Overview: Numerics of Hamilton-Jacobi-Bellman Equations in Finance

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Outline

1. General formulation of stochastic control problems
2. Two examples: illustrate basic numerical methods
   (a) Gas storage
       $\rightarrow$ semi-Lagrangian timestepping
   (b) Optimal portfolio allocation with stochastic volatility
       $\rightarrow$ Monotone discretization (wide stencil)
       $\rightarrow$ Policy iteration
3. Many HJB equations can be solved using these techniques

Basic Forms of Optimal Stochastic Control

Definitions:

- $X(t)$: a set of state variables, usually specified by a set of SDEs
- $q(X(t), t)$: a control, which allows us to steer $X(t)$

$$q(\cdot) \in \mathcal{Z}(X) : \text{ the set of admissible controls}$$

- An objective function $F(X(T))$ at terminal time T.
- $C(X(t), t)$ cash flows at time t

And in addition:

$$\begin{aligned} E_{t,x}^{q(\cdot)}[\cdot] \quad = \quad & \text{Expectation conditional on } X(t) = x, \\ & \text{at time } t, \text{ under control } q(\cdot) \end{aligned}$$

Value Function

Value function $V(x,t)$ defined by

$$V(x,t) = \sup_{q(\cdot)} E_{t,x}^{q(\cdot)} \left[\int_t^T \overbrace{C(X(s),s)\ ds}^{cash\ \ flows} + \overbrace{F(X(T))}^{payoff} \right]$$

- Solve for value function $\rightarrow$ get optimal control $q(\cdot)$

HJB equation: standard arguments ($\tau = T - t$):

$$\begin{aligned} V_\tau &= \sup_{q \in \mathcal{Z}} \left(\mathcal{L}^q V \right) \\ & \quad \mathcal{L}^q V = \text{ degenerate parabolic operator} \end{aligned}$$

Viscosity Solution

Original problem posed as a dynamic program

$\rightarrow$ Value function defined in terms of integration

$\rightarrow$ Value function need not be differentiable

- We seek the weak form solution of the HJB equation
  - $\rightarrow$ *viscosity* solution

Need to guarantee numerical scheme converges to viscosity solution

- Sufficient conditions (Barles, Souganidis (1991))
  - Monotone, consistent (in the viscosity sense) and ℓ_∞ stable

Example I: control appears only in 1st derivative terms

Natural gas storage facilities: buy low (summer), sell high (winter)[1]

- Long term storage: underground caverns
- Objective: determine no-arbitrage value of leasing a storage facility for a fixed term
- By-product of valuation: optimal operating strategy (i.e. when to inject, produce or do nothing)
- Control = injection strategy

[1] Economists describe the situation where a large deposit of natural gas crowds out other industries as the *Dutch Disease*

State Variables

Risk neutral process for spot price of gas P

$$dP = \alpha(K(t) - P)\,dt + \sigma P\,dZ$$

$$\sigma = \text{ volatility } \; ; \; dZ = \text{ increment of Wiener process}$$

$$K(t) = K_0 + \beta_{SA} \sin(4\pi(t - t_{SA}))$$

Current amount of working gas inventory I

$$I \in [0, I_{\max}]$$

$$c = \text{ Rate of gas production from storage}$$

$$c \in C(I) = [c_{\min}(I), c_{\max}(I)]$$

$$c > 0 \rightarrow \text{ production } \; ; \; c < 0 \rightarrow \text{ injection}$$

Gas inventory satisfies

$$\frac{dI}{dt} = -(c + a(I, c))$$

$$a(I, c) = \text{ production/injection losses}$$

More Definitions

Max/min production rates are nