for example LETFs as discussed in this paper. Since the expense ratios of LETFs range typically between 80 and 150 basis points, whereas standard leveraged positions are subject to substantial margin rates of borrowing which can easily exceed 5% for smaller institutional investors even during periods of low interest rates (DeVault et al. (2021)) while hedge funds charge hefty management fees, LETFs are certainly cost effective. In addition, LETFs can offer great upside returns in combination with limited liability without the need to manage short positions, and as discussed in this paper, positions in LETFs can be *infrequently* rebalanced while still obtaining competitive investment outcomes relative to standard investment benchmarks.

## 1.1 Main Contributions

While the academic literature offers a sophisticated and careful treatment of optimal rebalancing for hedging purposes in the case of LETFs (Dai et al. (2023)), or optimal replication policies for LETF construction (Guasoni and Mayerhofer (2023)), in this paper we therefore aim to make progress towards closing the observed gap between the literature and investment practice by showing that there is a role for LETFs in *infrequently rebalanced* (e.g. quarterly rebalanced) portfolios designed for *long-term* institutional or retail investors wishing to outperform some investment benchmark.

Since LETFs are a relatively recent invention, historical LETF returns since 1926 are obtained by constructing a proxy LETF replicating<sup>5</sup>  $\beta = 2$  times the daily returns of a broad US equity market index, deflating

---

<sup>1</sup>For example, daily rebalancing. Note that in some cases, the general investment literature actually advises investors not to hold LETFs for longer than a single trading session. See for example Forbes Advisor, accessed 10 March 2024. Michael Adams. *Eight best leveraged ETFs of March 2024*. <https://www.forbes.com/advisor/investing/best-leveraged-etfs>.

<sup>2</sup>Financial Times, November 14, 2022. Steven Johnson. *Investors pump record sums into leveraged ETFs*. <https://www.ft.com/content/b98ab360-2506-44f2-8e08-9d434df5f15d>

<sup>3</sup>Financial Times, May 4, 2024. George Steer and Will Schmitt. *Retail investors snap up triple-leveraged US equity ETFs*. <https://on.ft.com/3WsWTom>

<sup>4</sup>For example, as at 9 October 2023, four out of the five most popular ETFs as measured by the average daily trading volume over the preceding three months were LETFs. <https://etfdb.com/compare/volume>. Accessed 9 October 2023

<sup>5</sup>We make the standard assumption (see e.g. Bansal and Marshall (2015), Leung and Sircar (2015)) that the LETF managers do not have challenges in replicating the underlying index. Note that given improvements in designing replication strategies for LETFs that remain robust even during periods of market volatility (see for example Guasoni and Mayerhofer (2023)), this appears to be a reasonable assumption for ETFs (VETFs and LETFs) written on major equity market indices as considered in this paper.

84 the returns by a typical LETF expense ratio and interest rates (see Appendix B). All returns time series are  
85 inflation-adjusted to reflect real returns.

86 To ensure the practical relevance of our conclusions, illustrative investment results are based on data sets  
87 generated using (i) stochastic differential equations calibrated to historical data since 1926 for closed-form  
88 solutions and (ii) block bootstrap resampling of historical data since 1926 (Anarkulova et al. (2022); Cogneau  
89 and Zakalmouline (2013); Politis and Romano (1994)) for neural network-based numerical solutions. Note  
90 that the bootstrap resampling of historical data captures *all* empirical qualities of actual returns, including  
91 potentially sophisticated volatility dynamics, which may not be reflected in the stylized settings of stochastic  
92 differential equations.

93 In more detail, the main contributions of this paper are as follows:

- 94 (i) We construct dynamic (multi-period) investment strategies which maximize the information ratio (IR)  
95 of the active portfolio manager (or simply “investor”) relative to standard investment benchmarks using  
96 a LETF or a VETF on the same underlying equity index as well as bonds. The IR is chosen due to  
97 its popularity in investment practice (Bajeux-Besnainou et al. (2013); Bolshakov and Chincarini (2020);  
98 Hassine and Roncalli (2013); Israelsen and Cogswell (2007)), so that the results are not just of academic  
99 interest but also of practical relevance to institutional investors.
- 100 (ii) Under stylized assumptions including parametric dynamics for the underlying assets, we present closed-  
101 form IR-optimal dynamic investment strategies which enable us to obtain intuition regarding the expected  
102 behavior of IR-optimal investment strategies in more general settings. Note that the closed-form solutions  
103 allow for jump-diffusion dynamics of the equity index, which as noted above is crucial to consider in the  
104 case of LETFs, so that our results contributes to the existing literature which is almost exclusively based  
105 on diffusion dynamics (see for example Giese (2010), Jarrow (2010) Leung and Santoli (2012), Leung et al.  
106 (2017), Leung and Sircar (2015), Wagalath (2014), Guasoni and Mayerhofer (2023)). However, in the  
107 context of  $\mathbb{Q}$  measure option pricing, Ahn et al. (2015) consider jump processes for LETFs.
- 108 (iii) Relaxing the stylized assumptions to allow for more general and practical conclusions, we implement  
109 a data-driven neural network approach to obtain optimal investment strategies using stationary block  
110 bootstrap resampled historical data (including proxy LETF returns) since 1926. This investment setting  
111 considers multi-asset portfolios, infrequent rebalancing, as well as multiple investment constraints including  
112 leverage restrictions and borrowing premiums. We analyze the dynamic IR-optimal investment strategies  
113 for different scenarios, including: (i) investing in a VETF and bonds but with no leverage allowed, (ii)  
114 investing in a VETF and bonds with different levels of maximum leverage allowed and different levels of  
115 borrowing premiums being applicable, and (iii) investing in a LETF on the same equity market index as  
116 well as bonds with no leverage being allowed.
- 117 (iv) We find that IR-optimal investment strategies involving LETFs are fundamentally *contrarian*. This finding  
118 aligns to the empirical asset allocation behavior observed by DeVault et al. (2021) in their analysis of the  
119 SEC filings by institutional fund managers, whereby managers seem to decrease their holdings in LETFs  
120 after observing strong recent investment performance. In terms of investment performance, we find that  
121 IR-optimal strategies including the LETF are not only more likely to outperform the benchmark than  
122 IR-optimal strategies derived using the corresponding VETF, but are able to achieve partial stochastic  
123 dominance over the investment benchmark in terms of portfolio value<sup>6</sup> (wealth).

124 Our results therefore encourage the reconsideration of the role of broad equity market LETFs within more so-  
125 phisticated dynamic investment strategies, and provide a potential additional motivation regarding the enduring  
126 popularity of LETFs observed in practice.

## 127 1.2 Intuition

128 In this section, we provide some insight into the potential advantages of including LETFs in optimal dynamic  
129 asset allocation. We will give an overview here, leaving the technical details to Section 3.1.

130 Suppose an investor allocates their initial wealth  $W(0)$  to US 30-day T-bills and either a LETF ( $\beta = 2$ ) or  
131 a VETF on a broad US equity market index  $S$  at time  $t_0 = 0$ , and does not rebalance the portfolio over the  
132 holding time horizon  $\Delta t > 0$ . We discuss in more detail in Section 3 how the LETF and VETF can be viewed

---

<sup>6</sup>For a definition of partial stochastic dominance, see Atkinson (1987); Van Staden et al. (2021)

133 as derivative contracts on underlying  $S$  costing  $F_\ell(0)$  (LETF) and  $F_v(0)$  (VETF) to purchase, with payoffs of  
 134  $F_\ell(\Delta t)$  and  $F_v(\Delta t)$ , respectively.

135 Assume that the underlying index price follows a geometric Brownian motion, which implies that the price  
 136 is always positive. In addition, assume a constant risk-free rate for short term bonds. It can be easily shown  
 137 that (see e.g. (Avellaneda and Zhang, 2010))

$$138 \quad \frac{F_\ell(\Delta t)}{F_\ell(0)} = \exp\{-c_\ell \cdot \Delta t\} \cdot f_\ell(\Delta t; \beta) \cdot \left(\frac{S(\Delta t)}{S(0)}\right)^\beta, \quad (1.1)$$

139 where

$$140 \quad f_\ell(\Delta t; \beta) = \exp\left\{-\left[(\beta - 1)r + \frac{1}{2}(\beta - 1)\beta\sigma^2\right] \cdot \Delta t\right\} \quad (1.2)$$

141  $c_\ell > 0$  is the expense ratio,  $r$  is the risk-free interest and  $\sigma$  is the volatility of return. In other words, payoff  
 142  $F_\ell(\Delta t)$  is a deterministic function of the terminal value of the underlying index. Of course,  $S(\Delta t)$  itself is  
 143 stochastic.

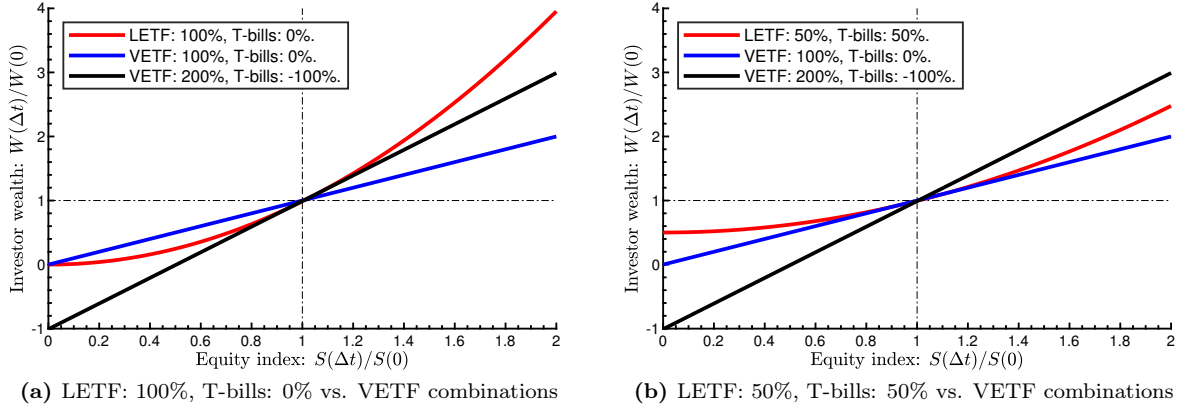
144 Since small values of maturity  $\Delta t$  can be undesirable due to frequent trading, and large values of  $\Delta t$  emphasize  
 145 the time- and volatility decay of simply holding the LETF  $F_\ell$ , suppose the investor chooses a convenient maturity  
 146 of  $\Delta t = 0.25$  years (one quarter). Figure 1.1 illustrates the payoff diagrams for the investor's wealth at maturity  
 147  $W(\Delta t)$  under different combinations of T-bills and an ETF, as a function of the value at maturity  $S(\Delta t)$  of  
 148 the underlying equity index. Similar payoff diagrams can be seen in Knapp (2023) and for the unleveraged case  
 149 in Bertrand and Prigent (2022).

150 For simplicity, we assume parametric asset dynamics for the T-bills and index  $S$  ( $S$  follows geometric  
 151 Brownian motion) in Figure 1.1 calibrated to US market data over 1926 to 2023. We impose realistic ETF  
 152 expense ratios, and a borrowing premium of 3% over the T-bill rate for short positions. Note that the assumption  
 153 of parametric asset dynamics is for purposes of intuition only, since the investment results in Section 5 do not  
 154 rely on any parametric assumptions. Leaving rigorous derivation for subsequent sections, we make the following  
 155 qualitative observations regarding the payoff diagrams for purposes of intuition:

- 156 • Figures 1.1 illustrate that we can characterize the payoffs of LETF investments as *call-like*. This suggests  
 157 that the addition of an LETF to a portfolio can be a useful way to add inexpensive leverage while preserving  
 158 downside protection, much like a usual call option. Provided that the investment in the LETF is itself  
 159 not funded by borrowing (i.e. the LETF position itself is not leveraged), the LETF payoff is always  
 160 non-negative due to limited liability even in the case of significant drops in the value of the underlying  
 161 equity index, in contrast with leveraged VETF positions.
- 162 • For calibrated geometric Brownian motion (GBM) dynamics for the equity index  $S$ , Figure 1.1 illustrates  
 163 that investing all wealth in the LETF with  $\beta = 2$  (Figure 1.1(a)) dominates the 2x leveraged investment  
 164 in the VETF (200% of wealth in VETF funded by borrowing and amount equal to 100% of wealth)  
 165 almost everywhere, but underperforms a 100% investment in the VETF for negative underlying index  
 166 returns (i.e. when  $S(\Delta t)/S(0) < 1$ ). By contrast, investing 50% of wealth in the LETF (Figure 1.1(a))  
 167 and the remaining 50% in T-bills dominates the payoff of investing 100% of wealth in the VETF almost  
 168 everywhere, but underperforms a 2x leveraged investment in the VETF for positive underlying index  
 169 returns ( $S(\Delta t)/S(0) > 1$ ). Note that under GBM dynamics for  $S$ , the terminal wealth  $W(\Delta t)$  conditional  
 170 on the terminal value  $S(\Delta t)$  is deterministic for both VETF and LETF investments (see Section 3.1).

171 When the underlying index is modelled by a jump process, due to limited liability, a correction to the index  
 172 price is needed so that LETF remains positive, see Section 3.1 for a more detailed discussion. This makes the  
 173 payoff relation between LETF and the underlying index stochastic. While qualitatively similar observations as  
 174 in the case of no jumps (Figure 1.1) apply to the *median* payoffs of the LETF investments, allowing for jumps  
 175 in the underlying asset dynamics can affect the LETF payoff significantly, and jumps are therefore critical to  
 176 incorporate in the investor's strategy. However, most of the existing literature on investment strategies with  
 177 LETFs only allows for pure diffusion processes for the equity index underlying the LETF (Giese (2010) Jarrow  
 178 (2010), Leung and Santoli (2012), Leung et al. (2017), Leung and Sircar (2015), Wagalath (2014), Guasoni and  
 180 Mayerhofer (2023)).

181 We emphasize that while Figure 1.1 is for the purposes of intuition only, it is nevertheless based on asset  
 182 dynamics calibrated to empirical US market data. Since even long-term investments (e.g. 10 years) can be  
 183 managed effectively using a *dynamic* investment strategy with for example quarterly rebalancing (i.e. at the



**Figure 1.1:** Payoffs when equity market index  $S$  follows calibrated GBM dynamics: Investor wealth gross return  $W(\Delta t)/W(0)$  as a function of underlying equity index gross return  $S(\Delta t)/S(0)$ ,  $\Delta t = 0.25$  (1 quarter), for different proportions of initial wealth  $W(0)$  invested in the LETF, VETF and T-bills at time  $t_0 = 0$ . Asset parameters are calibrated to US equity and bond market data over the period 1926:01 to 2023:12 (Appendix B), LETF and VETF expense ratios are assumed to be 0.89% and 0.06% respectively, and a borrowing premium of 3% over the T-bill rate is applicable to short positions. See Section 3.1 for a rigorous treatment of the illustrated relationships.

184 beginning of each quarter, the investor faces investment choices and associated outcomes such as those in Figure  
 185 1.1 ), this suggests that the benefits of LETFs could potentially be harnessed without being unduly affected by  
 186 the compounding effects as well as time- and volatility-decay. Our results show that this is indeed the case,  
 187 even if no parametric form of the underlying dynamics is assumed.

### 188 1.3 Organization

189 The remainder of the paper is organized as follows. Section 2 provides the general problem formulation, with  
 190 Section 3 presenting closed-form results obtained under stylized assumptions. Section 4 discusses a neural  
 191 network-based solution approach to obtain the optimal investment strategies numerically under multiple invest-  
 192 ment constraints. Finally, Section 5 presents indicative investment results and Section 6 concludes the paper,  
 193 with additional analytical and numerical results presented in Appendices A, B and C.

## 194 2 General problem formulation

195 In this section we formulate, in general terms, the dynamic portfolio optimization problem to be solved by an  
 196 active portfolio manager (simply “investor”) over a given time horizon  $[t_0 = 0, T]$ , where  $T > 0$  can be large  
 197 (e.g. 10 years). We assume that the investment performance of the investor is measured relative to that of a  
 198 given benchmark portfolio, as is typically the case for professional asset managers (see for example Alekseev and  
 199 Sokolov (2016); Kashyap et al. (2021); Korn and Lindberg (2014); Lehalle and Simon (2021); Zhao (2007)). To  
 200 this end, for any  $t \in [t_0 = 0, T]$ , let  $W(t)$  and  $\hat{W}(t)$  denote the portfolio value (or informally, simply the “wealth”)  
 201 of the investor and benchmark portfolios, respectively. The same initial wealth  $w_0 := W(t_0) = \hat{W}(t_0) > 0$  is  
 202 assumed to ensure that the performance comparison remains fair. The investor’s strategy is based on investing  
 203 in any of a set of  $N_a$  candidate assets, while the benchmark is defined in terms of  $\hat{N}_a$  potentially different  
 204 underlying assets.

205 In more detail, if  $\hat{\mathbf{X}}(t)$  denotes the state (or informally, the information) used in obtaining the benchmark  
 206 asset allocation strategy at time  $t \in [t_0, T]$ , let  $\hat{p}_j(t, \hat{\mathbf{X}}(t))$  denotes the proportion of the benchmark wealth  
 207  $\hat{W}(t)$  invested in asset  $j \in \{1, \dots, \hat{N}_a\}$ . The vector  $\hat{\mathbf{p}}(t, \hat{\mathbf{X}}(t)) = (\hat{p}_j(t, \hat{\mathbf{X}}(t)) : j = 1, \dots, \hat{N}_a) \in \mathbb{R}^{\hat{N}_a}$  then  
 208 denotes the asset allocation or investment strategy of the benchmark at time  $t \in [t_0, T]$ .

209 Similarly, if  $\mathbf{X}(t)$  denotes the state or information incorporated by the investor in making their asset  
 210 allocation decision, let  $p_i(t, \mathbf{X}(t))$  denote the proportion of the investor’s wealth  $W(t)$  invested in asset  $i \in$   
 211  $\{1, \dots, N_a\}$ , with  $\mathbf{p}(t, \mathbf{X}(t)) = (p_i(t, \mathbf{X}(t)) : i = 1, \dots, N_a) \in \mathbb{R}^{N_a}$  denoting the investor’s asset allocation or  
 212 investment strategy at time  $t \in [t_0, T]$ .

213 The set of portfolio rebalancing events is denoted by  $\mathcal{T} \subseteq [t_0, T]$ , where we consider  $\mathcal{T} = [t_0, T]$  in the case of  
 214 continuous rebalancing (Section 3), or a discrete subset  $\mathcal{T} \subset [t_0, T]$  in the case of discrete rebalancing (Section

215 4). Given the set  $\mathcal{T}$ , the investor and benchmark investment strategies are respectively defined as

$$216 \quad \mathcal{P} = \{\mathbf{p}(t, \mathbf{X}(t)), t \in \mathcal{T}\}, \quad \text{and} \quad \hat{\mathcal{P}} = \{\hat{\mathbf{p}}(t, \hat{\mathbf{X}}(t)), t \in \mathcal{T}\}. \quad (2.1)$$

217 It is typical for the investor to be subject to investment constraints, which are encoded by the set  $\mathcal{A}$  of admissi-  
 218 ble controls. In the simplest case, admissible investor strategies  $\mathcal{P} \in \mathcal{A}$  are such that  $\mathcal{P} = \{\mathbf{p}(t, \mathbf{X}(t)) \in \mathcal{Z} : t \in \mathcal{T}\}$ ,  
 219 with  $\mathcal{Z}$  denoting the admissible control space. More complex constraints require a more careful formulation of  
 220  $\mathcal{A}$  and  $\mathcal{Z}$ , see for example Section 4.

221 Since the investor aims to construct  $\mathcal{P}$  to *outperform* the benchmark strategy  $\hat{\mathcal{P}}$ , Assumption 2.1 below  
 222 outlines some general assumptions regarding the investment benchmark  $\hat{\mathcal{P}}$ . Note that Assumption 2.1 aligns  
 223 with investment practice and is important for assessing the relevance of LETFs when constructing portfolios for  
 224 outperforming a benchmark - see further discussion in Remark 2.1 below.

225 **Assumption 2.1.** (General assumptions regarding the benchmark strategy  $\hat{\mathcal{P}}$ ) We make the following general  
 226 assumptions regarding the benchmark strategies considered in this paper:

- 227 (i) The investor can observe the asset allocation  $\hat{\mathbf{p}}(t, \hat{\mathbf{X}}(t))$  of the benchmark strategy at each  $t \in \mathcal{T}$ .
- 228 (ii) The sets of investable assets available to the investor and benchmark, respectively, do not necessarily  
 229 correspond. In particular, the benchmark strategy may invest in assets which the investor is unwilling or  
 230 unable to invest in, or the investor might consider investing in a much larger universe of investable assets  
 231 than those included in the benchmark.  $\square$

232 Remark 2.1 highlights some key observations regarding Assumption 2.1.

233 **Remark 2.1.** (Clarification of benchmark assumptions) With regards to Assumption 2.1, we observe the fol-  
 234 lowing:

- 235 (i) Observable benchmarks play a key role in performance reporting for many institutional investors, since  
 236 active portfolio managers often explicitly pursue the outperformance of a *predetermined* investment bench-  
 237 mark (see for example Alekseev and Sokolov (2016); Kashyap et al. (2021); Korn and Lindberg (2014);  
 238 Lehalle and Simon (2021); Zhao (2007)). As a result, the benchmark is clearly defined and transpar-  
 239 ent in the sense of the underlying asset allocation, which often incorporates broad market indices and  
 240 bonds. In the case of pension funds, the benchmark (or “reference”) portfolios are usually constructed  
 241 using traded assets in fixed proportions. Examples include the Canadian Pension Plan (CPP) with a base  
 242 reference portfolio of 15% Canadian government bonds and 85% global equity (Canadian Pension Plan  
 243 (2022)), or the Norwegian government pension plan (“Government Pension Fund Global”, or GPFPG) using  
 244 a benchmark portfolio of 70% equities and 30% bonds (Government Pension Fund Global (2022)).
- 245 (ii) Active portfolio managers often consider not only different but indeed larger/broader sets of assets than  
 246 the benchmark. For example, pension funds might include private equity whereas the benchmark might be  
 247 based on publicly traded assets only (see for example Canadian Pension Plan (2022)). In the assessment  
 248 of the effect of replacing VETFs with LETFs discussed in Section 3, we consider scenarios where the  
 249 benchmark strategy is defined in terms of a broad stock market index, but the investor might not be able  
 250 to invest directly in the index itself, and invests instead in an ETF (VETF or LETF) replicating the index  
 251 returns. Since the ETFs only replicate (a multiple of) the index returns *before* costs, the existence of a  
 252 non-zero ETF expense ratios implies that investing in ETFs is not exactly the same as investing in the  
 253 underlying index, i.e. an ETF and its underlying index can be viewed as different assets. Assumption  
 254 2.1(ii) is therefore relevant to an assessment of the role of a VETF or LETF within portfolios designed to  
 255 beat a broad equity index-based investment benchmark.  $\square$

256 Let  $E_{\mathcal{P}}^{t_0, w_0}[\cdot]$  and  $Var_{\mathcal{P}}^{t_0, w_0}[\cdot]$  denote the expectation and variance, respectively, given initial wealth  $w_0 =$   
 257  $W(t_0) = \hat{W}(t_0)$  at time  $t_0 = 0$  and using admissible investor strategy  $\mathcal{P} \in \mathcal{A}$  over  $[t_0, T]$ . As a result of  
 258 Assumption 2.1, the benchmark strategy  $\hat{\mathcal{P}}$  remains implicit and fixed for notational simplicity.

259 As discussed in the Introduction, for an investment objective measuring outperformance, we wish to maximize  
 260 the information ratio (IR) of the investor relative to the benchmark. In the context of dynamic trading with  
 261 strategies of the form (2.1), the information ratio (IR) is defined as (Bajeux-Besnainou et al. (2013); Goetzmann  
 262 et al. (2002))

$$\mathcal{IR}_{\mathcal{P}}^{t_0, w_0} = \frac{E_{\mathcal{P}}^{t_0, w_0} [W(T) - \hat{W}(T)]}{Stdev_{\mathcal{P}}^{t_0, w_0} [W(T) - \hat{W}(T)]}. \quad (2.2)$$

Maximizing the IR (2.2) can be achieved by solving a mean-variance (MV) optimization problem (Bajeux-Besnainou et al. (2013)),

$$\sup_{\mathcal{P} \in \mathcal{A}} \left\{ E_{\mathcal{P}}^{t_0, w_0} [W(T) - \hat{W}(T)] - \rho \cdot Var_{\mathcal{P}}^{t_0, w_0} [W(T) - \hat{W}(T)] \right\}, \quad \rho > 0, \quad (2.3)$$

where  $\rho$  denotes a scalarization parameter.

Using the embedding technique of Li and Ng (2000); Zhou and Li (2000), we solve (2.3) by formulating the equivalent problem (Van Staden et al. (2023))

$$(IR(\gamma)) : \quad \inf_{\mathcal{P} \in \mathcal{A}} E_{\mathcal{P}}^{t_0, w_0} \left[ \left( W(T) - [\hat{W}(T) + \gamma] \right)^2 \right], \quad \gamma > 0, \quad (2.4)$$

where  $\gamma > 0$  denotes the embedding parameter. As discussed in Van Staden et al. (2023), the parameter  $\gamma$  can be viewed as the investor's (implicit) target for benchmark outperformance formulated in terms of terminal wealth.

**Remark 2.2.** (Time consistency.) Note that the control for problem (2.3) is formally the pre-commitment policy, i.e. not time consistent. However, the pre-commitment policy solution of Problem (2.3) is identical to the strategy for an induced time consistent policy (Forsyth, 2020; Strub et al., 2019), and hence it is implementable.<sup>7</sup> The induced time consistent strategy in this case is the target based Problem (2.4), with a fixed value of  $\gamma, \forall t > 0$ . The relationship between pre-commitment and implementable target-based schemes in the mean-variance context is discussed in Menoncin and Vigna (2017); Vigna (2014, 2020, 2022). We consider the policy followed by the investor for  $t > 0$  to be the implementable solution of Problem (2.4) with a fixed value of  $\gamma$ . This is identical to the solution of Problem (2.3) as seen at  $t = 0$ .

### 3 Closed-form solutions

To obtain the valuable intuition regarding the characteristics of IR-optimal investment strategies incorporating a LETF or VETF on a broad equity market index, we present closed-form solutions to the IR problem (2.4) under stylized assumptions. Remark 3.1 emphasizes that these assumptions are required for the derivation of closed-form solutions in this section only.

**Remark 3.1.** (Relaxing closed-form assumptions) The closed-form results presented in this section require stylized assumptions (Assumption 3.1 and Assumption 3.2 below), but we will use numerical techniques (Section 4) and present indicative investment results (Section 5) where these assumptions are not required. The investment problem is solved numerically in a setting where the following is allowed: (i) no restrictions on the number of underlying assets, (ii) no parametric assumptions are required for the dynamics of the underlying assets, (iii) discrete portfolio rebalancing is used, (iv) leverage is restricted and in some scenarios not allowed at all, (v) nonzero borrowing premiums over the risk-free rate are applicable when funding leveraged positions, (vi) no trading in the event of insolvency can occur, and (vii) more general benchmark strategies are allowed, though for illustrative purposes we will use constant proportion strategies due to their popularity in practical applications.  $\square$

The first set of general assumptions for the derivation of the closed-form solution in this section is outlined in Assumption 3.1.

**Assumption 3.1.** (Stylized assumptions for closed-form solutions) For the purposes of obtaining closed-form solutions in this section, we assume the following:

<sup>7</sup>An implementable strategy has the property that the investor has no incentive to deviate from the strategy computed at time zero at later times (Forsyth, 2020).

- 301 (i) The benchmark investment strategy (asset allocation) is a deterministic function of time defined in terms  
302 of the 30-day T-bills (“risk-free” asset) denoted by  $B$  and a broad equity market index (“risky” asset)  
303 denoted by  $S$ . Note that any known deterministic benchmark strategy clearly satisfies Assumption 2.1,  
304 and includes as a special case the constant proportion strategies which are popular benchmarks used in  
305 practice (see for example Canadian Pension Plan (2022), Government Pension Fund Global (2022)).
- 306 (ii) We consider two investors, each optimizing their respective portfolios relative to the same benchmark. Both  
307 investors are assumed to be unable or unwilling to invest directly the underlying broad equity market index  
308 itself (i.e. replicate the index with individual stocks), and instead invests in ETFs referencing the index.  
309 The first investor, informally referred to as the “VETF investor”, allocates wealth to two underlying assets,  
310 namely 30-day T-bills  $B$  and a VETF  $F_v$  with expense ratio  $c_v > 0$ , where the VETF simply replicates  
311 the instantaneous returns of the index  $S$  before costs. The second investor, informally referred to as the  
312 “LETF investor”, allocates wealth to two underlying assets, namely 30-day T-bills  $B$  and a LETF  $F_\ell$  with  
313 expense ratio  $c_\ell > 0$ , where the LETF returns  $\beta > 1$  times the instantaneous returns of the index  $S$  before  
314 costs.
- 315 (iii) Parametric dynamics for all underlying assets are assumed, including jump-diffusion dynamics for the  
316 broad equity market index  $S$  - see (3.1)-(3.2), (3.4) and (3.8) below.  $\square$

317 Table 3.1 provides an example of an investment scenario consistent with Assumption 3.1(i)-(ii), which will  
318 be used for the illustration of the closed-form solutions of this section.

**Table 3.1:** Closed-form solutions - Candidate assets and benchmark: Example of the investment scenario outlined in Assumption 3.1(i)-(ii), which will be used when illustrating the closed-form solutions in this section. The constant proportion benchmark has been chosen to align with typical benchmarks used by pension funds, while the indicative expense (or cost) ratios are chosen from the range of expense ratios of VETFs and LETFs on broad equity market indices typically observed in practice.

Underlying assets			Benchmark	Investor candidate assets	
Label	Value	Asset description		Using VETF	Using LETF
T30	$B(t)$	30-day Treasury bill	30%	✓	✓
Market	$S(t)$	Market portfolio (broad equity market index)	70%	-	-
VETF	$F_v(t)$	Vanilla or standard/unleveraged ETF (VETF) replicating the returns of the market portfolio $S(t)$ , with expense ratio $c_v = 0.06\%$	0%	✓	-
LETF	$F_\ell(t)$	Leveraged ETF (LETF) with daily returns replicating $\beta = 2$ times the daily returns of the market portfolio $S(t)$ , with expense ratio $c_\ell = 0.89\%$	0%	-	✓

319 We assume that the underlying index  $S$  can follow any of the commonly-encountered jump diffusion processes  
320 in finance (see for example Kou (2002); Merton (1976)), resulting in the following assumed dynamics for  $B$  and  
321  $S$ , respectively,  
322

$$323 \frac{dB(t)}{B(t)} = r \cdot dt, \quad (3.1)$$

$$324 \frac{dS(t)}{S(t^-)} = (\mu - \lambda \kappa_1^s) dt + \sigma \cdot dZ + d \left( \sum_{i=1}^{\pi(t)} (\xi_i^s - 1) \right). \quad (3.2)$$

325 In (3.1)-(3.2),  $r$  denotes the continuously compounded risk-free rate,  $\pi(t)$  denotes a Poisson process with  
326 intensity  $\lambda \geq 0$ , while  $\mu$  and  $\sigma$  denote the drift and volatility coefficients, respectively, under the objective (real-  
327 world) probability measure.  $\xi_i^s$  are i.i.d. random variables with the same distribution as  $\xi^s$ , which represents  
328 the jump multiplier associated with the  $S$ -dynamics, and we define

$$329 \kappa_1^s = \mathbb{E}[\xi^s - 1], \quad \kappa_2^s = \mathbb{E}[(\xi^s - 1)^2], \quad (3.3)$$

330 which can be obtained using a given probability density function (pdf) of  $\xi^s$ , denoted by  $G(\xi^s)$ . Finally,  
331 for any functional  $\psi(t), t \in [t_0, T]$ , we use  $\psi(t^-)$  and  $\psi(t^+)$  as shorthand notation for the one-sided limits

332  $\psi(t^-) = \lim_{\epsilon \downarrow 0} \psi(t - \epsilon)$  and  $\psi(t^+) = \lim_{\epsilon \downarrow 0} \psi(t + \epsilon)$ , respectively. Note that we can recover the assumption of  
 333 geometric Brownian motion (GBM) dynamics for  $S$  by simply setting the intensity  $\lambda \equiv 0$  in (3.2).

334 Since the VETF  $F_v$  with expense ratio  $c_v$  is a vanilla/standard ETF simply replicating the returns of  $S$   
 335 before costs, it has dynamics given by

$$336 \quad \frac{dF_v(t)}{F_v(t^-)} = \frac{dS(t)}{S(t^-)} - c_v \cdot dt$$

$$337 \quad = (\mu - \lambda \kappa_1^s - c_v) \cdot dt + \sigma \cdot dZ + d \left( \sum_{i=1}^{\pi(t)} (\xi_i^s - 1) \right). \quad (3.4)$$

338 In contrast, the LETF  $F_\ell$  with expense ratio  $c_\ell > 0$  aims at replicating  $\beta > 1$  times the instantaneous returns  
 339 of the underlying broad stock market index  $S$  before costs, and therefore has dynamics *approximately* given by

$$340 \quad \frac{dF_\ell(t)}{F_\ell(t^-)} \simeq \beta \frac{dS(t)}{S(t^-)} - [(\beta - 1)r + c_\ell] dt. \quad (3.5)$$

341 It should be emphasized that (3.5) is only an approximation. Since the investor in an LETF has limited liability,  
 342 exact equality in (3.5) only holds in the case where there are no jumps in the dynamics of  $S$ . In the case of pure  
 343 GBM dynamics,  $F_\ell$  can never become negative, hence limited liability is irrelevant. As a result, (3.5) is indeed  
 344 used to model LETF dynamics in the literature where the analysis is limited to GBM dynamics for  $S$  (see  
 345 for example Avellaneda and Zhang (2010), Jarrow (2010), Guasoni and Mayerhofer (2023)), with the notable  
 346 exception of Ahn et al. (2015) in the context of  $\mathbb{Q}$  measure dynamics.

347 However, in the case of jump-diffusion dynamics for  $S$ , (3.5) is not quite correct since the LETF investor  
 348 is protected by limited liability. In more detail, if there is a jump with multiplier  $\xi^s$  in the underlying index  
 349 (3.2) at a specific time  $t$ , then the approximation (3.5) suggests that the value of the LETF would jump to  
 350  $F_\ell(t) = F_\ell(t^-) \cdot [1 + \beta(\xi^s - 1)]$ .

351 Therefore, in the case of a large downward jump in the underlying index, in particular where the jump  
 352 multiplier satisfies  $\xi^s < (\beta - 1)/\beta$ , the approximation (3.5) implies that  $F_\ell(t) < 0$ , which cannot occur due  
 353 to limited liability of the LETF. Instead, if it is indeed the case that  $\xi^s \leq (\beta - 1)/\beta$ , the value of the LETF  
 354 simply drops to zero, i.e.  $F_\ell(t) \equiv 0$ .

355 We can therefore model the limited liability of the LETF  $F_\ell$  by observing that  $F_\ell$  therefore experiences  
 356 jumps which are related to, but not necessarily exactly the same as the jumps experienced by the underlying  
 357 index  $S$ . To this end, we define a jump multiplier  $\xi^\ell$  for the  $F_\ell$ -dynamics in terms of the jump multiplier  $\xi^s$  in  
 358 the  $S$ -dynamics as

$$359 \quad \xi^\ell = \begin{cases} \xi^s & \text{if } \xi^s > (\beta - 1)/\beta, \\ \frac{(\beta - 1)}{\beta} & \text{if } \xi^s \leq (\beta - 1)/\beta. \end{cases} \quad (3.6)$$

360 The second case in (3.6) enforces the limited liability of the LETF investor in the case of large downward jumps,  
 361 i.e.  $F_\ell(t) \equiv 0$  if  $\xi^s \leq (\beta - 1)/\beta$ . For subsequent use, we also define the following quantities involving the LETF  
 362 jump multiplier  $\xi^\ell$ ,

$$363 \quad \kappa_1^\ell = \mathbb{E}[\xi^\ell - 1], \quad \kappa_2^\ell = \mathbb{E}[(\xi^\ell - 1)^2], \quad \kappa_{\chi^s}^{\ell,s} = \mathbb{E}[(\xi^\ell - 1)(\xi^s - 1)]. \quad (3.7)$$

364 Given (3.2), (3.5) and (3.6), the LETF dynamics correctly incorporating jumps is therefore given by

$$365 \quad \frac{dF_\ell(t)}{F_\ell(t^-)} = [\beta(\mu - \lambda \kappa_1^s) - (\beta - 1)r - c_\ell] \cdot dt + \beta \sigma \cdot dZ + \beta \cdot d \left( \sum_{i=1}^{\pi(t)} (\xi_i^\ell - 1) \right), \quad (3.8)$$

366 where  $\xi_i^\ell$  are i.i.d. random variables with the same distribution as  $\xi^\ell$ , which represents the jump multiplier  
 367 associated with the  $F_\ell$ -dynamics (3.6). We highlight the following observations regarding the dynamics (3.2),  
 368 (3.4) and (3.8).

369 (i) The jumps in the underlying index  $S$ , the VETF  $F_v$  and LETF  $F_\ell$  occur at the same time, so the Poisson  
 370 process  $\pi(t)$  and intensity  $\lambda$  are the same in (3.2), (3.4) and (3.8). More formally, the processes (3.2),

(3.4) and (3.8) have the same Poisson random measures, but the compensated Poisson random measure is different for the  $F_\ell$ -dynamics since the LETF jump sizes are slightly different due to (3.6).

(ii) The dynamics (3.4) and (3.8) implicitly assume that the ETFs have negligible tracking errors, while having non-negligible expense ratios. While ETF expense ratios can indeed be material, especially for LETFs, the assumption that tracking errors are negligible are often employed in the literature (see for example Bansal and Marshall (2015), Leung and Sircar (2015)). Given the recent developments in designing replication strategies for LETFs that remain robust even during periods of market volatility (see for example Guasoni and Mayerhofer (2023)), this appears to be a reasonable assumption especially in the case of the most popular VETFs and LETFs written on the major stock market indices.

(iii) As noted in the Introduction, we limit the analysis to the case of LETFs where  $\beta > 1$  (i.e. “bullish” LETFs). However, the dynamics (3.8) could also be applicable to inverse or “bearish” ETFs where  $\beta < 1$  (see for example Jarrow (2010)), but adjustments are usually required to incorporate the time-dependent borrowing cost involved in short-selling the particular components of the underlying replication basket each time  $t$  (see for example Avellaneda and Zhang (2010)).

**Remark 3.2.** (Relationship to Ahn et al. (2015)) In Ahn et al. (2015), the authors model the limited liability of the LETF by taking the point of view of the manager of the LETF. The manager must purchase insurance to handle the cases where the manager’s position becomes negative. In this work, we simply take the point of view of the holder the LETF (not the manager), who has no exposure to the possible negative value of the manager’s position. The cost of this insurance (to the manager) is assumed to be passed on to the LETF investor as part of the fee  $c_\ell$  charged by the manager, which is easily observable.

### 3.1 Intuition: lump sum investment scenario

As a simple and intuitive illustration of the potential and risks of using LETFs vs. VETFs, we consider a simple version of the general formulation of the problem as outlined in Section 2. Specifically, we consider a lump sum investment scenario as discussed in the Introduction (see Figures 1.1 and 3.1), where the initial wealth  $w_0 = W(t_0) = \tilde{W}(t_0) > 0$  is invested at time  $t_0 = 0$  with no intermediate intervention/rebalancing until the terminal time  $T = \Delta t = 0.25$  years (i.e. one quarter). In the notation of Section 2, we therefore have a trivial set of rebalancing events  $\mathcal{T} = [t_0]$ .

First, we consider the implications of the underlying asset dynamics without referencing the investment strategy (i.e. wealth allocation to assets). The dynamics (3.1)-(3.2) imply that

$$\frac{B(\Delta t)}{B(0)} = \exp\{r \cdot \Delta t\}, \quad (3.9)$$

$$\frac{S(\Delta t)}{S(0)} = \exp\left\{\left(\mu - \lambda \kappa_1^s - \frac{1}{2}\sigma^2\right) \cdot \Delta t + \sigma \cdot Z(\Delta t) + \sum_{i=1}^{\pi(\Delta t)} \log \xi_i^s\right\}, \quad (3.10)$$

where the values  $B(0)$  and  $S(0)$  are observable at time  $t_0 = 0$ . In the case of the VETF, we simply have

$$\frac{F_v(\Delta t)}{F_v(0)} = \exp\{-c_v \cdot \Delta t\} \cdot \left(\frac{S(\Delta t)}{S(0)}\right). \quad (3.11)$$

In the case of the LETF, we have

$$\frac{F_\ell(\Delta t)}{F_\ell(0)} = \exp\{-c_\ell \cdot \Delta t\} \cdot f_\ell(\Delta t; \beta) \cdot \tilde{Y}_\ell(\Delta t; \beta) \cdot \left(\frac{S(\Delta t)}{S(0)}\right)^\beta, \quad (3.12)$$

where

$$f_\ell(\Delta t; \beta) = \exp\left\{-\left[(\beta - 1)r + \frac{1}{2}(\beta - 1)\beta\sigma^2\right] \cdot \Delta t\right\}, \quad \text{and} \quad \tilde{Y}_\ell(\Delta t; \beta) = \prod_{i=1}^{\pi(\Delta t)} \left[\frac{1 + \beta(\xi_i^\ell - 1)}{(\xi_i^s)^\beta}\right]. \quad (3.13)$$

Expression (3.12) for the case where there are no jumps, is given in Avellaneda and Zhang (2010).

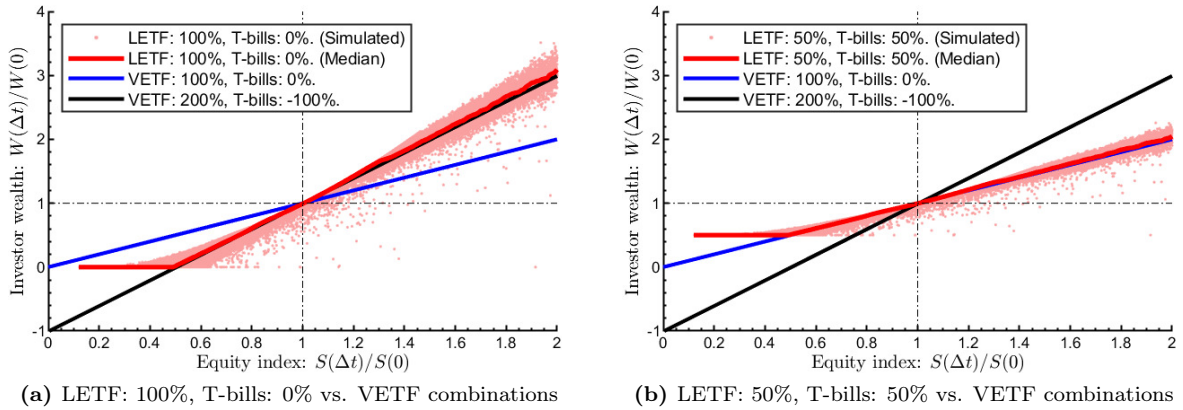
For purposes of intuition, consider the *special case* where the underlying index  $S$  experiences *zero growth/decline* over the time horizon  $\Delta t$ . If  $S(\Delta t) = S(0)$ , it is clear that the LETF will perform worse than the VETF, i.e. assuming  $\beta > 1$ ,  $F_\ell(\Delta t) < F_v(\Delta t)$ , due to the following:

- 410 • Decay due to volatility: The term  $f_\ell(\Delta t; \beta) < 1$ , which only affects the LETF, is dominated by the  
411 diffusive volatility  $\sigma$  in the underlying  $S$ -dynamics. All else being equal, the larger volatility of  $S$ , the  
412 worse the performance of the LETF relative to the VETF, with the limiting case  $\lim_{\sigma \rightarrow \infty} f_\ell(\Delta t; \beta) = 0$ .
- 413 • Time decay: Even if there is no change in the value of the underlying index,  $S(\Delta t) = S(0)$ , the value of  
414 the LETF tends to zero if held for a long time, since  $\lim_{\Delta t \rightarrow \infty} f_\ell(\Delta t; \beta) = 0$ .
- 415 • Costs and interest rates: Expense ratios for LETFs are typically substantially higher than for VETFs,  
416  $0 < c_v \ll c_\ell$ . In addition, all else being equal, increasing interest rates  $r > 0$  also decreases  $f_\ell(\Delta t; \beta)$ .  
417 However, while these effects further decrease the value of the VETF relative to the LETF, they are  
418 expected to be comparatively small compared to the other effects.
- 419 • Decay due to jumps: It can be shown that the term  $\tilde{Y}_\ell(\Delta t; \beta)$ , which only affects the LETF, satisfies  
420  $\tilde{Y}_\ell(\Delta t; \beta) \leq 1$ , where the maximum value ( $\tilde{Y}_\ell(\Delta t; \beta) = 1$ ) is achieved either when there are no jumps in  
421 the value of the underlying over  $[0, \Delta t]$  (i.e.  $\pi(\Delta t) = 0$ ), or if there are jumps but they all satisfy  $\xi_i^s = 1$   
422 which has probability of almost surely zero. In other words, when  $S(\Delta t) = S(0)$ , the mere presence of  
423 jumps in the underlying  $S$  decreases the value of the LETF relative to the VETF via the term  $\tilde{Y}_\ell(\Delta t; \beta)$ .  
424 This is illustrated in Figure 3.1 at the point where  $S(\Delta t)/S(0) = 1$ .

425 The preceding observations summarize what are effectively the standard objections to LETFs that can be found  
426 in the literature, the only addition being the rigorous treatment of jumps in the LETF dynamics and the  
427 associated jump decay component.

428 Next, we discuss lump sum investment strategies, i.e. wealth allocation to assets at time  $t_0 = 0$  with no  
429 subsequent rebalancing prior to maturity  $T = \Delta t$ . For simplicity, we consider a constant proportion benchmark  
430 strategy  $\hat{\mathcal{P}} = \hat{\mathcal{p}}(t_0) := (1 - \hat{p}_s, \hat{p}_s)$ , where  $\hat{p}_s$  denotes the proportion of benchmark wealth  $\hat{W}(t_0) = w_0$  invested  
431 in the broad equity market index  $S$  at time  $t_0$ , with the remaining proportion  $(1 - \hat{p}_s)$  invested in 30-day T-bills.  
432 To emphasize that the benchmark wealth at the end of the investment time horizon depends on  $\hat{p}_s$ , we use the  
433 notation  $\hat{W}(\Delta t; \hat{p}_s)$ , and observe that

$$434 \frac{\hat{W}(\Delta t; \hat{p}_s)}{w_0} = (1 - \hat{p}_s) \cdot \exp\{r \cdot \Delta t\} + \hat{p}_s \cdot \frac{S(\Delta t)}{S(0)}. \quad (3.14)$$



**Figure 3.1:** Payoffs when equity market index  $S$  follows calibrated jump-diffusion dynamics (Kou (2002) model): Investor wealth gross return  $W(\Delta t)/W(0)$  as a function of underlying equity index gross return  $S(\Delta t)/S(0)$ ,  $\Delta t = 0.25$  (1 quarter), for different proportions of initial wealth  $W(0)$  invested in the LETF, VETF and T-bills at time  $t_0 = 0$ . Asset parameters are calibrated to US equity and bond market data over the period 1926:01 to 2023:12 (Appendix B), LETF and VETF expense ratios are assumed to be 0.89% and 0.06% respectively, and a borrowing premium of 3% over the T-bill rate is applicable to short positions.

435  
436 The investor, being unable to invest directly in  $S$ , can combine an ETF investment with T-bills. We will  
437 assume that the investor does not short-sell the LETF or VETF<sup>8</sup>, but might short-sell the T-bills (i.e. borrow  
438 funds) to leverage their investment in the ETF, in which case a constant borrowing premium  $b \geq 0$  is added

<sup>8</sup>As can be seen from the relationship between the objective functions (2.3) and (2.4), optimizing the IR essentially places us within a variant of the Mean-Variance (MV) framework with a constant risk aversion parameter. In MV optimization (see for example Bensoussan et al. (2014); Van Staden et al. (2018)) with a constant risk aversion parameter, it is never optimal to short-sell

439 to the T-bill returns. In more detail, if  $p$  denotes the fraction of wealth  $W(t_0) = w_0$  that the LETF or VETF  
440 investors invest in their respective ETFs, an investment fraction  $p > 1$  in the ETF is funded by borrowing the  
441 amount  $(1 - p) \cdot w_0$  at an interest rate of  $(r + b)$ , so the T-bill dynamics applicable to the investors can be  
442 modified as

$$443 \quad \frac{\bar{B}(\Delta t)}{\bar{B}(0)} = \exp\{\bar{r}(p) \cdot \Delta t\}, \quad \text{where} \quad \bar{r}(p) = r + b \cdot \mathbb{I}_{[p > 1]}, \quad (3.15)$$

444 with  $\mathbb{I}_{[A]}$  denoting the indicator of the event  $A$ .

445 The VETF investor (see Assumption 3.1(ii)) specifies an investment strategy  $\mathcal{P}_v = \mathbf{p}_v(t_0) = (1 - p_v, p_v)$ ,  
446 where  $p_v$  denotes the fraction of wealth  $W(t_0) = w_0$  invested in the VETF  $F_v$  at time  $t_0 = 0$ , and the remaining  
447 fraction of wealth  $(1 - p_v)$  invested in 30-day T-bills. The VETF investor's wealth at the end of the investment  
448 time horizon,  $W_v(\Delta t; p_v)$  therefore satisfies

$$449 \quad \frac{W_v(\Delta t; p_v)}{w_0} = (1 - p_v) \cdot \exp\{\bar{r}(p_v) \cdot \Delta t\} + p_v \cdot \exp\{-c_v \cdot \Delta t\} \cdot \left(\frac{S(\Delta t)}{S(0)}\right). \quad (3.16)$$

450 Similarly, the LETF investor specifies investment strategy  $\mathcal{P}_\ell = \mathbf{p}_\ell(t_0) = (1 - p_\ell, p_\ell)$ , where  $p_\ell$  is the  
451 fraction of wealth  $W(t_0) = w_0$  invested in the LETF  $F_\ell$  at time  $t_0 = 0$ , and the remaining fraction of wealth  
452  $(1 - p_\ell)$  invested in 30-day T-bills. Using (3.12), the LETF investor's wealth at the end of the investment time  
453 horizon,  $W_\ell(\Delta t; p_\ell)$  therefore satisfies

$$454 \quad \frac{W_\ell(\Delta t; p_\ell)}{w_0} = (1 - p_\ell) \cdot \exp\{\bar{r}(p_\ell) \cdot \Delta t\} + p_\ell \cdot \exp\{-c_\ell \cdot \Delta t\} \cdot f_\ell(\Delta t; \beta) \cdot \tilde{Y}_\ell(\Delta t; \beta) \cdot \left(\frac{S(\Delta t)}{S(0)}\right)^\beta. \quad (3.17)$$

455 Varying  $p_v$  and  $p_\ell$  in (3.16) and (3.17) therefore trace out different payoffs for  $W_v(\Delta t; p_v)$  and  $W_\ell(\Delta t; p_\ell)$   
456 as a function of the (random) underlying index outcome  $S(\Delta t)$ , with specific choices of  $p_v$  and  $p_\ell$  illustrated  
457 in Figures 1.1 and 3.1. Note however that the wealth of the VETF investor  $W_v(\Delta t; p_v)$  is linear in  $S(\Delta t)$ , and  
458 conditional on  $S(\Delta t)$  the outcome  $W_v(\Delta t; p_v)$  is deterministic. However, this is not the case for the wealth  
459  $W_\ell(\Delta t; p_\ell)$  of the LETF investor, which has a power call-like payoff due to the  $[S(\Delta t)]^\beta$  term of (3.17) in  
460 conjunction with limited liability. Note that even if we condition on the value of  $S(\Delta t)$ , the wealth outcome  
461  $W_\ell(\Delta t; p_\ell)$  is *not* deterministic due to the presence of the jump term  $\tilde{Y}_\ell(\Delta t; \beta)$  in (3.17). However, if no jumps  
462 are present then  $W_\ell(\Delta t; p_\ell)$  conditional on  $S(\Delta t)$  is linear in  $[S(\Delta t)]^\beta$ , compare Figures 1.1 and 3.1.

463 Suppose the LETF and VETF investors want to choose values  $p_v^*$  and  $p_\ell^*$ , respectively, to maximize the IR  
464 (2.2) subject to an implicit target  $\gamma > 0$  in (2.4). In this setting of parametric asset dynamics, we can simulate  
465  $N_d$  paths of the underlying equity index using (3.10) use each path's information together.

466 First, we discretize possible values of the fractions  $p_v$  and  $p_\ell$  using a fine grid, so that using each discretized  
467 value of  $p_v$  and  $p_\ell$ , we can obtain the corresponding values of  $W_v^{(j)}(\Delta t; p_v)$  and  $W_\ell^{(j)}(\Delta t; p_\ell)$  respectively, along  
468 each path  $j = 1, \dots, N_d$ . Next, using a discretization of the objective (2.4) in this setting, we can find the  
469 approximate IR-optimal values  $p_v^*$  and  $p_\ell^*$  by exhaustive search over the grid by solving

$$470 \quad p_k^* = \arg \min_{p_k} \left\{ \frac{1}{N_d} \sum_{j=1}^{N_d} \left( W_k^{(j)}(\Delta t; p_k) - \left[ \hat{W}^{(j)}(\Delta t; \hat{p}_s) + \gamma \right] \right)^2 \right\}, \quad k \in \{v, \ell\}. \quad (3.18)$$

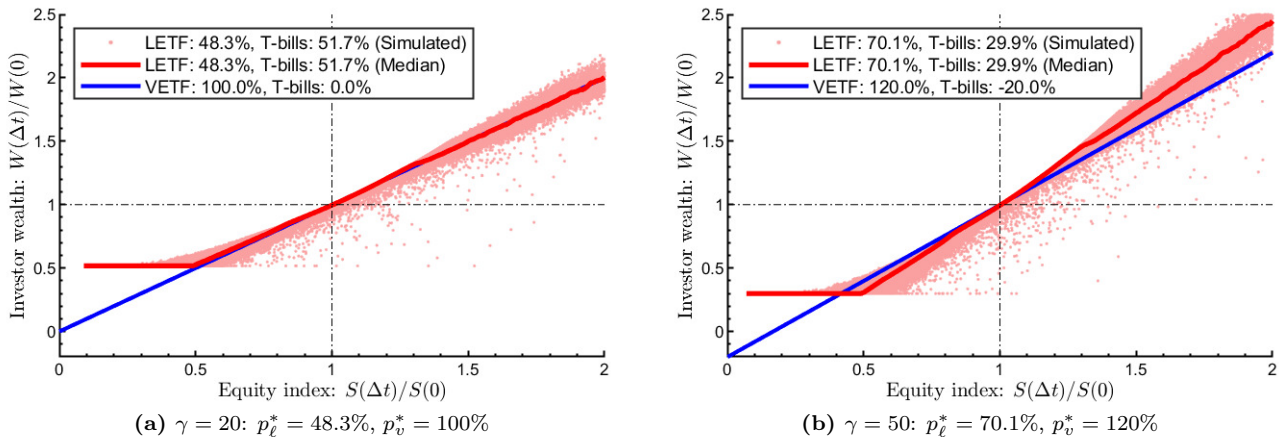
471 Figure 3.2 illustrates the results of this procedure for two different values of the implicit target,  $\gamma = 20$  and  
472  $\gamma = 50$ , where we observe the following:

- 473 • In the case of  $\gamma = 20$  in Figure 3.2(a), the VETF investor simply invests all wealth in the VETF ( $p_v^* =$   
474 100%), whereas the LETF investor invests slightly less than half of total wealth in the LETF ( $p_\ell^* = 48.3\%$ ).  
475 Note that the IR-optimal strategies in this case satisfy  $p_v^*/p_\ell^* = 2.070$ .
- 476 • With a significantly more aggressive benchmark outperformance target of  $\gamma = 50$  in Figure 3.2(b), the  
477 LETF investor now invests  $p_\ell^* = 70.1\%$  in the LETF, i.e. there is no need to leverage the LETF investment  
478 itself, whereas the VETF investor borrows 20% of wealth to invest  $p_v^* = 120\%$  in the VETF. This leverage  
479 is costly for the VETF investor due to the lack of downside protection and borrowing premiums, which  
480 can be seen in both the upside and extreme downside outcomes of Figure 3.2(b). In this case, we have  
481  $p_v^*/p_\ell^* = 1.712$ .

---

the risky asset. The subsequent results of Section 3 and the results of Ni et al. (2024) suggest that this observation also holds our investment scenario, i.e. it is never expected to be IR-optimal to short-sell high return/high volatility assets given the existence of low return/low volatility assets.

482 Comparing IR-optimal investment strategies implemented using a LETF or VETF with the same benchmark  
483 outperformance target  $\gamma > 0$ , the observation that  $p_v^*/p_\ell^* \approx \beta = 2$  holds in the cases illustrated in Figure 3.2  
484 is not an accident. This relationship is more rigorously discussed in the subsequent results of this section, but  
485 for now we note that exact equality  $p_v^*/p_\ell^* \equiv \beta$ , where  $\beta$  is the returns multiplier of the LETF, only holds  
486 for the IR-optimal investment strategies in the case of continuous rebalancing ( $\Delta t \downarrow 0$ ), zero expense ratios  
487 ( $c_v = c_\ell = 0$ ), zero borrowing premium over the risk-free rate  $r$  ( $b = 0$ ), and when no leverage restrictions are  
488 applicable. While this is a very restrictive set of assumptions,  $p_v^*/p_\ell^* \approx \beta$  is nevertheless a useful rule-of-thumb  
489 to keep in mind when comparing IR-optimal investment strategies in more general cases, a simple example  
490 being Figure 3.2. However, it should be emphasized that while the *strategies* might satisfy  $p_v^*/p_\ell^* \approx \beta$ , this does  
491 *not* mean that the ultimate investment *outcomes* for the LETF and VETF investors (such as investor wealth,  
492 benchmark outperformance etc.) have straightforward relationships except under very restrictive conditions,  
493 since the outcomes are affected by limited liability and the path-dependent role of jumps (see Figure 3.2).



**Figure 3.2:** Payoffs when equity market index  $S$  follows calibrated jump-diffusion dynamics (Kou (2002) model): Investor wealth gross return  $W(\Delta t)/W(0)$  as a function of underlying equity index gross return  $S(\Delta t)/S(0)$ ,  $\Delta t = 0.25$  (1 quarter), for different proportions of initial wealth  $W(0)$  invested in the LETF, VETF and T-bills at time  $t_0 = 0$ . Asset parameters are calibrated to US equity and bond market data over the period 1926:01 to 2023:12 (Appendix B), LETF and VETF expense ratios are assumed to be 0.89% and 0.06% respectively, and a borrowing premium of 3% over the T-bill rate is applicable to short positions.

494

### 495 3.2 Dynamically-optimal strategies under continuous rebalancing

496 Section 3.1 treated the lump-sum investment scenario with no subsequent intervention over  $(t_0, T]$ , i.e. the set  
497 of rebalancing times being simply  $\mathcal{T} = [t_0]$ . We now consider the other extreme, namely that of continuous  
498 rebalancing, where the set of rebalancing times is given by  $\mathcal{T} = [t_0, T]$ .

499 Derivation of closed-form optimal strategies necessarily requires stylized assumptions, in this case outlined in  
500 Assumption 3.2. As per Remark 3.1, we emphasize that these assumptions are not required for the subsequent  
501 results discussed in Section 5.

502 **Assumption 3.2.** (Stylized assumptions - continuous rebalancing) In the case of continuous rebalancing, for  
503 the purposes of obtaining closed-form solutions in this section, we assume the following:

- 504 (i) Assumption 3.1 holds, including parametric dynamics (3.1)-(3.2), (3.4) and (3.8) for the underlying assets.
- 505 (ii) The investor injects cash into the portfolio at a constant rate of  $q \geq 0$  per year. To ensure the wealth  
506 processes remain comparable, the identical rate of cash injection is assumed for the benchmark portfolio.
- 507 (iii) We assume continuous portfolio rebalancing ( $\mathcal{T} = [t_0, T]$ ), no investment constraints (i.e. no leverage  
508 limits or short-selling constraints), zero borrowing premium so that both borrowing and lending occurs  
509 at the risk-free rate  $r$ , and trading is allowed to continue in the event of insolvency. Note that these  
510 assumptions are standard in the derivation of closed-form solutions of multi-period portfolio optimization  
511 problems (see for example Zhou and Li (2000)). This implies that the investment in the LETF can itself  
512 be leveraged, which is plausible since even retail investors can borrow and invest in LETFs. However, the

513 degree to which leverage is required by either the LETF or VETF investors depends on the aggressiveness  
 514 of the outperformance target  $\gamma$ , as shown by the subsequent results.  $\square$

515 Since Assumption 3.1 remains applicable (see Assumption 3.2(i)), the deterministic benchmark strategy  
 516 allocates wealth to two assets, namely the T-bills  $B$  and the broad equity market index  $S$ . For the special case  
 517 of continuous rebalancing, let  $t \rightarrow \hat{\varrho}_s(t)$  be a deterministic function of time denoting the proportional allocation  
 518 to  $S$  at time  $t \in \mathcal{T} = [t_0, T]$ , with  $(1 - \hat{\varrho}_s(t))$  denoting the corresponding allocation to T-bills. The benchmark  
 519 strategy in this section is therefore given by

$$520 \quad \hat{\mathcal{P}} = \left\{ \hat{\mathbf{p}}(t, \hat{W}(t)) = (1 - \hat{\varrho}_s(t), \hat{\varrho}_s(t)) : t \in [t_0, T] \right\}. \quad (3.19)$$

521 By Assumption 3.1(ii), in the case of the VETF investor, let  $\varrho_v(t, \mathbf{X}_v(t))$  denote the proportional al-  
 522 location of wealth  $W_v(t)$  at time  $t$  to the VETF  $F_v$  in the case of continuous rebalancing, with  $\mathbf{X}_v(t) =$   
 523  $(W_v(t), \hat{W}(t), \hat{\varrho}_s(t))$ . The VETF investor strategy is therefore of the form

$$524 \quad \mathcal{P}_v = \left\{ \mathbf{p}_v(t, \mathbf{X}_v(t)) = (1 - \varrho_v(t, \mathbf{X}_v(t)), \varrho_v(t, \mathbf{X}_v(t))) : t \in [t_0, T] \right\}. \quad (3.20)$$

525 In the case of the LETF investor, let  $\varrho_\ell(t, \mathbf{X}_\ell(t))$  denote the proportional allocation of wealth  $W_\ell(t)$  at  
 526 time  $t$  to the LETF  $F_\ell$  in the case of continuous rebalancing, with  $\mathbf{X}_\ell(t) = (W_\ell(t), \hat{W}(t), \hat{\varrho}_s(t))$ , to obtain  
 527 the LETF investor strategy as

$$528 \quad \mathcal{P}_\ell = \left\{ \mathbf{p}_\ell(t, \mathbf{X}_\ell(t)) = (1 - \varrho_\ell(t, \mathbf{X}_\ell(t)), \varrho_\ell(t, \mathbf{X}_\ell(t))) : t \in [t_0, T] \right\}. \quad (3.21)$$

529 Given investment strategies of the form (3.19) and (3.21), as well as dynamics (3.1), (3.2), (3.4) and (3.8),  
 530 we therefore have the following wealth dynamics in the case of continuous rebalancing:

$$531 \quad d\hat{W}(t) = \left\{ \hat{W}(t^-) \cdot [r + \hat{\varrho}_s(t)(\mu - \lambda\kappa_1^s - r)] + q \right\} \cdot dt \\
 532 \quad + \hat{W}(t^-) \hat{\varrho}_s(t) \sigma \cdot dZ(t) + \hat{W}(t^-) \hat{\varrho}_s(t) \cdot d \left( \sum_{i=1}^{\pi(t)} (\xi_i^s - 1) \right), \quad (3.22)$$

$$533 \\
 534 \quad dW_v(t) = \left\{ W_v(t^-) \cdot [r + \varrho_v(t, \mathbf{X}_v(t^-)) \{(\mu - \lambda\kappa_1^s - r) - c_v\}] + q \right\} \cdot dt \\
 535 \quad + W_v(t^-) \varrho_v(t, \mathbf{X}_v(t^-)) \sigma \cdot dZ(t) + W_v(t^-) \varrho_v(t, \mathbf{X}_v(t^-)) \cdot d \left( \sum_{i=1}^{\pi(t)} (\xi_i^s - 1) \right), \quad (3.23)$$

$$536 \\
 537 \quad dW_\ell(t) = \left\{ W_\ell(t^-) \cdot [r + \varrho_\ell(t, \mathbf{X}_\ell(t^-)) \{\beta(\mu - \lambda\kappa_1^s - r) - c_\ell\}] + q \right\} \cdot dt \\
 538 \quad + W_\ell(t^-) \varrho_\ell(t, \mathbf{X}_\ell(t^-)) \beta \sigma \cdot dZ(t) + W_\ell(t^-) \varrho_\ell(t, \mathbf{X}_\ell(t^-)) \beta \cdot d \left( \sum_{i=1}^{\pi(t)} (\xi_i^\ell - 1) \right), \quad (3.24)$$

539 for  $t \in (t_0, T]$ , with initial wealth  $W_v(t_0) = W_\ell(t_0) = \hat{W}(t_0) = w_0$ . As a reminder,  $q \geq 0$  denotes the constant  
 540 rate per year at which cash is contributed to each portfolio (see Assumption 3.2(ii)), and  $\beta > 1$  in (3.24) denotes  
 541 the multiplier of the LETF.

542 Due to Assumption 3.2(iii), the set of admissible investor strategies is given by  $\varrho_k(t, \mathbf{X}_k(t)) \in \mathcal{A}_0$  for  
 543  $k \in \{v, \ell\}$ , where

$$544 \quad \mathcal{A}_0 = \left\{ \varrho_k(t, w, \hat{w}, \hat{\varrho}_s(t)) \mid \varrho_k : [t_0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R} \right\}, \quad k \in \{v, \ell\}. \quad (3.25)$$

545 The IR optimization problem (2.4) in this setting can therefore be written as

$$546 \quad (IR(\gamma)) : \quad \inf_{\varrho_k \in \mathcal{A}_0} E_{\varrho_k}^{t_0, w_0} \left[ \left( W_k(T) - [\hat{W}(T) + \gamma] \right)^2 \right], \quad \gamma > 0, \quad \text{for } k \in \{v, \ell\}, \quad (3.26)$$

547 with wealth dynamics (3.22), (3.23) and (3.24) respectively.

548 The following theorem describes the HJB partial integro-differential equation (PIDE) satisfied by the value

549 function of (3.26) for the LETF investor.

550 **Theorem 3.1.** (IR optimization for the LETF investor: Verification theorem) Let  $\gamma > 0$ , and assume a given  
551 benchmark strategy  $t \rightarrow \hat{\varrho}_s(t)$  that is deterministic and integrable. Suppose that for all  $(t, w, \hat{w}, \hat{\varrho}_s) \in [t_0, T] \times \mathbb{R}^3$ ,  
552 there exist functions  $V(t, w, \hat{w}, \hat{\varrho}_s) : [t_0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R}$  and  $\varrho_\ell^*(t, w, \hat{w}, \hat{\varrho}_s; \gamma) : [t_0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R}$  such that: (i)  
553  $V$  and  $\varrho_\ell^*$  are sufficiently smooth and solve the HJB PIDE (3.27)-(3.28), and (ii) the pointwise supremum in  
554 (3.27) is attained by the function  $\varrho_\ell^*(t, w, \hat{w}, \hat{\varrho}_s; \gamma)$ .

$$555 \quad \frac{\partial V}{\partial t} + \inf_{\varrho_\ell \in \mathbb{R}} \left\{ \mathcal{H}(\varrho_\ell; t, w, \hat{w}, \hat{\varrho}_s) \right\} = 0, \quad (3.27)$$

$$556 \quad V(T, w, \hat{w}, \hat{\varrho}_s) = (w - \hat{w} - \gamma)^2, \quad (3.28)$$

557 where

$$558 \quad \begin{aligned} \mathcal{H}(\varrho_\ell; t, w, \hat{w}, \hat{\varrho}_s) &= (w \cdot [r + \{\beta(\mu - \lambda\kappa_1^s - r) - c_\ell\} \cdot \varrho_\ell] + q) \cdot \frac{\partial V}{\partial w} \\ 559 &+ (\hat{w} \cdot [r + (\mu - \lambda\kappa_1^s - r) \cdot \hat{\varrho}_s] + q) \cdot \frac{\partial V}{\partial \hat{w}} \\ 560 &+ \frac{1}{2} (\varrho_\ell \cdot w\beta\sigma)^2 \cdot \frac{\partial^2 V}{\partial w^2} + \frac{1}{2} (\hat{\varrho}_s \hat{w}\sigma)^2 \cdot \frac{\partial^2 V}{\partial \hat{w}^2} + (\varrho_\ell \cdot w\beta\sigma) (\hat{\varrho}_s \hat{w}\sigma) \cdot \frac{\partial^2 V}{\partial w \partial \hat{w}} \\ 561 &- \lambda \cdot V + \lambda \cdot \int_0^\infty V(w + \varrho_\ell \cdot w\beta(\xi^\ell - 1), \hat{w} + \hat{\varrho}_s \hat{w}(\xi^s - 1), t) G(\xi^s) d\xi^s. \end{aligned} \quad (3.29)$$

562 Then given Assumption 3.2 and wealth dynamics (3.22) and (3.24),  $V$  is the value function and  $\varrho_\ell^*$  is the  
563 optimal control (i.e. optimal proportion of the investor's wealth to be invested in the LETF with  $\beta > 1$ ) for the  
564 IR( $\gamma$ ) problem (3.26) for the LETF investor.

565 *Proof.* See Appendix A.1. Note that since  $\xi^\ell$  is a function of  $\xi^s$  (see (3.6)), the integral in (3.29) is only written  
566 with respect to values of  $\xi^s$  with associated PDF  $G(\xi^s)$ .  $\square$

567 Solving the HJB PIDE (3.27)-(3.28), we obtain the IR-optimal investment strategy for the LETF investor  
568 as per Proposition 3.2.

569 **Proposition 3.2.** (IR-optimal investment strategy using the LETF) Let  $\gamma > 0$  be fixed. Suppose that Assump-  
570 tion 3.2 and wealth dynamics (3.22) and (3.24) apply. Let  $W_\ell^*(t)$  denote the LETF investor's wealth process  
571 (3.24) under the optimal strategy  $\varrho_\ell^*$ , and let  $\mathbf{X}_\ell^*(t) = (W_\ell^*(t), \hat{W}(t), \hat{\varrho}_s(t))$ . Then the IR-optimal fraction of  
572 the investor's wealth invested in the LETF,  $\varrho_\ell^*$ , satisfies

$$573 \quad \begin{aligned} &\varrho_\ell^*(t, \mathbf{X}_\ell^*(t^-)) \cdot W_\ell^*(t^-) \\ 574 &= \left( \frac{\beta [\mu + \lambda(\kappa_1^\ell - \kappa_1^s) - r] - c_\ell}{\beta^2 (\sigma^2 + \lambda\kappa_2^\ell)} \right) \cdot \left[ h_\ell(t) + \gamma e^{-r(T-t)} - (W_\ell^*(t^-) - g_\ell(t) \cdot \hat{W}(t^-)) \right] \\ 575 &+ \frac{1}{\beta} g_\ell(t) \left( \frac{\sigma^2 + \lambda\kappa_\chi^{\ell,s}}{\sigma^2 + \lambda\kappa_2^\ell} \right) \cdot \hat{\varrho}_s(t) \hat{W}(t^-), \end{aligned} \quad (3.30)$$

576 where  $g_\ell$  and  $h_\ell$  are the following deterministic functions,

$$577 \quad g_\ell(t) = \exp \left\{ K_\beta^{\ell,s} \cdot \int_t^T \hat{\varrho}_s(u) du \right\}, \quad (3.31)$$

$$578 \quad h_\ell(t) = -\frac{q}{r} \left( 1 - e^{-r(T-t)} \right) + q e^{-r(T-t)} \cdot \int_t^T \exp \left\{ r(T-y) + K_\beta^{\ell,s} \cdot \int_y^T \hat{\varrho}_s(u) du \right\} dy, \quad (3.32)$$

579 with constant  $K_\beta^{\ell,s}$  given by

$$580 \quad K_\beta^{\ell,s} = \mu - r - \frac{(\beta [\mu + \lambda(\kappa_1^\ell - \kappa_1^s) - r] - c_\ell) (\sigma^2 + \lambda\kappa_\chi^{\ell,s})}{\beta (\sigma^2 + \lambda\kappa_2^\ell)}. \quad (3.33)$$

581 *Proof.* See Appendix A.2.  $\square$

582 Note that values of  $\kappa_1^\ell$ ,  $\kappa_2^\ell$  and  $\kappa_\chi^{\ell,s}$  (see (3.7)), which depend on the multiplier  $\beta$  through the LETF jumps  
583 (3.6), are required by the optimal strategy (3.30). Expressions for these quantities can be derived in terms of the  
584 calibrated parameters of the underlying asset dynamics (3.1)-(3.2) without difficulty. As an illustration, Lemma  
585 A.1 in Appendix A.3 presents expressions for  $\kappa_1^\ell$ ,  $\kappa_2^\ell$  and  $\kappa_\chi^{\ell,s}$  in the case of the double-exponential Kou model  
586 (Kou (2002)) used for illustrating the results of this section, but we note that this can also be done similarly  
587 for other jump diffusion models (e.g. Merton (1976)).

588 The IR-optimal investment strategy for the VETF investor is given in Corollary 3.3.

589 **Corollary 3.3.** (*IR-optimal investment strategy using the VETF*) Let  $\gamma > 0$  be fixed. Suppose that Assumption  
590 3.2 and wealth dynamics (3.22) and (3.23) apply. Let  $W_v^*(t)$  denote the VETF investor's wealth process (3.23)  
591 under the optimal strategy  $\varrho_v^*$ , and let  $\mathbf{X}_v^*(t) = (W_v^*(t), \hat{W}(t), \hat{\varrho}_s(t))$ . Then the IR-optimal fraction of the  
592 investor's wealth invested in the VETF,  $\varrho_v^*$ , satisfies

$$593 \quad \varrho_v^*(t, \mathbf{X}_v^*(t^-)) \cdot W_v^*(t^-) = \left( \frac{\mu - r - c_v}{\sigma^2 + \lambda \kappa_2^s} \right) \cdot \left[ h_v(t) + \gamma e^{-r(T-t)} - \left( W_v^*(t^-) - g_v(t) \cdot \hat{W}(t^-) \right) \right] \\ 594 \quad + g_v(t) \cdot \hat{\varrho}_s(t) \hat{W}(t^-), \quad (3.34)$$

595 where  $g_v$  and  $h_v$  are the following deterministic functions,

$$596 \quad g_v(t) = \exp \left\{ c_v \cdot \int_t^T \hat{\varrho}_s(u) du \right\}, \quad (3.35)$$

$$597 \quad h_v(t) = -\frac{q}{r} \left( 1 - e^{-r(T-t)} \right) + q e^{-r(T-t)} \cdot \int_t^T \exp \left\{ r(T-y) + c_v \cdot \int_y^T \hat{\varrho}_s(u) du \right\} dy. \quad (3.36)$$

598 *Proof.* See Appendix A.4. □

599 The following remark relates the results of Proposition 3.2 and Corollary 3.3 to the available results in the  
600 literature.

601 **Remark 3.3.** (Relationship of Proposition 3.2 and Corollary 3.3 to results in the literature). In the special  
602 case of a VETF with zero expense ratio  $c_v = 0$ , the results of Corollary 3.3 imply that  $g_v(t) = 1$  and  $h_v(t) = 0$   
603 for all  $t \in [t_0 = 0, T]$ , so that (3.34) simplifies considerably to

$$604 \quad \varrho_v^*(t, \mathbf{X}_v^*(t^-)) \cdot W_v^*(t^-) = \left( \frac{\mu - r}{\sigma^2 + \lambda \kappa_2^s} \right) \cdot \left[ \gamma e^{-r(T-t)} - \left( W_v^*(t^-) - \hat{W}(t^-) \right) \right] + \hat{\varrho}_s(t) \hat{W}(t^-), \quad (3.37)$$

605 which corresponds to the IR-optimal investment strategy where direct investment in the underlying equity  
606 market index  $S$  is possible. This special case (3.37) can be found in Van Staden et al. (2023), where the results  
607 of Goetzmann et al. (2002, 2007) are extended to the case of jumps in the risky asset processes. Corollary 3.3  
608 therefore extends this to the case of investing in the equity index *indirectly* via a VETF with a non-negligible  
609 expense ratio, whereas Proposition 3.2 extends these results further to the case a LETF with multiplier  $\beta > 1$   
610 and expense ratio  $c_\ell$  referencing an equity index  $S$  with jump-diffusion dynamics. □

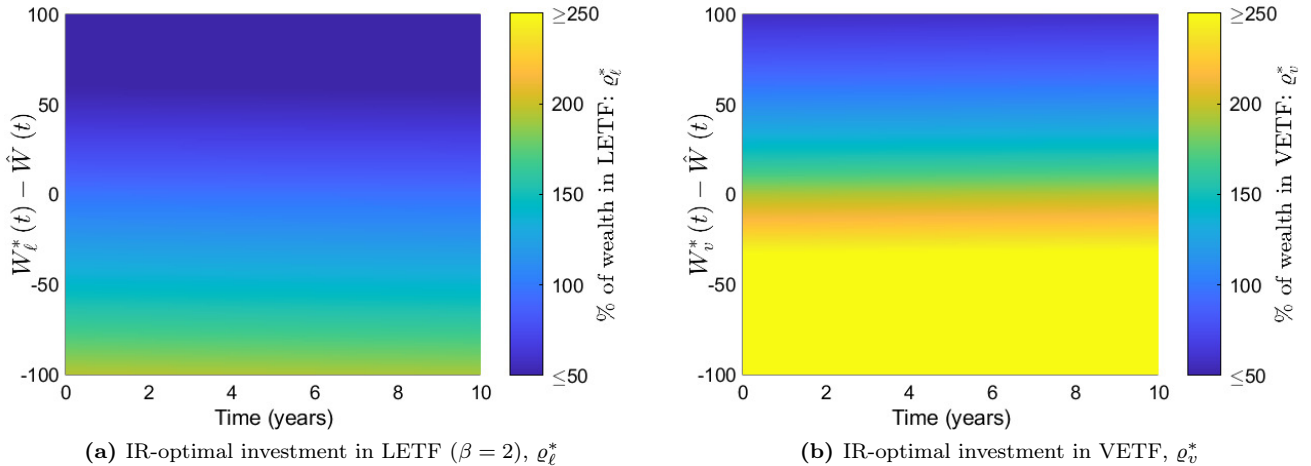
611 To gain some intuition regarding the behavior of the IR-optimal investment strategies in Proposition 3.2 and  
612 Corollary 3.3, we compare the illustrative investment results from implementing these strategies over a 10-year  
613 time horizon. We use a benchmark and assets as in Table 3.1, illustrative investment parameters as in Table  
614 3.2, and a Kou model (Kou (2002)) is assumed for the jump diffusion dynamics with calibrated parameters as  
615 in Appendix B (Table B.1). Since LETFs are a relatively recent invention, we follow the example of Bansal  
616 and Marshall (2015) in constructing a proxy LETF replicating  $\beta = 2$  times the daily returns of a broad stock  
617 market index, in this case using the CRSP VWD index, which is a capitalization-weighted index consisting of  
618 all domestic stocks trading on major US exchanges, with historical data available since January 1926. As in for  
619 example Bansal and Marshall (2015) and Leung and Sircar (2015), we assume that the managers of the LETF  
620 do not have challenges in replicating the underlying index, which is reasonable given the possibility of designing  
621 replication strategies for LETFs that remain robust even during periods of market volatility (see for example  
622 Guasoni and Mayerhofer (2023)). For more information on the source data and calibrated, inflation-adjusted  
623 parameters, please refer to Appendix B.

624

**Table 3.2:** Closed-form solutions - Investment parameters for illustrating the results. Note that the calibrated parameters for the jump-diffusion process are given in Appendix B (Table B.1), while the underlying assets, benchmark and ETF expense ratios are given in Table 3.1.

Parameter	$T$	$w_0$	$q$	$\gamma$
Value	10 years	\$ 100	\$ 5 per year	125

625 Figure 3.3 illustrates the IR-optimal proportion of wealth invested in the ETF as a function of time  $t$  and  
626 the wealth difference  $W_k^*(t) - \hat{W}(t)$ ,  $k \in \{v, \ell\}$ . In the case of the LETF investor, Figure 3.3(a) illustrates  $\varrho_\ell^*$ ,  
627 whereas for the VETF investor, Figure 3.3(b) illustrates  $\varrho_v^*$ , where both strategies use the same target  $\gamma = 125$ .

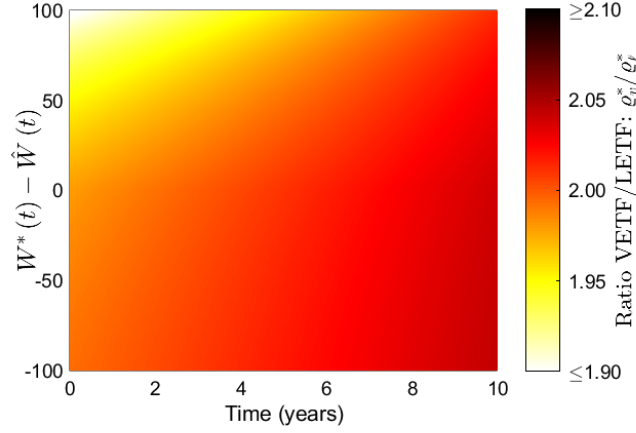


**Figure 3.3:** Closed-form IR-optimal investment strategies using the LETF ( $\varrho_\ell^*$  as per (3.30)) or the VETF ( $\varrho_v^*$  as per (3.34)) as a function of time  $t$  and the wealth difference  $W_k^*(t) - \hat{W}(t)$ ,  $k \in \{v, \ell\}$ , given the same implicit benchmark outperformance target  $\gamma$ . The underlying assets, benchmark, investment parameters and calibrated process parameters are as in Table 3.1, Table 3.2 and Table B.1, respectively. Note that the same color scale is used in both figures for comparison purposes.

628 Using the same strategies as illustrated in Figure 3.3, Figure 3.4 shows the *ratio* of IR-optimal proportions  
629 of wealth invested in the VETF relative to the investment in the LETF,  $\varrho_v^*/\varrho_\ell^*$ , given an otherwise identical  
630 wealth difference  $W_k^*(t) - \hat{W}(t)$ ,  $k \in \{v, \ell\}$  at time  $t$ . We emphasize that both Figure 3.3 and Figure 3.4 treat  
631 the IR-optimal strategies from Proposition 3.2 and Corollary 3.3 simply as functions of time and the wealth  
632 difference relative to the benchmark.  
633

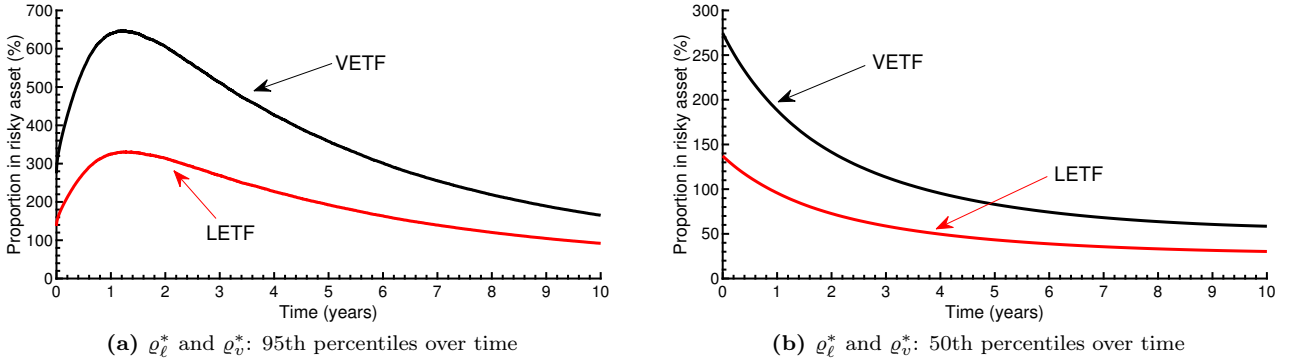
634 With regards to Figure 3.3 and Figure 3.4, we make the following observations regarding the IR-optimal  
635 investment strategies of the LETF investor vs. the VETF investor:  
636

- 637 (i) The IR-optimal investment strategy using a LETF (Figure 3.3(a)) is, like the strategy using a VETF  
638 (Figure 3.3(b)), fundamentally *contrarian*. Specifically, in the case of the LETF, we observe that the IR-  
639 optimal proportion of wealth  $\varrho_\ell^*$  in the LETF decreases as the wealth difference  $W_\ell^*(t) - \hat{W}(t)$  increases,  
640 which happens after a period of strong LETF return performance. In their analysis of reports to the SEC  
641 by institutional fund managers, DeVault et al. (2021) show that institutional investors indeed empirically  
642 tend to decrease their holdings in LETFs following periods of strong investment performance. While  
643 DeVault et al. (2021) concludes that this behavior might be explained as being a result of compensation-  
644 based incentives, our results show that strategies based on maximizing the IR (a widely-used investment  
645 metric) relative to a standard investment benchmark could also be related to this empirical investment  
646 behavior.
- 647 (ii) Figure 3.4 shows that the IR-optimal strategies under stylized assumptions (Assumption 3.1) and identical  
648 implicit benchmark outperformance target  $\gamma$  satisfy  $\varrho_v^*/\varrho_\ell^* \approx \beta = 2$ . Informally, the LETF and VETF  
649 investors therefore take on nearly identical “risk” exposure to the movements of the underlying index,  
650 which provides valuable intuition when interpreting the results Section 5 where Assumption 3.2 is relaxed.



**Figure 3.4:** Ratio of IR-optimal proportions of wealth in the VETF vs. the LETF,  $\varrho_v^*/\varrho_\ell^*$ , given identical wealth differences relative to the benchmark  $W^*(t) - \hat{W}(t) \equiv W_v^*(t) - \hat{W}(t) = W_\ell^*(t) - \hat{W}(t)$  at each time  $t$  and same target  $\gamma$ . The underlying assets, benchmark, investment parameters and calibrated process parameters are as in Table 3.1, Table 3.2 and Table B.1, respectively.

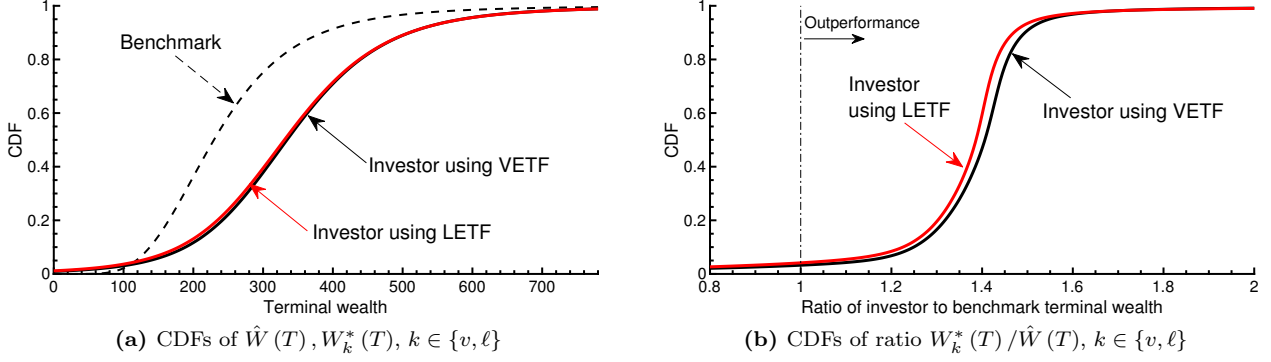
651 While Figure 3.3 and Figure 3.4 illustrate the IR-optimal strategy as a function of time and the wealth difference  
652 relative to the benchmark, implementing this strategy in a Monte Carlo simulation provides an additional  
653 perspective. Figure 3.5 shows the median and 95th percentiles of the IR-optimal proportion of wealth invested  
654 in the LETF and VETF, given the same benchmark outperformance target  $\gamma$ . Note that under the stylized  
655 assumptions (Assumption 3.2), the LETF and VETF positions can be leveraged without restriction, with  
656 leverage constraints only subsequently introduced in Section 4. We observe that the corresponding ETF exposure  
657 percentiles tend to approximately satisfy  $\varrho_v^*/\varrho_\ell^* \approx \beta = 2$ , with decreasing exposure over time due to the  
658 contrarian nature of both strategies.



**Figure 3.5:** Closed-form IR-optimal investment strategies: 95th and 50th percentiles over time of the IR-optimal proportion of wealth invested in the LETF ( $\varrho_\ell^*$  as per (3.30)) or the VETF ( $\varrho_v^*$  as per (3.34)) obtained using Monte Carlo simulation of the underlying dynamics (3.22)-(3.24), and same target  $\gamma$ . The underlying assets, benchmark, investment parameters and calibrated process parameters are as in Table 3.1, Table 3.2 and Table B.1, respectively.

659  
660 Figure 3.6 compares the simulated CDFs of IR-optimal terminal wealth  $W_k^*(T)$ ,  $k \in \{v, \ell\}$  and CDFs of the  
661 terminal wealth ratio relative to the benchmark  $W_k^*(T)/\hat{W}(T)$ ,  $k \in \{v, \ell\}$  for the LETF and VETF investors,  
662 respectively.

663  
664 Figure 3.6 shows that due to implicitly similar exposure levels to movements in the underlying equity index  
665 (since  $\varrho_v^* \simeq \beta \cdot \varrho_\ell^*$ ) given identical targets  $\gamma$  for the LETF and VETF investors, implementing the strategies  
666 illustrated in Figure 3.3 result in nearly identical terminal wealth and outperformance outcomes. In fact, under  
667 the stylized assumptions of this section, Proposition 3.4 shows that in the special case of (i) zero expense ratios  
668 and (ii) no jumps in the  $S$ -dynamics, we have  $\varrho_v^* \equiv \beta \cdot \varrho_\ell^*$ , the IR-optimal investor should be entirely indifferent  
669 as to whether the LETF-based strategy (3.30) or VETF-based strategy (3.34) is used for investment purposes.



**Figure 3.6:** Closed-form IR-optimal investment strategies: CDFs of the IR-optimal terminal wealth for the same target  $\gamma$  obtained using Monte Carlo simulation of the underlying dynamics and investing according to optimal strategies (3.30) and (3.34). The underlying assets, benchmark, investment parameters and calibrated process parameters are as in Table 3.1, Table 3.2 and Table B.1, respectively.

**Proposition 3.4.** (Special case: Zero expense ratios, no jumps) Let  $\gamma > 0$  be fixed, and let Assumption 3.2 and wealth dynamics (3.22)-(3.24) apply. If (i) both LETF ( $\beta > 1$ ) and the VETF have zero expense ratios, i.e.  $c_v = c_\ell = 0$ , and (ii) there are no jumps in the underlying  $S$ -dynamics (i.e.  $\lambda = 0$  in (3.2)), then following results hold:

(i) The IR-optimal proportion of wealth invested in the VETF (3.34) is equal to  $\beta$  times the IR-optimal proportion of wealth invested in the LETF (3.30),

$$\varrho_v^*(t, \mathbf{X}_v^*(t)) = \beta \cdot \varrho_\ell^*(t, \mathbf{X}_\ell^*(t)), \quad \forall t \in [t_0, T], \quad (3.38)$$

where  $\mathbf{X}_v^*(t) = (W_v^*(t), \hat{W}(t), \hat{\rho}_s(t))$  and  $\mathbf{X}_\ell^*(t) = (W_\ell^*(t), \hat{W}(t), \hat{\rho}_s(t))$ .

(ii) An IR-optimal investor with given target  $\gamma > 0$  would be indifferent as to whether the optimal strategy is executed with a LETF (3.30) or VETF (3.34), since wealth outcomes are identical,

$$W_\ell^*(t) = W_v^*(t), \quad \forall t \in [t_0, T], \quad (3.39)$$

and the same Information Ratio is obtained,

$$\frac{E_{\varrho_\ell^*}^{t_0, w_0} [W_\ell^*(T) - \hat{W}(T)]}{Stdev_{\varrho_\ell^*}^{t_0, w_0} [W_\ell^*(T) - \hat{W}(T)]} = \frac{E_{\varrho_v^*}^{t_0, w_0} [W_v^*(T) - \hat{W}(T)]}{Stdev_{\varrho_v^*}^{t_0, w_0} [W_v^*(T) - \hat{W}(T)]} = \left( \exp \left\{ \left( \frac{\mu - r}{\sigma} \right)^2 \cdot T \right\} - 1 \right)^{1/2}. \quad (3.40)$$

*Proof.* See Appendix A.5. □

We summarize the closed-form solutions results presented in this section as follows. In the case of continuous rebalancing (i.e. rebalancing times  $\mathcal{T} = [t_0, T]$ ) with no investment constraints, the IR-optimal investor should be largely indifferent whether a VETF or LETF is used on the same underlying equity index to execute the IR-optimal investment strategy involving the ETF and T-bills for a given outperformance target  $\gamma$ . Since the IR-optimal investment strategies satisfy the approximate relationship  $\varrho_v^* \simeq \beta \cdot \varrho_\ell^*$  (Figure 3.4), when continuously rebalancing the LETF and VETF investors effectively maintain similar implicit risk exposures at each time instant to movements of the underlying index, resulting in broadly similar investment outcomes (Figure 3.6), with differences between outcomes entirely driven by different ETF expense ratios and the presence of jumps (see Proposition 3.4).

At the other extreme, namely the lump-sum investment scenario with no subsequent intervention (i.e.  $\mathcal{T} = [t_0]$ ) and one-quarter time horizon ( $\Delta t = 0.25$ ), we still observe the approximate relationship  $p_v^* \approx \beta \cdot p_\ell^*$  between corresponding IR-optimal strategies (Figure 3.2). However, in this case the power call-like payoff of the LETF can clearly be observed, whereby the LETF investor benefits from an upside due to comparatively inexpensive leverage while simultaneously enjoying downside protection due to limited liability (Figures 1.1, 3.1 and 3.2).

698 Far from ignoring the standard criticisms of LETFs in the literature (see Section 3.1), the closed form  
699 results of this section - illustrated using parametric dynamics calibrated to empirical market data - suggest  
700 that we should not be entirely surprised that LETFs might have substantial appeal to investors *despite* these  
701 shortcomings. However, we emphasize that none of the trading strategies illustrated - not even the lump-sum  
702 investment scenario results - advocate for simplistic strategies like buy-and-hold positions in the LETF over  
703 indefinite time horizons, so a certain degree of sophistication on the part of the investor is implicitly assumed. In  
704 addition, while the closed-form results provide valuable intuition, they were derived under stylized assumptions,  
705 which we relax in the subsequent sections to model the potential of LETFs under more realistic conditions.

## 706 4 Numerical solutions

707 To allow for more general and practical conclusions than available in the stylized setting of Section 3, we  
708 use a data-driven neural network approach based on stationary block bootstrap resampling of historical data  
709 (including proxy LETF returns) since 1926. This ensures that the investment strategies and performance  
710 analysis will incorporate all empirical aspects of actual returns, including potentially sophisticated volatility  
711 dynamics, which may not be reflected in the closed-form solutions of Section 3.

712 We start by formulating a more realistic investment setting with the following characteristics: (i) Restrictions  
713 on the maximum leverage and limitations on short-selling. (ii) An optional borrowing premium applicable when  
714 short-selling an asset. (iii) Prohibition of trading in insolvency. (iv) Infrequent (discrete) rebalancing of the  
715 portfolio. This is followed by a brief overview of a neural network-based numerical solution approach to solve  
716 the IR problem (2.4) in this setting. Indicative investment results obtained by implementing these techniques  
717 on empirical market data are discussed in Section 5.

### 718 4.1 Investment constraints and discrete rebalancing

719 Recall from Section 2 that the investor's strategy is based on investing in a set of  $N_a$  candidate assets indexed  
720 by  $i \in \{1, \dots, N_a\}$ , while the benchmark is defined in terms of  $\hat{N}_a$  potentially different underlying assets indexed  
721 by  $j \in \{1, \dots, \hat{N}_a\}$ . With investor and benchmark strategies of the form (2.1), and the benchmark strategy  
722 satisfying only the general assumptions outlined in Assumption 2.1, we assume that both the investor and  
723 benchmark portfolios are rebalanced at each of  $N_{rb}$  discrete rebalancing times during the investment time  
724 horizon  $[t_0 = 0, T]$ . As a result,  $\mathcal{T}$  is now of the form

$$725 \quad \mathcal{T} = \{t_n = n\Delta t \mid n = 0, \dots, N_{rb} - 1\}, \quad \Delta t = T/N_{rb}. \quad (4.1)$$

726 The assumption of equally-spaced rebalancing times in (4.1) is only for convenience, and can be relaxed without  
727 difficulty. At each rebalancing time  $t_n \in \mathcal{T}$ , we assume a pre-specified cash contribution  $q(t_n)$  is made to the  
728 investor portfolio, with the contribution also being added to the benchmark portfolio to ensure the comparison  
729 in performance remains appropriate.

730 There is no need to specify any parametric dynamics for the underlying assets in the numerical solution  
731 approach, which only requires the availability of empirical market data for deriving the optimal strategy (see  
732 Subsection 4.2 below). Specifically, we assume that at each time  $t_{n+1} \in \mathcal{T} \cup T$  we can observe  $R_i(t_n)$  and  
733  $\hat{R}_j(t_n)$ , the returns on investor asset  $i \in \{1, \dots, N_a\}$  and benchmark asset  $j \in \{1, \dots, \hat{N}_a\}$ , respectively, over  
734 the time interval  $[t_n, t_{n+1}]$ . Note that these returns might be inflation-adjusted and might include a borrowing  
735 premium applicable to assets that have been shorted (see e.g. Assumption 4.1 below). As a result, for the  
736 purposes of numerical solutions, the investor and benchmark wealth dynamics are respectively of the form

$$737 \quad W(t_{n+1}^-) = [W(t_n^-) + q(t_n)] \cdot \sum_{i=1}^{N_a} p_i(t_n, \mathbf{X}(t_n^-)) \cdot [1 + R_i(t_n)], \quad (4.2)$$

$$738 \quad \hat{W}(t_{n+1}^-) = [\hat{W}(t_n^-) + q(t_n)] \cdot \sum_{j=1}^{\hat{N}_a} \hat{p}_j(t_n, \hat{\mathbf{X}}(t_n^-)) \cdot [1 + \hat{R}_j(t_n)], \quad (4.3)$$

739 where  $W(t_0^-) = \hat{W}(t_0^-) := w_0 > 0$  and  $n = 0, \dots, N_{rb} - 1$ .

740 Since active funds often have restrictions on leverage and short-selling (see for example Forsyth et al. (2019);  
741 Ni et al. (2024)), these constraints are included in the formulation.

742 The investor's candidate asset  $i = 1$ , assumed to be 30-day T-bills in the indicative investment results of  
743 Section 5, plays a special role in leveraged portfolios. The investor is assumed to be able to short-sell this asset  
744 with a potential borrowing premium payable, i.e. the investor can borrow funds at an approximation of the  
745 prevailing short-term interest rate plus a borrowing premium to fund leveraged investments in the other assets.  
746 In addition, in the case of insolvency, defined as occurring when the investor wealth is negative,  $W(t_n) < 0$ , we  
747 will assume that the negative wealth (i.e. the outstanding debt) is placed in asset  $i = 1$ , where it grows at the  
748 rate of return of this asset with an addition of a possible borrowing premium. Note that this effectively implies  
749 that trading ceases when  $W < 0$ , either until maturity  $T$  or until such a time where the cash injections pay off  
750 the debt resulting in  $W > 0$ , in which case trading can resume. A maximum leverage ratio at a portfolio level  
751 of  $p_{max}$  is also assumed, where typical values are in the range  $p_{max} \in [1.0, 1.5]$ . Assumption 4.1 outlines the  
752 details more formally.

753 **Assumption 4.1.** (Investor strategy: Leverage restrictions, borrowing premium and no trading in insolvency)  
754 The following assumptions and restrictions apply to the investor strategy, where the investor considers invest-  
755 ment in  $N_a \geq 2$  candidate assets. As discussed, the investor's set of candidate assets may not correspond to the  
756 assets included in the benchmark strategy.

757 (i) **Shortable and long-only assets:** Only investor candidate asset  $i = 1$  will (potentially) be shorted, with  
758 the remaining investor candidate assets  $i \in \{2, \dots, N_a\}$  being long only. In other words, at any rebalancing  
759 time  $t_n \in \mathcal{T}$ , we have

$$760 \quad (\text{Shortable asset } i = 1) : \quad p_1(t_n, \mathbf{X}(t_n^-)) \in \mathbb{R}, \quad t_n \in \mathcal{T}, \quad (4.4)$$

$$761 \quad (\text{Long-only assets } i \in \{2, \dots, N_a\}) : \quad p_i(t_n, \mathbf{X}(t_n^-)) \geq 0, \quad i \in \{2, \dots, N_a\}, t_n \in \mathcal{T}, \quad (4.5)$$

$$762 \quad (\text{All wealth invested}) : \quad \sum_{i=1}^{N_a} p_i(t_n, \mathbf{X}(t_n^-)) = 1, \quad t_n \in \mathcal{T}. \quad (4.6)$$

763 (ii) **Borrowing premium  $b \geq 0$ :** If investor candidate asset  $i = 1$  is shorted at time  $t_n$  (i.e. if  $p_1(t_n, \mathbf{X}(t_n^-)) <$   
764  $0$ ), then a constant borrowing premium  $b \geq 0$  is added to the returns on asset  $i = 1$  over the time interval  
765  $[t_n, t_{n+1}]$  to be paid by the investor. In other words, for asset  $i = 1$ , the return  $R_1(t_n)$  incorporated in  
766 (4.2) is of the form

$$767 \quad (\text{Borrowing premium}) : \quad R_1(t_n) = \begin{cases} \bar{R}_1(t_n), & \text{if } p_1(t_n, \mathbf{X}(t_n^-)) \geq 0 \\ \bar{R}_1(t_n) + b, & \text{if } p_1(t_n, \mathbf{X}(t_n^-)) < 0, \end{cases} \quad (4.7)$$

768 where  $\bar{R}_1(t_n)$  is the (possibly inflation-adjusted) return on underlying asset  $i = 1$  over  $[t_n, t_{n+1}]$  without  
769 any added premiums. For long-only assets, we simply have  $R_i(t_n) = \bar{R}_i(t_n)$ ,  $i \in \{2, \dots, N_a\}, t_n \in \mathcal{T}$ .  
770 Note that in the case of the benchmark strategy, no borrowing premium is applicable to any asset due to  
771 Assumption 4.2 below.

772 (iii) **Maximum leverage ratio  $p_{max}$ :** The total allocated proportion of wealth to the long-only assets  $i \in$   
773  $\{2, \dots, N_a\}$  cannot exceed the maximum leverage ratio  $p_{max}$ ,

$$774 \quad (\text{Maximum leverage ratio}) : \quad \sum_{i=2}^{N_a} p_i(t_n, \mathbf{X}(t_n^-)) \leq p_{max}, \quad t_n \in \mathcal{T}. \quad (4.8)$$

775 (iv) **No trading in insolvency:** If the investor wealth is negative, i.e. if  $W(t_n) < 0$  at any  $t_n \in \mathcal{T}$ , then all  
776 long asset positions (4.5) are liquidated and the total debt (the amount  $W(t_n) < 0$ ) is allocated to the  
777 shortable asset (4.4). In such a scenario, no further trading occurs for the remainder of the investment  
778 time horizon ( $t_m \in \mathcal{T}, t_m > t_n$ ), unless cash injections pay off the debt, and the portfolio wealth becomes  
779 positive. Total debt accumulates at a rate (4.7) which possibly includes a borrowing premium. More  
780 formally,

$$781 \quad (\text{No trading in insolvency}) : \quad \text{If } W(t_n^-) < 0 \quad \Rightarrow \quad \mathbf{p}(t_n, \mathbf{X}(t_n^-)) = \mathbf{e}_1, \quad t_n \in \mathcal{T}, \quad (4.9)$$

where  $\mathbf{e}_1 = (1, 0, \dots, 0) \in \mathbb{R}^{N_a}$  is the standard basis vector  $\mathbb{R}^{N_a}$  with 1 in the first position (corresponding to  $i = 1$ , the shortable asset as per (4.4)) and all other entries are zero.  $\square$

Note that (4.9) also implies  $\mathbf{p}(t_m, \mathbf{X}(t_m^-)) = \mathbf{e}_1$  for all  $t_m > t_n$ , so that no further trading does indeed occur in the case of insolvency as required by Assumption 4.1(iv).

Recalling that  $\mathcal{A}$  denotes the set of admissible controls and  $\mathcal{Z}$  denoting the admissible control space, Assumption 4.1 implies that we have the following form for  $\mathcal{Z}$  and  $\mathcal{A}$ , respectively:

$$\mathcal{Z} = \left\{ \mathbf{z} \in \mathbb{R}^{N_a} \left| \begin{array}{l} z_1 \in \mathbb{R}, \\ z_i \geq 0, \forall i \in \{2, \dots, N_a\}, \\ \sum_{i=1}^{N_a} z_i = 1, \\ \sum_{i=2}^{N_a} z_i \leq p_{max}. \end{array} \right. \right\}, \quad (4.10)$$

and

$$\mathcal{A} = \left\{ \mathcal{P} = \left\{ \mathbf{p}(t_n, \mathbf{X}(t_n)), t_n \in \mathcal{T} \left| \begin{array}{l} \mathbf{p}(t_n, \mathbf{X}(t_n)) \in \mathcal{Z}, \text{ if } W(t_n^-) \geq 0, \\ \mathbf{p}(t_n, \mathbf{X}(t_n)) = \mathbf{e}_1, \text{ if } W(t_n^-) < 0. \end{array} \right. \right\} \right\}. \quad (4.11)$$

Note that with slight abuse of notation in (4.11),  $\mathcal{Z}$  is the admissible control space in the case of solvency only.

Finally, in order ensure that the benchmarks align with typical investment benchmarks used in practice (see Remark 2.1) and to avoid pathological examples, Assumption 4.2 below specifies that no short-selling is allowed in the case of the benchmark strategy.

**Assumption 4.2.** (Benchmark: leverage restrictions) The benchmark strategy does not engage in the short-selling of any asset.

$$\text{(Long-only benchmark)} \quad \hat{p}_j(t_n, \hat{\mathbf{X}}(t_n^-)) \geq 0, \quad \forall j = 1, \dots, \hat{N}_a. \quad (4.12)$$

As a result, the benchmark strategy has an implicit maximum leverage ratio of  $p_{max} = 1$ , with no borrowing premium being applicable, while benchmark insolvency is ruled out in the sense that  $\hat{W}(t_n) \geq 0$  for all  $t_n \in \mathcal{T}$  given dynamics (4.3).  $\square$

## 4.2 Neural network solution approach

The objective function in the case of the numerical solutions remains of the form (2.4),

$$(IR(\gamma)) : \inf_{\mathcal{P} \in \mathcal{A}} E_{\mathcal{P}}^{t_0, w_0} \left[ \left( W(T) - [\hat{W}(T) + \gamma] \right)^2 \right], \quad \gamma > 0, \quad (4.13)$$

with the main differences from the treatment in Section 3 being the following: (i) The set of admissible controls  $\mathcal{A}$  is now given by (4.11). (ii) Rebalancing occurs at a strict discrete subset of times  $t_n \in \mathcal{T} \subset [t_0 = 0, T]$ . (iii) As discussed below, we no longer need the assumption of parametric models for the underlying assets, but can use market data directly.

To solve (4.13) numerically to obtain the optimal investment strategy  $\mathcal{P}^* \in \mathcal{A}$ , we follow the neural network-based solution approach of Ni et al. (2024), where a ‘‘leverage-feasible neural network’’ (LFNN) is constructed to approximate the investment strategy directly as a feedback control  $(t_n, \mathbf{X}(t_n)) \rightarrow \mathcal{P}(t_n, \mathbf{X}(t_n)) := \mathbf{p}(t_n, \mathbf{X}(t_n)), \forall t_n \in \mathcal{T}$  in the case of admissible sets of the form (4.10)-(4.11). This approach forms part of a class of methods (see, for example, Buehler et al. (2019); Han and Weinan (2016); Mäkinen and Toivanen (2024); Reppen and Sonner (2023); Reppen et al. (2023); Van Staden et al. (2023, 2024), ) that does not require dynamic programming to solve problems such as (2.4), thereby avoiding the typical challenges such as evaluating high-dimensional conditional expectations and error amplifications over time-stepping.

Since more detailed information, including a convergence analysis, can be found in Ni et al. (2024), we give only a very short overview of the application of the LFNN approach in our setting. In this approach, the control function  $(t_n, \mathbf{X}(t_n)) \rightarrow \mathbf{p}(t_n, \mathbf{X}(t_n))$  is approximated by a single neural network (NN) with at least 3 features (inputs), namely  $(t_n, \mathbf{X}(t_n)) = (t_n, W(t_n), \hat{W}(t_n))$ . Note that additional features such as trading signals can be incorporated in the NN inputs in settings where this is considered valuable. Let  $\mathbf{F}(t, \mathbf{X}(t); \boldsymbol{\theta}) \equiv \mathbf{F}(\cdot, \boldsymbol{\theta})$  denote the NN, where  $\boldsymbol{\theta} \in \mathbb{R}^{\eta_\theta}$  is the NN parameters, i.e. the NN weights and biases. Since the time  $t_n$  is used

823 is an input into the NN, a single parameter vector  $\theta$  (equivalently, a single NN) is applicable to all rebalancing  
824 times, identifying this as a “global-in-time” approach in the taxonomy of Hu and Laurière (2023). One of the  
825 key contributions of Ni et al. (2024) is to construct the NN  $F(\cdot, \theta)$  with an output layer that guarantees, for all  
826 inputs  $(t, \mathbf{X}(t)) = (t, W(t), \hat{W}(t))$ , that

$$827 \quad (t, \mathbf{X}(t)) \rightarrow F(t, \mathbf{X}(t); \theta) \begin{cases} F(t, \mathbf{X}(t); \theta) \in \mathcal{Z}, & \text{if } W(t) \geq 0, \\ F(t, \mathbf{X}(t); \theta) = e_1, & \text{if } W(t) < 0. \end{cases} \quad (4.14)$$

828 As a result of (4.11), by using the approximation

$$829 \quad p(t, \mathbf{X}(t)) \simeq F(t, \mathbf{X}(t); \theta) \equiv F(\cdot, \theta), \quad (4.15)$$

830 we can therefore approximate the investor strategies as  $\mathcal{P} = \{F(t_n, \mathbf{X}(t_n); \theta), t_n \in \mathcal{T}\}$  while being assured that  
831  $\mathcal{P} \in \mathcal{A}$  where  $\mathcal{A}$  is as per (4.11), without the need to impose constraints on the optimization problem itself. As  
832 a result, (4.13) can be solved as an *unconstrained* optimization problem over  $\theta \in \mathbb{R}^{\eta_\theta}$ ,

$$833 \quad \inf_{\theta \in \mathbb{R}^{\eta_\theta}} E_{F(\cdot; \theta)}^{t_0, w_0} \left[ \left( W(T; \theta) - [\hat{W}(T) + \gamma] \right)^2 \right], \quad (4.16)$$

834 where the approximation (4.15) is used to obtain the asset allocation in the investor wealth dynamics (4.2)  
835 which depend on  $\theta$ .

836 Parametric models for the underlying asset dynamics are no longer required. Instead, we use a finite set  
837 of samples from the set  $Y = \{Y^{(j)} : j = 1, \dots, N_d\}$ , where each element  $Y^{(j)}$  denotes a time series of *joint*  
838 asset return observations  $R_i, i \in \{1, \dots, N_a\}$ , possibly adjusted for inflation and the application of a borrowing  
839 premium, observed at each  $t_n \in \mathcal{T}$ .  $Y$  represents the training data of the NN, so any  $\theta \in \mathbb{R}^{\eta_\theta}$  and returns  
840 path  $Y^{(j)} \in Y$ , the wealth dynamics (4.2) with approximation (4.15) generates a terminal wealth outcome  
841  $W^{(j)}(T; \theta)$ . The expectation in (4.16) is then approximated simply by

$$842 \quad \min_{\theta \in \mathbb{R}^{\eta_\theta}} \left\{ \frac{1}{N_d} \sum_{j=1}^{N_d} \left( W^{(j)}(T; \theta) - [\hat{W}^{(j)}(T) + \gamma] \right)^2 \right\}, \quad (4.17)$$

843 where the optimal parameter vector  $\theta^*$  is obtained using stochastic gradient descent. The resulting IR-  
844 optimal strategy for (4.13) consistent with the constraints as outlined in Assumption 4.1 is therefore given  
845 by  $p^*(\cdot, \mathbf{X}(\cdot)) \simeq F(\cdot, \theta^*)$ .

846 While the details underlying the construction of the data set  $Y$  are clearly of practical significance, we note  
847 that the approach of Ni et al. (2024) remains agnostic as to the how  $Y$  is constructed. It can be obtained using  
848 for example GAN-generated data sets (see e.g. Van Staden et al. (2024); Yoon et al. (2019)), or using Monte  
849 Carlo simulations if the underlying dynamics are specified for ground truth analysis purposes (see e.g. Van  
850 Staden et al. (2023)), or a version of bootstrap resampling of empirical market data, as we now discuss.

851 In practical applications, the use of empirical market data might be preferred for the construction of  $Y$ .  
852 However, since only a single historical path of asset returns is available, some form of data augmentation is  
853 typically used to obtain sufficiently rich training and testing data. For illustrative purposes, in Section 5 we use  
854 stationary block bootstrap resampling (Politis and Romano (1994)) to construct  $Y$ . This technique, designed  
855 for weakly stationary time series with serial dependence, is both popular in academic settings (Anarkulova  
856 et al. (2022)) and practical applications (Cavaglia et al. (2022); Cogneau and Zakalmouline (2013); Dichtl et al.  
857 (2016); Scott and Cavaglia (2017); Simonian and Martirosyan (2022)). Note that bootstrap resampling methods  
858 have been proposed for non-stationary time series (Politis (2003), Politis et al. (1999)), but this is not used in  
859 the illustrative investment results of Section 5.

## 860 5 Indicative investment results

861 This section demonstrates the potential of including LETFs within long-term, diversified, dynamic, IR-optimal  
862 investment strategies subject to the investment constraints outlined in Section 4. To ensure that the results  
863 are realistic and practical, we focus entirely on using bootstrapped historical returns (including the proxy  
864 ETF returns series dating back to 1926, see Appendix B), and we apply the numerical approach discussed  
865 in Subsection 4.2 to obtain the IR-optimal strategies based on the combination of T-bills and T-bonds with a

866 LETF or a VETF on a broad equity market index.

## 867 5.1 Investment scenarios

868 The key investment parameters used for illustrative purposes throughout this section are outlined in Table  
 869 5.1. Note in particular that we use a relatively long investment time horizon (10 years) coupled with relatively  
 870 infrequent (quarterly) rebalancing, and that the same implicit outperformance target  $\gamma$  is used in each of the  
 871 scenarios to facilitate a fair comparison. This value of  $\gamma$  is chosen for general illustrative purposes only, and the  
 872 conclusions remain qualitatively similar for different choices of  $\gamma$ .

**Table 5.1:** Key investment parameters for the illustrative results of Section 5.

Parameter	$T$	# rebalancing events ( $N_{rb}$ )	Initial wealth ( $w_0$ )	Contributions ( $q_n$ )	Target ( $\gamma$ )
Value	10 years	40 (quarterly rebalancing)	\$ 100	\$ 5 per year (\$1.25/quarter)	125

873 Table 5.2 provides an overview of the benchmark and the investor’s candidate assets. A 70/30 benchmark  
 874 strategy is again used, since it aligns to the definition of popular investment benchmarks used in practice (see  
 875 Remark 2.1). Note that the benchmark is defined in terms of the broad equity market index (“Market”) with  
 876 70% of the wealth allocation, with the remaining 30% split between 30-day T-bills and 10-year T-bonds. As  
 877 in Section 3, we assume that the investor cannot invest directly in the broad equity market index (“Market”),  
 878 but can gain exposure to this index via a VETF (expense ratio  $c_v = 0.06\%$ ) or a LETF with multiplier  $\beta = 2$   
 879 (expense ratio  $c_\ell = 0.89\%$ ).  
 880

**Table 5.2:** Candidate assets and benchmark for the illustrative results of Section 5. A mark “✓” indicates that an asset is available for inclusion. Note that the investor cannot invest directly in the market portfolio (“Market”), but only indirectly via either the VETF or LETF, whereas the benchmark is defined directly in terms of “Market” in alignment with popular investment benchmarks.

Underlying assets		Benchmark	Investor candidate assets	
Label	Asset description		Using VETF	Using LETF
T30	30-day Treasury bill	15%	✓	✓
B10	10-year Treasury bond	15%	✓	✓
Market	Market portfolio (broad equity market index)	70%	-	-
VETF	Vanilla (unleveraged) ETF replicating the returns of the market portfolio, with expense ratio $c_v = 0.06\%$	-	✓	-
LETF	Leveraged ETF with daily returns replicating $\beta = 2$ times the daily returns of the market portfolio, with expense ratio $c_\ell = 0.89\%$	-	-	✓

881 Table 5.3 provides more detail on the leverage and borrowing premium scenarios considered, where we  
 882 highlight the following:  
 883

- 884 • Investor portfolios formed with a LETF are never leveraged ( $p_{max} = 1.0$ ), whereas portfolios formed  
 885 with a VETF can use leverage up to a portfolio maximum of  $p_{max} \in \{1.0, 1.2, 1.5, 2.0\}$  via the short-  
 886 selling of 30-day T-bills (i.e. borrowing funds to invest in the VETF) with a borrowing premium  $b \in$   
 887  $\{0, 0.03\}$  potentially being applicable. This is done in order to compare the performance of an IR-optimal  
 888 portfolio with a LETF and no portfolio-level leverage with that of an IR-optimal portfolio formed with a  
 889 (potentially) leveraged VETF under various leverage assumptions.
- 890 • In terms of the selection of values for  $p_{max} \in \{1.0, 1.2, 1.5, 2.0\}$  in the case of the VETF investor, note that  
 891 Regulation T of the US Federal Reserve board requires at least 50% of the initial price of a stock position  
 892 to be available on deposit, while brokerage firms are free to establish more stringent requirements. For  
 893 the VETF investor, for illustrative purposes we will therefore mostly focus on the cases of  $p_{max} = 1.0$  (no  
 894 leverage) or  $p_{max} = 1.5$ , and for comparison purposes provide the additional examples using  $p_{max} = 1.2$   
 895 and  $p_{max} = 2.0$  in Appendix C.

- In terms of the selection of borrowing premiums  $b \in \{0, 0.03\}$  for the VETF investor, we first note that all returns are inflation-adjusted (see Appendix B), and so these quantities should be interpreted net of inflation. The case of zero borrowing premium ( $b = 0$ ) is provided for comparison purposes only, while the value of  $b = 3\%$  is obtained from the examples in Ni et al. (2024), where it is based on a consideration of the average real return for T-bills and the average inflation-adjusted corporate bond yields for Moody’s Aaa and Baa-rated bond issues.

**Table 5.3:** Maximum leverage and borrowing premium scenarios for the indicative investment results of Section 5.

Component of leverage scenario	Benchmark	Investor candidate assets	
		Using VETF	Using LETF
Maximum portfolio-level leverage ratio $p_{max}$	No leverage allowed ( $p_{max} = 1.0$ )	No leverage allowed ( $p_{max} = 1.0$ ) as well as scenarios $p_{max} \in \{1.2, 1.5, 2.0\}$	No leverage allowed ( $p_{max} = 1.0$ )
Shortable asset to fund leveraged position (if applicable)	-	T30	-
Borrowing premiums: Scenarios for premium $b$ over T30 return on leveraged positions (if applicable)	N/a	$b = 0$ or $b = 0.03$	N/a

The underlying data sets for the training and testing of the neural network giving the IR-optimal investment strategies using stationary block bootstrap resampling of empirical market data (see Section 4 and Appendix B) instead of calibrated process dynamics. In particular, we use all available inflation-adjusted market data over the time period January 1926 to December 2023, together with an expected block size of 3 months, to obtain 500,000 jointly bootstrapped asset return paths (see Li and Forsyth (2019); Van Staden et al. (2024) and Appendix B for more information). As in Ni et al. (2024), we use a shallow NN (2 hidden layers) with only the minimal input features  $(t, \mathbf{X}(t)) = (t, W(t), \hat{W}(t))$ , since that has been found sufficient to obtain a stable and accurate IR-optimal investment strategy in a setting where no additional market signals are used as inputs.

For illustrative purposes, we also present the investment results obtained from investing according to the IR-optimal investment strategy on selected historical data paths. This is discussed in more detail in Remark 5.1.

**Remark 5.1.** (Performance on single historical data paths) Since the future evolution of asset returns are not expected to replicate the past evolution of returns *precisely*, we consider illustrative investment results based on bootstrapped data sets as significantly more informative than using a single historical data path of asset returns to illustrate performance.

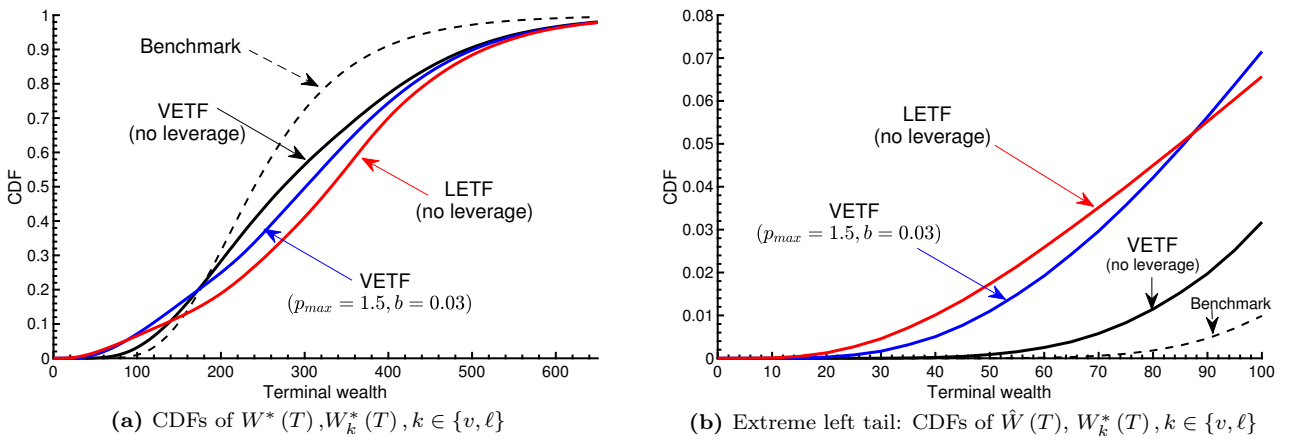
However, for purposes of concreteness and intuition, we do show the evolution of the LETF and VETF investor wealth obtained by implementing the corresponding IR-optimal portfolios and the benchmark on four historical data paths each spanning a period equal to the investment time horizon of 10 years:

- (i) January 2000 until December 2009, which illustrates the impact on the portfolio wealth of both the DotCom bubble crash as well as the GFC period.
- (ii) January 2005 until December 2014, which focuses on the GFC and the subsequent period of relatively slow market recovery.
- (iii) January 2010 until December 2019, which illustrates the performance during the bull market of the 2010s, a period of very low interest rates and therefore cheap leverage.
- (iv) January 2014 until December 2023, which combines an initial period of strong growth and low interest rates with the Covid-19 period and subsequent recovery, only to be followed by the bear market for stocks lasting from January to October 2022 and higher interest rates.

Note that while the historical path of returns enter the training data of the NN indirectly via bootstrap resampling, the probability that the actual historical data path itself appearing in the resulting bootstrapped data sets is vanishingly small (see Ni et al. (2022) for a proof), so that the historical data paths can themselves be considered as effectively “out-of-sample” for testing purposes. However, we emphasize that in this section the main focus remains on the investment results based on the much richer bootstrapped data set of returns data, which ensures a meaningful discussion of the implications for wealth *distributions*, for example, rather than individual wealth values from a single historical path.  $\square$

## 937 5.2 Comparison of investment results

938 Figure 5.1 illustrates the distributions of the IR-optimal terminal investor wealth  $W_k^*(T)$ ,  $k \in \{v, \ell\}$  for different  
 939 portfolios formed under the leverage scenarios as per Table 5.3, as well as the distribution of the benchmark  
 940 terminal wealth  $\hat{W}(T)$ . With regards to Figure 5.1(a), we see that the IR-optimal strategy using the LETF  
 941 (and bonds) achieves *partial stochastic dominance* (Atkinson (1987); Ni et al. (2024); Van Staden et al. (2021))  
 942 over the IR-optimal strategies using the VETF (and bonds), even if the VETF investment can be leveraged<sup>9</sup>.  
 943 The main driver of the difference in performance of the LETF-based strategy relative to that of the VETF-  
 944 based strategy in this setting is a combination of (i) the call-like payoff of the LETF as underlying asset over  
 945 a relatively short time horizon (e.g. 1 quarter) and (ii) the contrarian nature of the discretely-rebalanced  
 946 IR-optimal investment strategy locking in the gains from rebalancing. The observation that holding a LETF  
 947 position for a quarter amounts to holding a ‘‘continuously rebalanced’’ position in the equity index and bonds  
 948 which results in a power law-type payoff (see Section 3.1), also holds for the historical data underlying these  
 949 results (see Appendix C.2 for additional analyses).



**Figure 5.1:** CDFs of IR-optimal terminal wealth  $W_k^*(T)$ ,  $k \in \{v, \ell\}$  and the benchmark terminal wealth  $\hat{W}(T)$ .

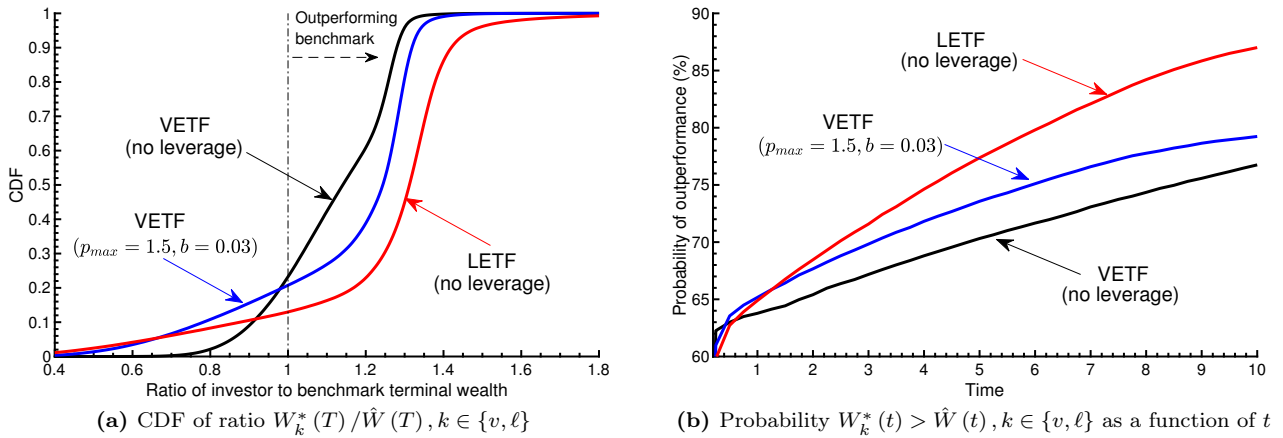
950 However, as Figure 5.1(b) illustrates, there is no free lunch with regards to leverage, similar to what we  
 951 observed in Figures 1.1, 3.1 and 3.2. In more detail, when considering the extreme left tails of the IR-optimal  
 952 terminal wealth CDFs, whether leveraging an investment implicitly (via the LETF) or explicitly (via a leveraged  
 953 VETF investment), the downside wealth outcomes are worse than using the VETF with no leverage (or  
 954 simply the benchmark). Note that this is based on the distribution of empirical market data together with  
 955 the implementation of IR-optimal investment strategies, and is consistent with the observations regarding the  
 956 downside protection offered by the LETFs in truly extreme cases (illustrated in Figures 3.2).

957 While Figure 5.1 considers a maximum leverage ratio of  $p_{max} = 1.5$  with a borrowing premium for short-  
 958 selling of  $b = 3\%$  (applicable to the leveraged VETF position), the results of Figure C.1 in Appendix C show  
 959 that qualitatively similar results are obtained even if leverage is allowed to increase to  $p_{max} = 2.0$ . However,  
 960 Appendix C shows that in the unrealistic case where the borrowing premium on short-selling is reduced to zero  
 961 (i.e. if the investor pays interest on short positions at exactly the T-bill rate), then a leveraged VETF-based  
 962 strategy with  $p_{max} = 2.0$  generates results that are comparable to (though *slightly* better than) the LETF-based  
 963 strategy that uses no additional leverage. These results are to be expected given the difference in VETF and  
 964 LETF expense ratios, since in the case of continuous rebalancing, no investment constraints and zero costs,  
 965 Proposition 3.4 shows that the IR-optimal investment results are identical, regardless of whether the investor  
 966 uses a LETF with multiplier  $\beta$  or a VETF leveraged  $\beta$  times. While the underlying assumptions of Proposition  
 967 3.4 are clearly violated in the setting of this section and the results in Appendix C, this observation underscores  
 968 the practical relevance of the closed-form solutions in the stylized setting of Section 3.

970 Figure 5.2 focuses on different measures of benchmark outperformance rather than investor wealth, with  
 971 Figure 5.2(a) illustrating the CDF of the terminal pathwise wealth ratio  $W_k^*(T)/\hat{W}(T)$ ,  $k \in \{v, \ell\}$  and Figure  
 972 5.2(b) illustrating the probability of benchmark outperformance over time. It is clear that IR-optimal port-  
 973 folios formed using the LETF and no further leverage significantly improves the benchmark outperformance

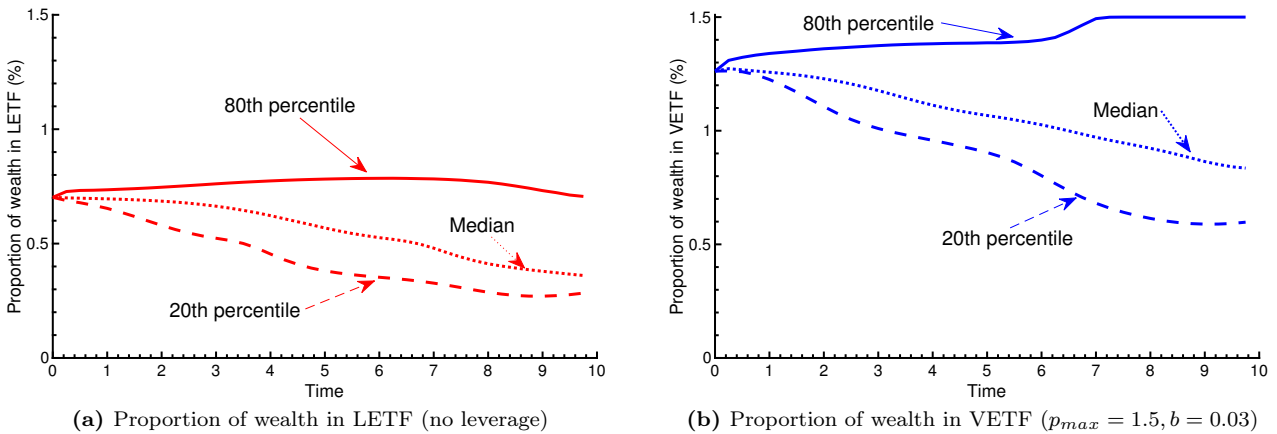
<sup>9</sup>Note that IR-optimal strategies incorporating either the VETF or the LETF achieve partial stochastic dominance over the benchmark, which is to be expected considering the results of Van Staden et al. (2023).

974 characteristics of the resulting strategy. Note that the results of Appendix C (Figure C.2 and Figure C.3) show  
 975 that the conclusions of Figure 5.2 remain qualitatively applicable for different leverage scenarios for the VETF  
 976 strategy, with the LETF strategy being slightly outperformed by a VETF-based strategy only in the specific  
 977 and unrealistic case of a zero borrowing premium and maximum leverage of  $p_{max} = 2.0$ .



**Figure 5.2:** CDFs of IR-optimal terminal wealth ratios  $W_k^*(T)/\hat{W}(T)$ ,  $k \in \{v, \ell\}$ , and probability  $W_k^*(t) > \hat{W}(t)$ ,  $k \in \{v, \ell\}$  of benchmark outperformance as a function of  $t$ .

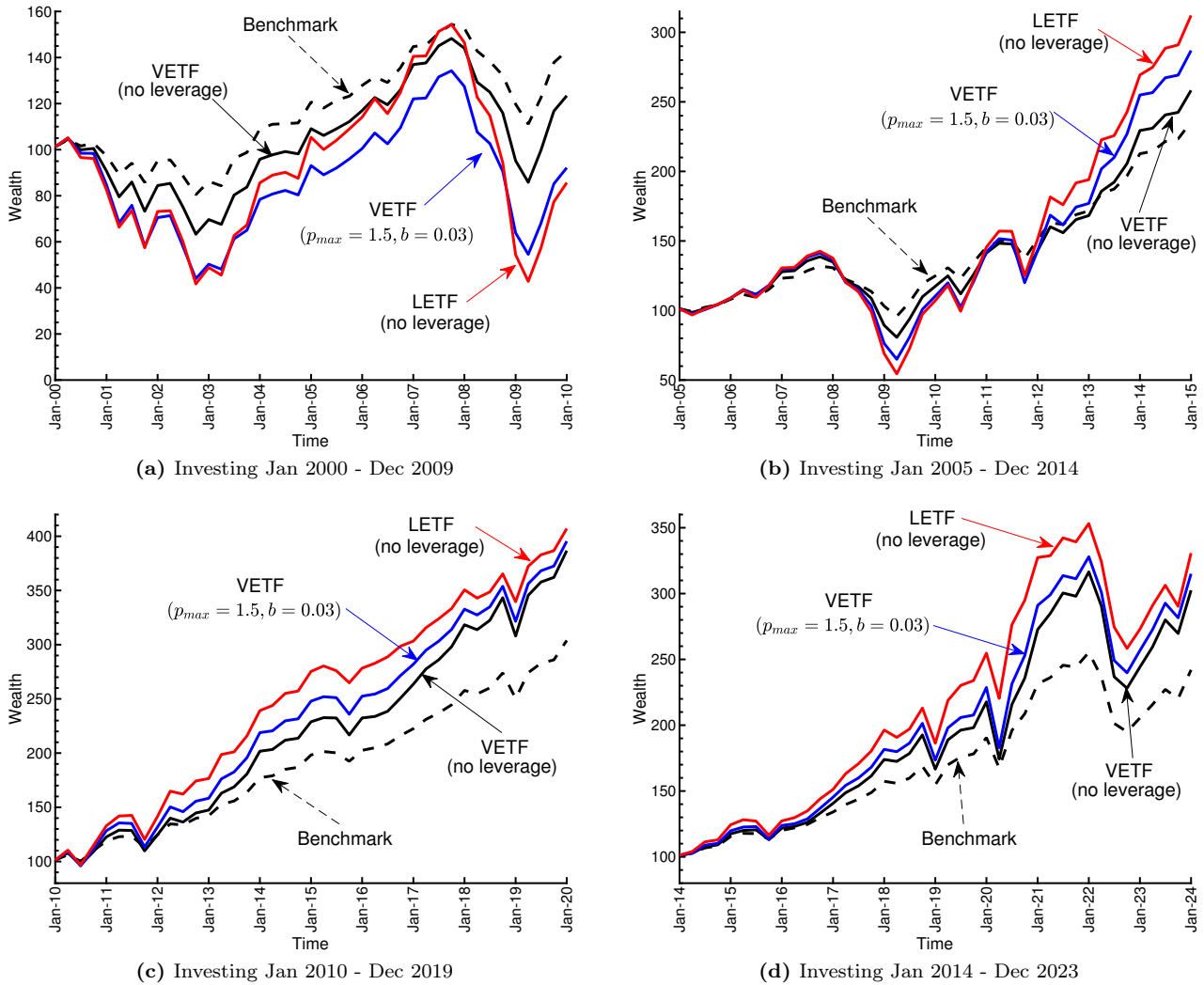
978  
 979 Figure 5.3 compares selected percentiles of the IR-optimal proportion of wealth in the LETF (no leverage)  
 980 or leveraged VETF ( $p_{max} = 1.5, b = 0.03$ ) over time, using the same scale in Figure 5.3(a) and Figure 5.3(b)  
 981 for illustrative purposes. Figure 5.3(a) shows that the LETF investor initially allocates around 70% of wealth  
 982 allocated to LETF, which falls to around 40% or less for both the 20th and 50th percentiles. In the case of the  
 983 portfolio with a leveraged position in the VETF ( $p_{max} = 1.5$ ), Figure 5.3(b) shows that the median allocation  
 984 to the VETF exceeds 100% of wealth for more than half the investment time horizon.



**Figure 5.3:** Selected percentiles of the IR-optimal proportion of wealth in the LETF (no leverage) compared to the corresponding percentiles invested in the leveraged VETF ( $p_{max} = 1.5, b = 0.03$ ) over time. Note the same scale on the y-axis has been used to facilitate comparison.

985  
 986 Figure 5.3 shows that executing the IR-optimal strategy using a LETF offers the investor more flexibility:  
 987 with lower levels of wealth tied up in the LETF compared to the allocation to a VETF, together with the higher  
 988 volatility of LETF-based returns, the investor can “lock in” periods of good past returns by implementing a  
 989 systematic de-risking of the portfolio to a lower allocation to LETF over time, increasing the allocation to bonds.  
 990 While the leveraged VETF-based strategy essentially follows the same contrarian pattern, the 80th percentile  
 991 in Figure 5.3(b) shows that it significantly harder for the leveraged VETF strategy to recover from periods of  
 992 poor past returns in a setting of maximum leverage restrictions, borrowing costs and no trading in the event of  
 993 bankruptcy.

994 In addition to the preceding results which are based on the bootstrap resampling of historical data, for  
 995 illustrative purposes we consider the investment performance on selected single historical paths (see Remark  
 996 5.1) illustrated in Figure 5.4. Figures 5.4(a) and (b) show that both the LETF and VETF investors (regardless  
 997 of VETF leverage) underperform the benchmark during the lowest points of the DotCom and GFC crashes,  
 998 with the LETF investor experiences larger peak-to-trough declines but also faster post-crash recovery. Figure  
 999 5.4(c) illustrate that the LETF-based strategy remains only slightly ahead of the VETF-based strategies during  
 1000 periods of strong equity market performance and low interest rates, while Figure 5.4(d) shows that the LETF  
 1001 investor stays ahead despite the significant impact on portfolio wealth of the Covid-19 period and subsequent  
 1002 bear market of 2022.



**Figure 5.4:** Evolution of portfolio wealth over time when investing according to the corresponding IR-optimal investment strategies on historical paths selected. See Remark 5.1 for a discussion regarding the paths selected.

1003

## 1004 6 Conclusion

1005 In this paper, we investigated the potential of including a broad stock market index-based leveraged ETF  
 1006 (LETF) in long-term, dynamically-optimal investment strategies designed to maximize the outperformance  
 1007 over standard performance benchmarks in terms of the information ratio (IR).

1008 Using both closed-form and numerical solutions, we showed that an investor can exploit the observation  
 1009 that LETFs offer call-like payoffs, and therefore could be a convenient way to add inexpensive leverage to the  
 1010 portfolio while providing extreme downside protection.

1011 Under stylized assumptions including continuous rebalancing and no investment constraints, we derived the

closed-form IR-optimal investment strategy for the LETF investor, which provided valuable intuition as to the contrarian nature of the strategy.

To allow for more general and practical conclusions, we use a data-driven neural network approach based on stationary block bootstrap resampling of historical data (including proxy LETF returns) since 1926. This ensures that the investment strategies and performance analysis incorporate all empirical aspects of actual returns, including potentially sophisticated volatility dynamics. We derive IR-optimal strategies that allow for quarterly trading, leverage restrictions, no trading in the event of insolvency and the presence of margin costs on borrowing. Our findings show that unleveraged IR-optimal strategies with a broad stock market LETF not only outperform the benchmark more often than possibly leveraged IR-optimal strategies derived using a VETF, but can achieve partial stochastic dominance over the benchmark and (leveraged or unleveraged) VETF-based strategies in terms of terminal wealth.

Two important caveats are to be kept in mind regarding our results demonstrating the potential of LETFs: (i) The results and conclusions are associated with dynamic IR-optimal investment strategies, which are most emphatically *not* naive strategies like the buy-and-hold strategies over long time horizons often considered in the literature (see the Introduction for a discussion). In particular, critical to the investment outcomes are the rebalancing of the portfolio within the context of a contrarian investment strategy. (ii) The results emphasize that there is no free lunch with regards to leverage. Specifically, the extreme left tails of the IR-optimal terminal wealth CDFs confirm that whether leveraging an investment implicitly (via the LETF) or explicitly (via a leveraged VETF investment), the downside wealth outcomes are worse than using the VETF without any leverage, and therefore the upside outcomes of leverage is not without significant risks. Nevertheless, bootstrap resampling tests indicate that use of an optimal strategy using LETFs outperforms the benchmark  $> 95\%$  of the time, which may make the extreme tail risk acceptable.

Despite the controversy surrounding the uses of LETFs for investment purposes in the literature, our results help to explain the empirical appeal of LETFs to institutional and retail investors alike, and encourage a reconsideration of the role of broad stock market LETFs within the context of more sophisticated investment strategies.

## 7 Declaration

Forsyth's work was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC) grant RGPIN-2017-03760. Li's work was supported by NSERC grant RGPIN-2020-04331.

The authors have no conflicts of interest to report.

## References

- Ahn, A., M. Haugh, and A. Jain (2015). Consistent pricing of options on leveraged ETFs. *SIAM Journal on Financial Mathematics* 6, 559–593.
- Alekseev, A. G. and M. V. Sokolov (2016). Benchmark-based evaluation of portfolio performance: a characterization. *Annals of Finance* 12, 409–440.
- Anarkulova, A., S. Cederburg, and M. S. O'Doherty (2022). Stocks for the long run? Evidence from a broad sample of developed markets. *Journal of Financial Economics* 143:1, 409–433.
- Applebaum, D. (2004). *Lévy processes and stochastic calculus*. Cambridge University Press.
- Atkinson, A. (1987). On the measurement of poverty. *Econometrica* 55(4), 749–764.
- Avellaneda, M. and S. Zhang (2010). Path-dependence of leveraged ETF returns. *SIAM Journal on Financial Mathematics* 1(1), 586–603.
- Bajeux-Besnainou, I., R. Portait, and G. Tergny (2013). Optimal portfolio allocations with tracking error volatility and stochastic hedging constraints. *Quantitative Finance* 13(10), 1599–1612.
- Bansal, V. K. and J. F. Marshall (2015). A tracking error approach to leveraged ETFs: Are they really that bad? *Global Finance Journal* 26, 47–63.
- Bednarek, Z. and P. Patel (2022). Just say no to leveraged ETFs. *Journal of Investment Management* 20(3), 53–69.
- Bensoussan, A., K. C. Wong, S. C. P. Yam, and S. P. Yung (2014). Time-consistent portfolio selection under short-selling prohibition: From discrete to continuous setting. *SIAM Journal on Financial Mathematics* 5, 153–190.
- Bertrand, P. and J.-L. Prigent (2022). On the diversification and rebalancing returns: performance comparison of constant mix versus buy and hold strategies SSRN 4153690.

- 1063 Bolshakov, A. and L. B. Chincarini (2020). Manager skill and portfolio size with respect to a benchmark. *European*  
1064 *Financial Management* 26(1), 176–197.
- 1065 Buehler, H., L. Gonon, J. Teichmann, and B. Wood (2019). Deep hedging. *Quantitative Finance* 19(8), 1271–1291.
- 1066 Canadian Pension Plan (2022). Our investment strategy. [www.cppinvestments.com/the-fund/how-we-invest/  
1067 our-investment-strategy](http://www.cppinvestments.com/the-fund/how-we-invest/our-investment-strategy).
- 1068 Carver, A. B. (2009). Do leveraged and inverse ETFs converge to zero? *ETFs and Indexing* 1, 144–149.
- 1069 Cavaglia, S., L. Scott, K. Blay, and S. Hixon (2022). Multi-asset class factor premia: A strategic asset allocation  
1070 perspective. *The Journal of Portfolio Management* 48:9, 14–32.
- 1071 Charupat, N. and P. Miu (2011). The pricing and performance of leveraged exchange-traded funds. *Journal of Banking  
1072 and Finance* 35(4), 966–977.
- 1073 Cheng, M. and A. Madhavan (2009). The dynamics of leveraged and inverse exchange-traded funds. *Journal of Investment  
1074 Management* 7, 43–62.
- 1075 Cogneau, P. and V. Zakalmouline (2013). Block bootstrap methods and the choice of stocks for the long run. *Quantitative  
1076 Finance* 13, 1443–1457.
- 1077 Dai, M., S. Kou, H. M. Soner, and C. Yang (2023). Leveraged exchange-traded funds with market closure and frictions.  
1078 *Management Science* 69(4), 2517–2535.
- 1079 Dang, D. and P. Forsyth (2016). Better than pre-commitment mean-variance portfolio allocation strategies: A semi-self-  
1080 financing Hamilton–Jacobi–Bellman equation approach. *European Journal of Operational Research* 250, 827–841.
- 1081 DeVault, L., H. Turtle, and K. Wang (2021). Blessing or curse? institutional investment in leveraged ETFs. *Journal of  
1082 Banking and Finance* 129(106169).
- 1083 Dichtl, H., W. Drobetz, and M. Wambach (2016). Testing rebalancing strategies for stock-bond portfolios across different  
1084 asset allocations. *Applied Economics* 48, 772–788.
- 1085 Forsyth, P. (2020). Multiperiod mean conditional value at risk asset allocation: Is it advantageous to be time consistent?  
1086 *SIAM Journal on Financial Mathematics* 11(2), 358–384.
- 1087 Forsyth, P. and K. Vetzal (2017). Dynamic mean variance asset allocation: Tests for robustness. *International Journal  
1088 of Financial Engineering* 4:2. 1750021 (electronic).
- 1089 Forsyth, P., K. Vetzal, and G. Westmacott (2019). Management of portfolio depletion risk through optimal life cycle  
1090 asset allocation. *North American Actuarial Journal* 23(3), 447–468.
- 1091 Giese, G. (2010). On the risk-return profile of leveraged and inverse ETFs. *Journal of Asset Management* 11, 219–228.
- 1092 Goetzmann, W., J. Ingersoll, M. Spiegel, and I. Welch (2002). Sharpening Sharpe ratios. *NBER Working Paper 9116* .
- 1093 Goetzmann, W., J. Ingersoll, M. Spiegel, and I. Welch (2007). Manipulation and manipulation-proof performance  
1094 measures. *Review of Financial Economics* 20, 1503–1546.
- 1095 Government Pension Fund Global (2022). Investment strategy. [www.nbim.no/en/the-fund/how-we-invest/  
1096 investment-strategy/](http://www.nbim.no/en/the-fund/how-we-invest/investment-strategy/).
- 1097 Guasoni, P. and E. Mayerhofer (2023). Leveraged funds: Robust replication and performance evaluation. *Quantitative  
1098 Finance* 23(7-8), 1155–1176.
- 1099 Han, J. and E. Weinan (2016). Deep learning approximation for stochastic control problems. *NIPS Deep Reinforcement  
1100 Learning Workshop* .
- 1101 Hassine, M. and T. Roncalli (2013). Measuring performance of exchange-traded funds. *The Journal of Index Investing*  
1102 4(3), 57–85.
- 1103 Hill, J. and G. Foster (2009). Understanding returns of leveraged and inverse funds. *Journal of Indexes* Septem-  
1104 ber/October, 40–58.
- 1105 Hu, R. and M. Laurière (2023). Recent developments in machine learning methods for stochastic control and games.  
1106 *Working paper* .
- 1107 Israelsen, C. K. and G. F. Cogswell (2007). The error of tracking error. *Journal of Asset Management* 7(6), 419–424.
- 1108 Jarrow, R. A. (2010). Understanding the risk of leveraged ETFs. *Finance Research Letters* 7(3), 135–139.
- 1109 Kashyap, A. K., N. Kovrijnykh, J. Li, and A. Pavlova (2021). The benchmark inclusion subsidy. *Journal of Financial  
1110 Economics* Forthcoming.
- 1111 Knapp, C. (2023). Leveraged ETFs: The data says we’re missing out. *LinkedIn* [https://linkedin.com/pulse/leveraged-  
1112 etfs-data-says-were-missing-out-christian-knapp](https://linkedin.com/pulse/leveraged-etfs-data-says-were-missing-out-christian-knapp).
- 1113 Korn, R. and C. Lindberg (2014). Portfolio optimization for an investor with a benchmark. *Decisions in Economics and  
1114 Finance* 37, 373–384.
- 1115 Kou, S. G. (2002). A jump-diffusion model for option pricing. *Management Science* 48(8), 1086–1101.
- 1116 Lehalle, C.-A. and G. Simon (2021). Portfolio selection with active strategies: how long only constraints shape convictions.  
1117 *Journal of Asset Management* .  
1118 URL <https://doi.org/10.1057/s41260-021-00219-z>
- 1119 Leung, T., M. Lorig, and A. Pascucci (2017). Leveraged ETF implied volatilities from ETF dynamics. *Mathematical  
1120 Finance* 27(4), 1035–1068.

- 1121 Leung, T. and M. Santoli (2012). Leveraged etfs: admissible leverage and risk horizon. *Journal of Investment Strategies*  
1122 2(1), 39–61.
- 1123 Leung, T. and R. Sircar (2015). Implied volatility of leveraged ETF options. *Applied Mathematical Finance* 22(2),  
1124 162–188.
- 1125 Li, D. and W.-L. Ng (2000). Optimal dynamic portfolio selection: multi period mean variance formulation. *Mathematical*  
1126 *Finance* 10, 387–406.
- 1127 Li, Y. and P. Forsyth (2019). A data-driven neural network approach to optimal asset allocation for target based defined  
1128 contribution pension plans. *Insurance: Mathematics and Economics* 86, 189–204.
- 1129 Mackintosh, P. (2008). Double trouble. *The Journal of Beta Investment Strategies* 1, 25–31.
- 1130 Mäkinen, R. A. E. and J. Toivanen (2024). Short communication: Monte carlo expected wealth and risk measure trade-off  
1131 portfolio optimization. *SIAM Journal on Financial Mathematics* 15(2), SC41–SC53.
- 1132 Menoncin, F. and E. Vigna (2017). Mean-variance target-based optimisation for defined contribution schemes in a  
1133 stochastic framework. *Insurance: Mathematics and Economics* 76, 172–184.
- 1134 Merton, R. (1976). Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics* 3,  
1135 125–144.
- 1136 Ni, C., Y. Li, P. Forsyth, and R. Carroll (2022). Optimal asset allocation for outperforming a stochastic benchmark  
1137 target. *Quantitative Finance* 22:9, 1595–1626.
- 1138 Ni, C., Y. Li, and P. A. Forsyth (2024). Neural network approach to portfolio optimization with leverage constraints: a  
1139 case study on high inflation investment. *Quantitative Finance* 24(6), 753–777.
- 1140 Oksendal, B. and A. Sulem (2019). *Applied Stochastic Control of Jump Diffusions*. Springer, 3rd edition.
- 1141 Politis, D. and J. Romano (1994). The stationary bootstrap. *Journal of the American Statistical Association* 89,  
1142 1303–1313.
- 1143 Politis, D., J. Romano, and M. Wolf (1999). *Subsampling*. Springer-Verlag. Chapter 4: Subsampling for nonstationary  
1144 time series.
- 1145 Politis, D. N. (2003). The impact of bootstrap methods on time series analysis. *Statistical Science* 18:2, 219–230.
- 1146 Reppen, A. M. and H. M. Soner (2023). Deep empirical risk minimization in finance: looking into the future. *Mathematical*  
1147 *Finance* 33(1), 116–145.
- 1148 Reppen, A. M., H. M. Soner, and V. Tissot-Daguette (2023). Deep stochastic optimization in finance. *Digital Finance*  
1149 5, 91–111.
- 1150 Scott, L. and S. Cavaglia (2017). A wealth management perspective on factor premia and the value of downside protection.  
1151 *The Journal of Portfolio Management* 43:3, 1–9.
- 1152 Simonian, J. and A. Martirosyan (2022). Sharpe parity redux. *The Journal of Portfolio Management* 48:9, 183–193.
- 1153 Strub, M., D. Li, and X. Cui (2019). An enhanced mean-variance framework for robo-advising applications SSRN 302111.
- 1154 Sullivan, R. N. (2009). The pitfalls of leveraged and inverse ETFs. *CFA Magazine* 20(3).
- 1155 Trainor, W., I. Chhachi, and C. Brown (2018). A portfolio of leveraged exchange traded funds. *Working paper* .
- 1156 Van Staden, P. M., D. Dang, and P. Forsyth (2018). Time-consistent mean-variance portfolio optimization: a numerical  
1157 impulse control approach. *Insurance: Mathematics and Economics* 83(C), 9–28.
- 1158 Van Staden, P. M., D. Dang, and P. Forsyth (2021). On the distribution of terminal wealth under dynamic mean-variance  
1159 optimal investment strategies. *SIAM Journal on Financial Mathematics* 12(2), 566–603.
- 1160 Van Staden, P. M., P. A. Forsyth, and Y. Li (2023). Beating a benchmark: dynamic programming may not be the right  
1161 numerical approach. *SIAM Journal on Financial Mathematics* 14(2).
- 1162 Van Staden, P. M., P. A. Forsyth, and Y. Li (2024). A global-in-time neural network approach to dynamic portfolio  
1163 optimization. *Applied Mathematical Finance* (online), 1–33.  
1164 URL <https://doi.org/10.1080/1350486X.2024.2410200>
- 1165 Vigna, E. (2014). On efficiency of mean-variance based portfolio selection in defined contribution pension schemes.  
1166 *Quantitative Finance* 14(2), 237–258.
- 1167 Vigna, E. (2020). On time consistency for mean-variance portfolio selection. *International Journal of Theoretical and*  
1168 *Applied Finance* 23(6).
- 1169 Vigna, E. (2022). Tail optimality and preferences consistency for intertemporal optimization problems. *SIAM Journal*  
1170 *on Financial Mathematics* 13(1).
- 1171 Wagalath, L. (2014). Modelling the rebalancing slippage of leveraged exchange-traded funds. *Quantitative Finance* 14(9),  
1172 1503–1511.
- 1173 Yoon, J., D. Jarrett, and M. Van der Schaar (2019). Time-series generative adversarial networks. *33rd Conference*  
1174 *on Neural Information Processing Systems (NeurIPS 2019)* <https://proceedings.neurips.cc/paper/2019/file/c9efe5f26cd17ba6216bbe2a7d26d490-Paper.pdf>.
- 1175  
1176 Zhao, Y. (2007). A dynamic model of active portfolio management with benchmark orientation. *Journal of Banking and*  
1177 *Finance* 31, 3336–3356.

## 1180 Appendix A: Proofs of main results

1181 The proofs of the key results of Section 3 are presented in this appendix.

### 1182 A.1: Proof of Theorem 3.1

1183 For a given deterministic benchmark strategy  $\hat{\varrho}_s(t)$ , consider an arbitrary admissible investor strategy  $\varrho_\ell(t) :=$   
 1184  $\varrho_\ell(t, \mathbf{X}_\ell(t)) \in \mathcal{A}_0$ , where we omit the dependence of  $\varrho_\ell$  on  $\mathbf{X}_\ell(t) = (W_\ell(t), \hat{W}(t), \hat{\varrho}_s(t))$  for notational  
 1185 simplicity. Considering the objective functional of the IR problem (3.26) at a given point  $(t, w, \hat{w}) \in [t_0, T] \times \mathbb{R}^2$   
 1186 for a given and fixed value of  $\gamma > 0$ , define

$$1187 \quad J(t, w, \hat{w}; \varrho_\ell) = E_{\varrho_\ell}^{t, w, \hat{w}} \left[ \left( W_\ell(T) - [\hat{W}(T) + \gamma] \right)^2 \right]. \quad (\text{A.1})$$

1188 If we proceed informally and assume that  $J$  sufficiently smooth, then the application of Itô's lemma for jump  
 1189 processes (Oksendal and Sulem (2019)) gives

$$\begin{aligned} 1190 & E_{\varrho_\ell}^{t, w, \hat{w}} \left[ \int_t^{t+h} dJ(u, W_\ell(u), \hat{W}(u); \varrho_\ell) \right] \\ 1191 &= E_{\varrho_\ell}^{t, w, \hat{w}} \left[ \int_t^{t+h} \left( \frac{\partial J}{\partial t} + \frac{\partial J}{\partial w} \cdot \{W_\ell(u) \cdot [r + \varrho_\ell(u) \{\beta(\mu - \lambda\kappa_1^s - r) - c_\ell\}] + q\} \right) \cdot du \right] \\ 1192 &+ E_{\varrho_\ell}^{t, w, \hat{w}} \left[ \int_t^{t+h} \frac{\partial J}{\partial \hat{w}} \cdot \{ \hat{W}(u) \cdot [r + (\mu - \lambda\kappa_1^s - r) \hat{\varrho}_s(u)] + q \} \cdot du \right] \\ 1193 &+ E_{\varrho_\ell}^{t, w, \hat{w}} \left[ \int_t^{t+h} \frac{1}{2} \left( \frac{\partial^2 J}{\partial \hat{w}^2} \cdot [\hat{\varrho}_s(u) \hat{W}(u) \sigma]^2 + \frac{\partial^2 J}{\partial w^2} \cdot [\varrho_\ell(u) W_\ell(u) \beta \sigma]^2 \right) \cdot du \right] \\ 1194 &+ E_{\varrho_\ell}^{t, w, \hat{w}} \left[ \int_t^{t+h} \frac{\partial^2 J}{\partial w \partial \hat{w}} \cdot [\varrho_\ell(u) W_\ell(u) \beta \sigma] [\hat{\varrho}_s(u) \hat{W}(u) \sigma] \cdot du \right] \\ 1195 &+ E_{\varrho_\ell}^{t, w, \hat{w}} \left[ \lambda \int_t^{t+h} \left[ \int_0^\infty \phi(u, W_\ell(u^-), \hat{W}(u^-), \xi^s; \varrho_\ell) G(\xi^s) d\xi^s - J(W_\ell(u^-), \hat{W}(u^-), u; \varrho_\ell) \right] d\mathbf{A} \right] \end{aligned} \quad (\text{A.2})$$

1196 where all partial derivatives are evaluated at  $(u, W_\ell(u), \hat{W}(u); \varrho_\ell)$ , and

$$\begin{aligned} 1197 & \phi(u, W_\ell(u^-), \hat{W}(u^-), \xi^s; \varrho_\ell) \\ 1198 &= J(W_\ell(u^-) + \varrho_\ell(u) W_\ell(u^-) \beta (\xi^\ell - 1), \hat{W}(u^-) + \hat{\varrho}_s(u) \hat{W}(u^-) (\xi^s - 1), u; \varrho_\ell). \quad (\text{A.3}) \end{aligned}$$

1200 Recall that the LETF jump multiplier  $\xi^\ell$  is a function (3.6) of the underlying index  $S$  jump multiplier  $\xi^s$ , so  $\phi$   
 1201 in (A.4) can be interpreted as a function of  $\xi^s$  if all other values are held fixed.

1202 Continuing to proceed informally, dividing (A.2) by  $h > 0$ , taking limits as  $h \downarrow 0$  and assuming the limits  
 1203 and expectations could be interchanged, an application of the dynamic programming principle results in the  
 1204 PIDE (3.27)-(3.28).

1205 While providing the necessary intuition, the preceding arguments are merely informal. However, since  
 1206 similar arguments (see Applebaum (2004); Oksendal and Sulem (2019)) can be applied to a suitably smooth  
 1207 test function instead of the objective functional in order to formally prove (3.27)-(3.28), the details are therefore  
 1208 omitted.

## 1209 A.2: Proof of Proposition 3.2

1210 The quadratic terminal condition (3.28) suggests an ansatz for the value function  $V$  of the form

$$1211 \quad V(t, w, \hat{w}, \hat{\varrho}_s) = A(t)w^2 + \hat{A}(t)\hat{w}^2 + D(t)w\hat{w} + F(t)w + \hat{F}(t)\hat{w} + C(t), \quad (\text{A.4})$$

1212 where  $A, \hat{A}, D, F, \hat{F}$  and  $C$  are deterministic but unknown functions of time. Since (A.4) implies partial deriva-  
1213 tives of the form

$$1214 \quad \frac{\partial V}{\partial w} = 2A(t)w + F(t) + D(t)\hat{w}, \quad \frac{\partial^2 V}{\partial w^2} = 2A(t), \quad \text{and} \quad \frac{\partial^2 V}{\partial w \partial \hat{w}} = D(t), \quad (\text{A.5})$$

1215 substituting (A.4)-(A.5) into the HJB PIDE (3.27) results in the pointwise supremum  $\varrho_\ell^* = \varrho_\ell^*(t, w, \hat{w}, \hat{\varrho}_s)$   
1216 obtained from the first-order condition that satisfies the relationship

$$1217 \quad \varrho_\ell^* \cdot w = - \left( \frac{\beta [\mu + \lambda (\kappa_1^\ell - \kappa_1^s) - r] - c_\ell}{\beta^2 (\sigma^2 + \lambda \kappa_2^\ell)} \right) \left[ w + \frac{F(t)}{2A(t)} + \frac{D(t)}{2A(t)} \cdot \hat{w} \right] \\ 1218 \quad - \left[ \frac{\sigma^2 + \lambda \kappa_\chi^{\ell,s}}{\beta (\sigma^2 + \lambda \kappa_2^\ell)} \right] \frac{D(t)}{2A(t)} \cdot \hat{\varrho}_s \hat{w}. \quad (\text{A.6})$$

1219 Substituting (A.6) into (3.29) to obtain  $\mathcal{H}(\varrho_\ell^*; t, w, \hat{w}, \hat{\varrho}_s)$ , the PIDE (3.27)-(3.28) implies the following set of  
1220 ordinary differential equations (ODEs) for the unknown functions  $A, D$  and  $F$  on  $t \in [t_0, T]$ ,

$$1221 \quad \frac{d}{dt}A(t) = - \left( 2r - \frac{\beta [\mu + \lambda (\kappa_1^\ell - \kappa_1^s) - r] - c_\ell}{\beta^2 (\sigma^2 + \lambda \kappa_2^\ell)} \right) A(t), \quad A(T) = 1, \quad (\text{A.7})$$

$$1222 \quad \frac{d}{dt}D(t) = - \left( 2r - \frac{\beta [\mu + \lambda (\kappa_1^\ell - \kappa_1^s) - r] - c_\ell}{\beta^2 (\sigma^2 + \lambda \kappa_2^\ell)} + K_\beta^{\ell,s} \cdot \hat{\varrho}_s(t) \right) D(t), \quad D(T) = -2, \quad (\text{A.8})$$

$$1223 \quad \frac{d}{dt}F(t) = -2qA(t) - \left( r - \frac{\beta [\mu + \lambda (\kappa_1^\ell - \kappa_1^s) - r] - c_\ell}{\beta^2 (\sigma^2 + \lambda \kappa_2^\ell)} \right) F(t) - qD(t), \quad F(T) = -2\gamma, \quad (\text{A.9})$$

1224 where the constant  $K_\beta^{\ell,s}$  is given by (3.33)

1225 Note that the derivation of (A.8) as an ODE requires the benchmark strategy  $\hat{\varrho}_s$  to be deterministic (in the  
1226 case of closed-form solutions) as per Assumption 3.2. Solving ODEs (A.7)-(A.9), we obtain

$$1227 \quad A(t) = \exp \left\{ \left( 2r - \frac{\beta [\mu + \lambda (\kappa_1^\ell - \kappa_1^s) - r] - c_\ell}{\beta^2 (\sigma^2 + \lambda \kappa_2^\ell)} \right) (T - t) \right\}, \quad (\text{A.10})$$

$$1228 \quad D(t) = -2 \cdot \exp \left\{ \left( 2r - \frac{\beta [\mu + \lambda (\kappa_1^\ell - \kappa_1^s) - r] - c_\ell}{\beta^2 (\sigma^2 + \lambda \kappa_2^\ell)} \right) (T - t) + K_\beta^{\ell,s} \cdot \int_t^T \hat{\varrho}_s(u) du \right\}, \quad (\text{A.11})$$

$$1229 \quad F(t) = 2 \exp \left\{ \left( r - \frac{\beta [\mu + \lambda (\kappa_1^\ell - \kappa_1^s) - r] - c_\ell}{\beta^2 (\sigma^2 + \lambda \kappa_2^\ell)} \right) (T - t) \right\} \times \\ 1230 \quad \left[ -\gamma + \frac{q}{r} (e^{r(T-t)} - 1) - q \cdot \int_t^T \exp \left\{ r(T-v) + K_\beta^{\ell,s} \cdot \int_v^T \hat{\varrho}_s(u) du \right\} dv \right]. \quad (\text{A.12})$$

1231 Substituting (A.10)-(A.12) into (A.6) and simplifying, we obtain the optimal fraction of wealth  $\varrho_\ell^*$  to invest in  
1232 the LETF (3.30) as per Proposition 3.2.

## 1233 A.3: Expressions for $\kappa_1^\ell$ , $\kappa_2^\ell$ and $\kappa_\chi^{\ell,s}$

For the purposes of illustrating the closed-form solutions of Section 3, the broad equity market index  $S$  is assumed to have dynamics (3.2) with jumps as modelled in Kou (2002). As a result, with  $p_{up}$  denoting the probability of an upward jump given that a jump occurs,  $y = \log \xi^s$  is assumed in Kou (2002) to follow an asymmetric double-exponential distribution with PDF  $g(y)$ ,

$$g(y) = p_{up} \eta_1 e^{-\eta_1 y} \mathbb{I}_{\{y \geq 0\}} + (1 - p_{up}) \eta_2 e^{\eta_2 y} \mathbb{I}_{\{y < 0\}}, \quad (\text{A.13})$$

1234 where  $p_{up} \in [0, 1]$  and  $\eta_1 > 1, \eta_2 > 0$ . Equivalently, the PDF of  $\xi^s$  is given by

$$1235 \quad G(\xi^s) = p_{up}\eta_1 (\xi^s)^{-\eta_1-1} \mathbb{I}_{[\xi^s \geq 1]}(\xi^s) + (1-p_{up})\eta_2 (\xi^s)^{\eta_2-1} \mathbb{I}_{[0 \leq \xi^s < 1]}(\xi^s). \quad (\text{A.14})$$

1236 Recall that we have defined  $\kappa_1^s$  and  $\kappa_1^s$  in (3.3), repeated here for convenience,

$$1237 \quad \kappa_1^s = \mathbb{E}[\xi^s - 1], \quad \kappa_2^s = \mathbb{E}\left[(\xi^s - 1)^2\right]. \quad (\text{A.15})$$

1238 From the results in Kou (2002), we can obtain (A.15) for the distribution (A.14) using the results

$$1239 \quad \mathbb{E}[\xi^s] = \frac{p_{up}\eta_1}{\eta_1 - 1} + \frac{(1-p_{up})\eta_2}{\eta_2 + 1}, \quad \mathbb{E}\left[(\xi^s)^2\right] = \frac{p_{up}\eta_1}{\eta_1 - 2} + \frac{(1-p_{up})\eta_2}{\eta_2 + 2}. \quad (\text{A.16})$$

1240 However, since the LETF experiences slightly different jumps as per (3.6), which we repeat here for convenience,

$$1241 \quad \xi^\ell = \begin{cases} \xi^s & \text{if } \xi^s > (\beta - 1)/\beta, \\ \frac{(\beta - 1)}{\beta} & \text{if } \xi^s \leq (\beta - 1)/\beta, \end{cases} \quad (\text{A.17})$$

1242 we cannot use the results (A.15) directly. Instead, expressions for  $\kappa_1^\ell$ ,  $\kappa_2^\ell$  and  $\kappa_\chi^{\ell,s}$  are obtained using the results  
1243 the following lemma.

1244 **Lemma A.1.** ( $\kappa_1^\ell$ ,  $\kappa_2^\ell$  and  $\kappa_\chi^{\ell,s}$  in the Kou (2002) model) Suppose the jump multiplier  $\xi^s$  in the S-dynamics  
1245 (3.2) has PDF  $G(\xi^s)$  given by (A.14), and a LETF with returns multiplier  $\beta > 1$  and dynamics (3.8) has jump  
1246 multiplier  $\xi^\ell$ , which is defined in terms of  $\xi^s$  as per (A.17). Then the quantities

$$1247 \quad \kappa_1^\ell = \mathbb{E}[\xi^\ell - 1], \quad \kappa_2^\ell = \mathbb{E}\left[(\xi^\ell - 1)^2\right], \quad \kappa_\chi^{\ell,s} = \mathbb{E}\left[(\xi^\ell - 1)(\xi^s - 1)\right], \quad (\text{A.18})$$

1248 required by the IR-optimal investment strategy in Proposition 3.2 can be obtained using the following expressions:

$$1249 \quad \mathbb{E}[\xi^\ell] = \frac{p_{up}\eta_1}{\eta_1 - 1} + (1-p_{up})\eta_2 \cdot \left[ \frac{\vartheta^{\eta_2+1}}{\eta_2} + \left( \frac{1 - \vartheta^{\eta_2+1}}{\eta_2 + 1} \right) \right], \quad (\text{A.19})$$

$$1250 \quad \mathbb{E}\left[(\xi^\ell)^2\right] = \frac{p_{up}\eta_1}{\eta_1 - 2} + (1-p_{up})\eta_2 \cdot \left[ \frac{\vartheta^{\eta_2+2}}{\eta_2} + \left( \frac{1 - \vartheta^{\eta_2+2}}{\eta_2 + 2} \right) \right], \quad (\text{A.20})$$

$$1251 \quad \mathbb{E}[\xi^\ell \xi^s] = \frac{p_{up}\eta_1}{\eta_1 - 2} + (1-p_{up})\eta_2 \cdot \left[ \frac{\vartheta^{\eta_2+2}}{\eta_2 + 1} + \left( \frac{1 - \vartheta^{\eta_2+2}}{\eta_2 + 2} \right) \right], \quad (\text{A.21})$$

1252 where  $\vartheta = (\beta - 1)/\beta$ .

1253 *Proof.* Consider (A.21). Since  $\beta > 1$  and  $\vartheta = (\beta - 1)/\beta$ , we have  $0 < \vartheta < 1$ . Therefore, using the definition of  
1254 the LETF jump multiplier (A.17) and the PDF  $G(\xi^s)$ , we have

$$1255 \quad \mathbb{E}[\xi^\ell \xi^s] = \int_0^\infty \xi^\ell \xi^s \cdot G(\xi^s) d\xi^s \\ 1256 \quad = \vartheta \cdot \int_0^\vartheta \xi^s \cdot G(\xi^s) d\xi^s + \int_\vartheta^1 (\xi^s)^2 \cdot G(\xi^s) d\xi^s + \int_1^\infty (\xi^s)^2 \cdot G(\xi^s) d\xi^s. \quad (\text{A.22})$$

1257 Standard results (see (A.16) and Kou (2002)) for the Kou model gives

$$1258 \quad \int_1^\infty (\xi^s)^2 \cdot G(\xi^s) d\xi^s = \frac{p_{up}\eta_1}{\eta_1 - 2}. \quad (\text{A.23})$$

1259 Using (A.23) and writing the first two terms of (A.22) in terms of the log jump multiplier  $y = \log \xi^s$  with PDF  
1260  $g(y)$  as per (A.13), we have

$$1261 \quad \mathbb{E}[\xi^\ell \xi^s] = \vartheta \cdot \int_{-\infty}^{\log \vartheta} e^y g(y) dy + \int_{\log \vartheta}^0 e^{2y} g(y) dy + \frac{p_{up}\eta_1}{\eta_1 - 2} \\ 1262 \quad = (1-p_{up})\eta_2 \vartheta \cdot \int_{-\infty}^{\log \vartheta} e^{(\eta_2+1)y} dy + (1-p_{up})\eta_2 \int_{\log \vartheta}^0 e^{(\eta_2+2)y} dy + \frac{p_{up}\eta_1}{\eta_1 - 2}. \quad (\text{A.24})$$

1263 Simplifying (A.24) gives (A.21). Since (A.19) and (A.20) can be obtained using similar arguments, the details  
1264 are omitted.  $\square$

### 1265 A.4: Proof of Corollary 3.3

1266 For purposes of intuition, we first give informal arguments as to how the results of Corollary 3.3 relate to the  
1267 results of Proposition 3.2. Recall that the VETF has returns multiplier  $\beta = 1$  (i.e. the VETF simply aims to  
1268 replicate the returns of  $S$  before costs) and expense ratio  $c_v > 0$ . Note that if we let  $\beta \downarrow 1$  in (3.6), we have

$$1269 \lim_{\beta \downarrow 1} \xi^\ell = \xi^s \quad \text{a.s.}, \quad (\text{A.25})$$

1270 from which it follows that

$$1271 \lim_{\beta \downarrow 1} \kappa_1^\ell = \kappa_1^s, \quad \lim_{\beta \downarrow 1} \kappa_2^\ell = \kappa_2^s, \quad \lim_{\beta \downarrow 1} \kappa_\chi^{\ell,s} = \kappa_2^s. \quad (\text{A.26})$$

1272 Therefore, comparing the VETF and LETF investor wealth dynamics (3.23)-(3.24) in the case of identical  
1273 expense ratios (i.e.  $c_\ell = c_v$ ), identical but not necessarily optimal investment strategies ( $\varrho_\ell = \varrho_v$ ) and the  
1274 identical initial wealth, we have

$$1275 \lim_{\beta \downarrow 1} W_\ell(t) = W_v(t) \quad \text{a.s. } \forall t \in [t_0, T], \quad \text{if } W_\ell(t_0) = W_v(t_0), c_\ell = c_v, \quad \text{and } \varrho_\ell = \varrho_v. \quad (\text{A.27})$$

1276 In other words, if we let  $\beta \downarrow 1$  in the LETF investor wealth dynamics (3.24), we recover the VETF investor  
1277 wealth dynamics (3.23). Continuing to proceed informally, the results of Corollary 3.3 can therefore be obtained  
1278 by letting  $\beta \downarrow 1$  in the results of Proposition 3.2, provided we use the VETF expense ratio  $c_v$  in both (3.24)  
1279 and (3.23). Note that the definition (3.33) of  $K_\beta^{\ell,s}$ , results (A.26) and setting  $c_\ell = c_v$  imply that

$$1280 \lim_{\beta \downarrow 1} K_\beta^{\ell,s} = \lim_{\beta \downarrow 1} \left[ \mu - r - \frac{(\beta [\mu + \lambda (\kappa_1^\ell - \kappa_1^s) - r] - c_v) (\sigma^2 + \lambda \kappa_\chi^{\ell,s})}{\beta (\sigma^2 + \lambda \kappa_2^\ell)} \right] \\ 1281 = c_v, \quad (\text{A.28})$$

1282 which confirms that the functions  $g_v$  and  $h_v$  ((3.35)-(3.36)) can be obtained from the functions the functions  $g_\ell$   
1283 and  $h_\ell$  ((3.31)-(3.32)) if identical expense ratios are used.

1284 The preceding discussions were merely informal. More formally, the proof of Corollary 3.3 proceeds along  
1285 the same lines as the proof of Proposition 3.2, except that VETF investor wealth dynamics (3.23) is used instead  
1286 of (3.24), and details are therefore omitted.

### 1287 A.5: Proof of Proposition 3.4

1288 Suppose we have zero expense ratios, i.e.  $c_v = c_\ell = 0$ , and there are no jumps in the underlying  $S$ -dynamics  
1289 (i.e.  $\lambda = 0$  in (3.2)). Substituting these values in the deterministic functions  $g_\ell$  and  $h_\ell$  ((3.31) and (3.32)) in  
1290 the case of a LETF and the deterministic functions  $g_v$  and  $h_v$  ((3.35) and (3.36)) in the case of the VETF, we  
1291 have

$$1292 g_\ell(t) = g_v(t) = 1, \quad (\text{A.29})$$

1293 and

$$1294 h_\ell(t) = h_v(t) = 0. \quad (\text{A.30})$$

1295 In the case of the LETF investor, the optimal control (3.30) now satisfies

$$1296 \beta \cdot W_\ell^*(t) \cdot \varrho_\ell^*(t, \mathbf{X}_\ell^*(t)) = \left( \frac{\mu - r}{\sigma^2} \right) \cdot \left[ \gamma e^{-r(T-t)} - \left( W_\ell^*(t) - \hat{W}(t) \right) \right] + \hat{\varrho}_s(t) \hat{W}(t), \quad (\text{A.31})$$

1297 whereas in the case of the VETF investor, the optimal control (3.30) now becomes

$$1298 W_v^*(t) \cdot \varrho_v^*(t, \mathbf{X}_v^*(t)) = \left( \frac{\mu - r}{\sigma^2} \right) \cdot \left[ \gamma e^{-r(T-t)} - \left( W_v^*(t) - \hat{W}(t) \right) \right] + \hat{\varrho}_s(t) \hat{W}(t). \quad (\text{A.32})$$

1299 Using (A.31) and (A.32), define the auxiliary process  $Q(t)$  as

$$1300 Q(t) = e^{-rt} \cdot [W_\ell^*(t) - W_v^*(t)], \quad t \in [t_0 = 0, T], \quad (\text{A.33})$$

1301 with  $Q(t_0) = e^{-rt_0} [W_\ell^*(t_0) - W_v^*(t_0)] = w_0 - w_0 = 0$ .

Substituting the optimal controls in this special case ((A.31) and (A.32)) into the wealth dynamics (3.23)-(3.24) and recalling that there are no jumps, we obtain the dynamics

$$\frac{dQ(t)}{Q(t)} = \left(\frac{\mu - r}{\sigma}\right)^2 \cdot dt - \left(\frac{\mu - r}{\sigma}\right) \cdot dZ(t). \quad (\text{A.34})$$

1302 Since  $Q(t_0) = 0$ , dynamics (A.34) therefore imply that  $Q(t) = 0, \forall t \geq t_0$ , so that in the special case of zero  
1303 costs, we have

$$1304 \quad W_\ell^*(t) = W_v^*(t), \quad \forall t \in [t_0, T], \quad (\text{A.35})$$

1305 which confirms (3.39).

Using (A.35) to write  $W^*(t) := W_\ell^*(t) = W_v^*(t)$  in this special case, the difference in controls (A.31) and (A.32) satisfy

$$\begin{aligned} [\beta \cdot \varrho_\ell^*(t, \mathbf{X}_\ell^*(t)) - \varrho_v^*(t, \mathbf{X}_v^*(t))] \cdot W^*(t) &= \beta \cdot W_\ell^*(t) \cdot \varrho_\ell^*(t, \mathbf{X}_\ell^*(t)) - W_v^*(t) \cdot \varrho_v^*(t, \mathbf{X}_v^*(t)) \\ &= - \left(\frac{\mu - r}{\sigma^2}\right) [W_\ell^*(t) - W_v^*(t)] \\ &= 0, \quad \forall t \geq t_0, \end{aligned} \quad (\text{A.36})$$

1306 thereby verifying (3.38).

1307 Finally, given the form of the optimal control (A.32) for the VETF in this special case, together with  
1308 Assumption 3.1 and wealth dynamics (3.23), imply that we can obtain the optimal Information Ratio in the  
1309 case of the VETF as given by (3.40) (see Van Staden et al. (2023)). However, since (A.35), this is also the  
1310 optimal IR using the LET in this special case, thereby confirming (3.40) and completing the proof of Proposition  
1311 3.4.

## 1312 Appendix B: Source data and parameters

1313 In this appendix, we provide details regarding the source data used to obtain the indicative investment results  
1314 presented in Section 3 and Section 5.

1315 Returns data for US Treasury bills and bonds, as well as the broad equity market index, were obtained from  
1316 the CRSP<sup>10</sup>. In more detail, the historical time series are as follows:

- 1317 (i) T30 (30-day Treasury bill): CRSP, monthly returns for 30-day Treasury bill.
- 1318 (ii) B10 (10-year Treasury bond): CRSP, monthly returns for 10-year Treasury bond.
- 1319 (iii) Market (broad equity market index): CRSP, monthly and daily returns, including dividends and distribu-  
1320 tions, for a capitalization-weighted index consisting of all domestic stocks trading on major US exchanges  
1321 (the VWD index).

1322 CRSP data was obtained for the historical time period 1926:01 to 2023:12. All time series were inflation-adjusted  
1323 using inflation data from the US Bureau of Labor Statistics<sup>11</sup>.

### 1324 B.1: Constructing VETF and LETF returns time series

1325 LETFs were only introduced in 2006 (Bansal and Marshall (2015)), whereas the first VETFs were listed in the  
1326 US in the 1990s. In order to obtain longer time series of returns for indicative investment results of Section 5,  
1327 we construct a proxy returns time series for a VETF and LETF referencing a broad equity market index as  
1328 follows:

- 1329 (i) Obtain daily returns for the underlying broad equity market index referenced by the VETF and LETF. For  
1330 this purpose, we used daily returns for the CRSP capitalization-weighted index consisting of all domestic

<sup>10</sup>Calculations were based on data from the Historical Indexes 2024©, Center for Research in Security Prices (CRSP), The University of Chicago Booth School of Business. Wharton Research Data Services was used in preparing this article. This service and the data available thereon constitute valuable intellectual property and trade secrets of WRDS and/or its third party suppliers.

<sup>11</sup>The annual average CPI-U index, which is based on inflation data for urban consumers, were used - see <http://www.bls.gov.cpi>

1331 stocks trading on major US exchanges (the VWD index - see above), with historical data that is available  
 1332 since January 1926. We prefer to use a time series that is as long as possible, since this would include  
 1333 additional periods of exceptional market volatility such as 1929-1933.

1334 (ii) Multiply each daily return by the returns multiplier  $\beta$ , where we used  $\beta = 2$  for the LETF and  $\beta = 1$  for  
 1335 the VETF, and construct a time series of monthly returns.

1336 (iii) Adjust the time series of VWD returns using (3.5) to reflect the ETF expense ratios  $c_k, k \in \{v, \ell\}$  and the  
 1337 observed T-bill rate  $r$ . As per Table 3.1, we assumed expense ratios of  $c_\ell = 0.89\%$  p.a. for the LETF and  
 1338  $c_v = 0.06\%$  p.a. for the VETF to reflect typical values observed in the market.

1339 (iv) Inflation-adjust the time series using inflation data from the US Bureau of Labor Statistics.

1340 Although we can construct a synthetic LETF that achieves the desired leveraged returns by explicitly  
 1341 assuming borrowing at the risk-free rate plus a spread, in practice, LETFs are managed by large institutional  
 1342 fund managers who typically achieve leverage more efficiently through total return swaps. Due to their significant  
 1343 scale and high creditworthiness, these institutions can negotiate swap agreements priced at levels very close to  
 1344 the risk-free (T-bill) rate. Consequently, the effective borrowing cost embedded within the LETF fee remains  
 1345 minimal (e.g. captured by the T-bill rate and the expense ratio  $c_\ell = 0.89\%$ ), which is significantly lower than the  
 1346 direct borrowing spreads (e.g.  $b = 3\%$ ) typically faced by other investors. This institutional advantage therefore  
 1347 justifies modelling the synthetic historical LETF returns as if borrowing effectively occurs at the T-bill rate  
 1348 with a modest LETF fee.

1349 Note that a proxy time series of LETF returns is similarly constructed in Bansal and Marshall (2015),  
 1350 although a number of details (such as inflation adjustment of returns and choice of underlying index) differ.  
 1351 As noted in Bansal and Marshall (2015), the construction of such a proxy returns time series assumes that the  
 1352 ETF managers achieve a negligible tracking error with respect to the underlying index. We observe that these  
 1353 assumptions are often made out of necessity in the literature concerning LETFs (e.g. Bansal and Marshall (2015),  
 1354 Leung and Sircar (2015)). Furthermore, given improvements in designing replication strategies for LETFs that  
 1355 remain robust even during periods of market volatility (see for example Guasoni and Mayerhofer (2023)), this  
 1356 appears to be a reasonable assumption for ETFs written on major stock market indices as considered in this  
 1357 paper.

1358 We emphasize that the proxy time series for VETF and LETF returns are only used for bootstrapping the  
 1359 data sets for the numerical solutions implementing the data-driven neural network approach (Section 4 and  
 1360 Section 5), and *not* for the closed-form solutions of Section 3. This follows since closed-form solutions in Section  
 1361 3 assume parametric dynamics for the underlying assets including the broad equity market index, from which  
 1362 the LETF and VETF dynamics can be constructed using (3.8) and (3.4), respectively.

## 1363 B.2: Calibrated parameters for closed-form solutions

1364 For the closed-form solutions of Section 3, using the CRSP data for 30-day T-bills and the broad equity market  
 1365 index (VWD index) for the period 1926:01 to 2023:12 as outlined above, the filtering technique as per Dang  
 1366 and Forsyth (2016); Forsyth and Vetzal (2017) for calibrating inflation-adjusted Kou (2002) jump-diffusion  
 1367 processes resulted in the calibrated process parameters as presented in Table B.1. Given the specified dynamics  
 1368 (3.1)-(3.2) of the risk-free asset  $B$  and equity market index  $S$  (with parameters as in Table B.1), we can obtain  
 1369 the dynamics of the LETF (3.8) and VETF (3.4).

**Table B.1:** Closed-form solutions: Calibrated, inflation-adjusted parameters for asset dynamics (3.1) and (3.2), assuming the Kou (2002) jump-diffusion model with  $G(\xi^s)$  given by (A.14). The calibration methodology of Dang and Forsyth (2016); Forsyth and Vetzal (2017) is used with a jump threshold parameter value of 3.

Assumption for $S$ -dynamics	Calibrated parameters						
	$r$	$\mu$	$\sigma$	$\lambda$	$p_{up}$	$\eta_1$	$\eta_2$
Jump-diffusion (Kou model)	0.0031	0.0873	0.1477	0.3163	0.2258	4.3591	5.5337
GBM dynamics (no jumps)	0.0031	0.0819	0.1850	-	-	-	-

1370  
 1371 Note that the values of the remaining parameters for the parametric dynamics can be calculated by sub-  
 1372 stituting the values of Table B.1 into the results of Appendix A.3. This gives  $\kappa_1^s = -0.0513$ ,  $\kappa_2^s = 0.0884$ ,  
 1373  $\kappa_1^\ell = -0.0500$ ,  $\kappa_2^\ell = 0.0870$  and  $\kappa_\chi^{\ell,s} = 0.0876$ .

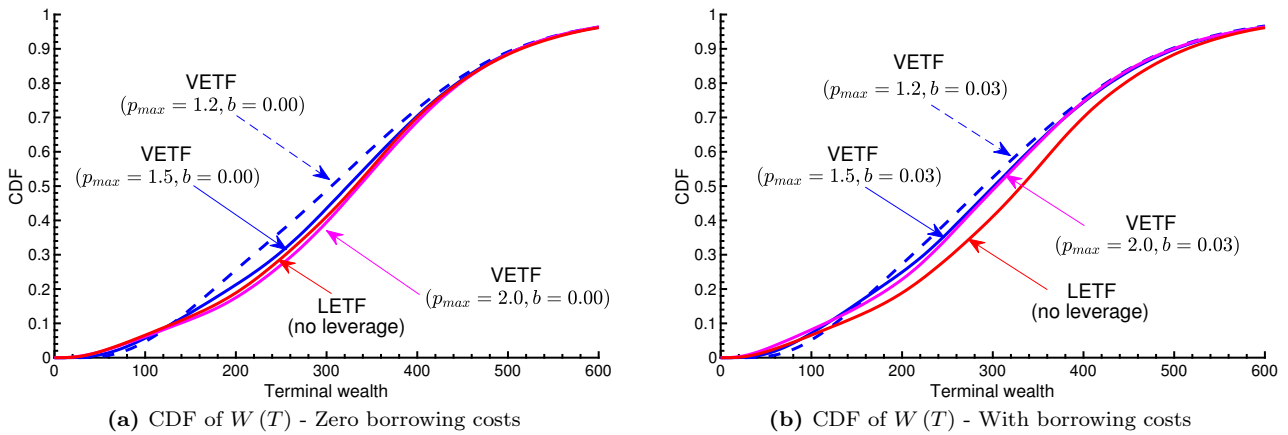
## 1374 Appendix C: Additional numerical results

1375 Additional leverage and borrowing cost scenarios are analyzed as a supplement to the results of Section 5. The  
 1376 results of this appendix confirm that the conclusions of Section 5 do not appear to be sensitive in a qualitative  
 1377 sense to the specific maximum leverage or borrowing costs parameters used in the numerical analysis, provided  
 1378 these values are within a reasonable range.

### 1379 C.1: Scenarios: Leverage and borrowing costs

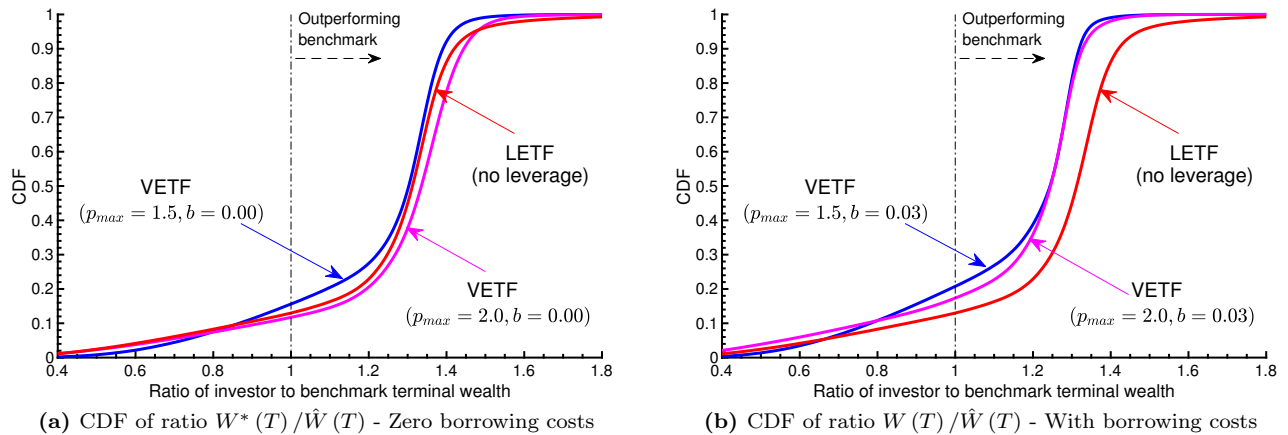
1380 We consider scenarios where the maximum leverage allowed decreases from  $p_{max} = 1.5$  to  $p_{max} = 1.2$ , or  
 1381 increases to  $p_{max} = 2.0$ , and where zero borrowing costs might be applicable (as opposed to borrowing costs of  
 1382  $b = 0.03$  throughout Section 5).

1383 As discussed in Section 5, we observe that the IR-optimal portfolio of the LETF investor achieves partial  
 1384 stochastic dominance over the corresponding IR-optimal portfolio of the VETF investor with the same out-  
 1385 performance target  $\gamma$ , even if the VETF investment can be leveraged 2x, provided borrowing premiums are  
 1386 applicable - see Figure C.1(b). However, when borrowing premiums drop to zero (i.e. the unrealistic scenario  
 1387 where the investor can borrow at the T-bill rate), a leveraged IR-optimal VETF-based strategy with  $p_{max} = 2.0$   
 1388 performs similarly, though *slightly* better, than the LETF-based strategy with no leverage (see Figure C.1(a)).  
 1389 Figure C.2 demonstrates that similar observations also hold if we consider benchmark outperformance instead  
 1390 of portfolio wealth. For a further explanation of the results of Figure C.1 and Figure C.2, see Appendix C.2.



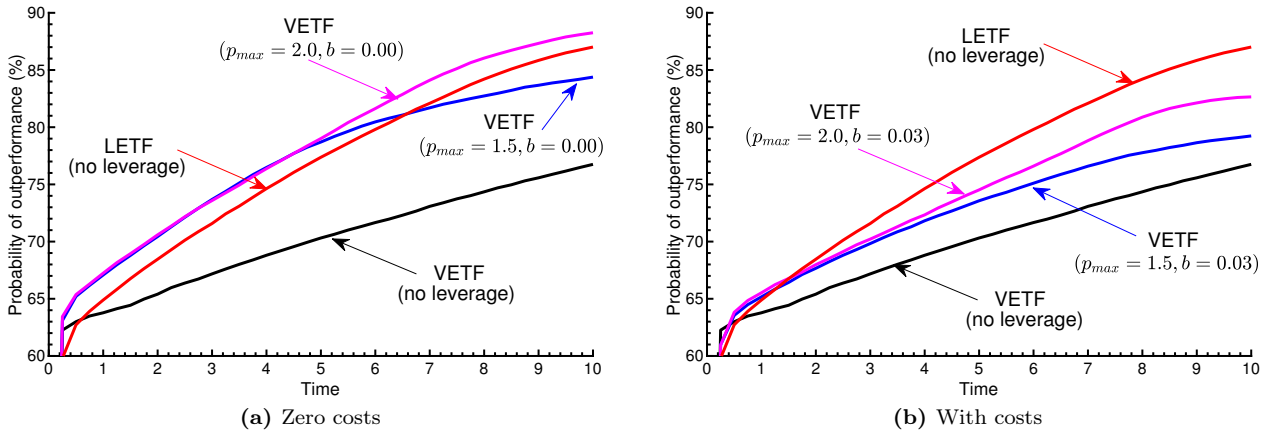
**Figure C.1:** Effect of leverage and borrowing cost assumptions on CDFs of IR-optimal terminal wealth  $W_k^*(T)$ ,  $k \in \{v, \ell\}$ , with  $T = 10$  years.

1391



**Figure C.2:** Effect of leverage and borrowing cost assumptions on the CDFs of IR-optimal terminal wealth ratios  $W_k^*(T)/\hat{W}(T)$ ,  $k \in \{v, \ell\}$ , with  $T = 10$  years.

1392



**Figure C.3:** Effect of leverage and borrowing cost assumptions on the probability  $W_k^*(t) > \hat{W}(t), k \in \{v, \ell\}$  of benchmark outperformance as a function of time  $t$ , over the time horizon of  $T = 10$  years.

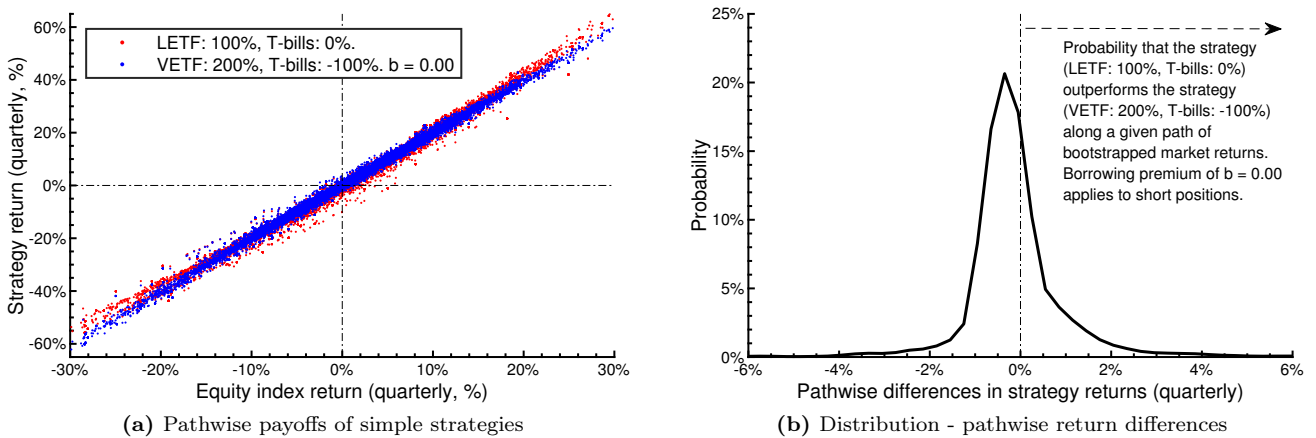
### 1393 C.2: Historical data: Bootstrapped quarterly returns

1394 We demonstrate that the observations from Section 3.1 (obtained using calibrated parametric models for the  
 1395 underlying assets) also hold in the case of the historical returns used to obtain the results of Section 5 and  
 1396 Appendix C.1. Recall that in Section 3.1 we observed that holding a LETF position for one quarter amounts to  
 1397 holding a “continuously rebalanced” position in the equity index and bonds, which results in a power law-type  
 1398 payoff from holding the LETF.

1399 Figure C.4 and Figure C.5 compare the pathwise quarterly returns of two simple strategies using the boot-  
 1400 strapped historical data. The strategies consist of (i) investing all wealth in the LETF at the start of a quarter,  
 1401 and (ii) investing 200% of wealth in the VETF at the start of a quarter funded by borrowing 100% of wealth  
 1402 at the T-bill rate plus a borrowing premium (where applicable). Outcomes are compared at the end of the  
 1403 quarter with no intermediate trading. The only difference between Figure C.4 and Figure C.5 is that Figure C.4  
 1404 applies a zero borrowing premium ( $b = 0.00$ ) to fund short positions in the VETF, whereas Figure C.5 applies  
 1405 a borrowing premium of  $b = 0.03$  to fund short positions in the VETF.

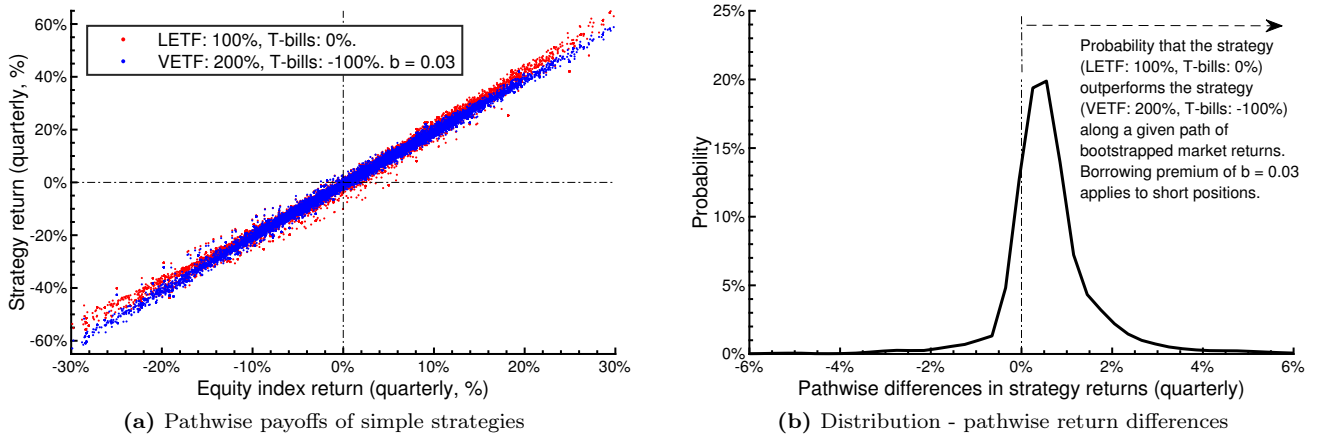
1406 Figure C.4(a) and Figure C.5(a) illustrate the quarterly returns of the simple strategies (y-axis) for a given  
 1407 level of equity index quarterly return (x-axis) over the quarter. Figure C.4(b) and and Figure C.5(b) illustrate  
 1408 the distribution of pathwise quarterly return differences where, for each value of the x-axis in the corresponding  
 1409 Figure C.4(a) and Figure C.5(a), and therefore for a particular given path of (joint) asset returns, we calculate  
 1410 the vertical difference between the returns of the two strategies.

Figure C.4(a) and Figure C.5(a) confirm that the call-like payoff of the LETF also holds in the bootstrapped



**Figure C.4:** Zero borrowing premium ( $b = 0.00$ ): Pathwise comparison of the bootstrapped historical quarterly inflation-adjusted returns of two simple strategies. See Appendix C.2 for more detail.

1411 historical data. However, this is translated into a slight advantage for the LETF relative to the 2x leveraged  
 1412 VETF strategy only when borrowing premiums are positive (compare Figure C.4(b) and Figure C.5(b)), i.e.  
 1413



**Figure C.5:** Positive borrowing premium ( $b = 0.03$ ): Pathwise comparison of the bootstrapped historical quarterly inflation-adjusted returns of two simple strategies. See Appendix C.2 for more detail.

1414 when the investor cannot fund short positions at only the T-bill rate.

1415 Given these payoff structures of the LETF, the IR-optimal LETF strategy responds to gains by reducing  
 1416 exposure to the LETF, thus locking in the results of prior quarters of good performance while reducing exposure  
 1417 to future possible losses by having lower exposure to the LETF (see Section 3 and Section 5). The compounding  
 1418 effect of applying the contrarian IR-optimal investment strategy quarter after quarter given returns as per Figure  
 1419 C.4(a) and Figure C.5(b) ultimately results in the terminal wealth results reported in Figures C.1, C.2 and C.3.

1420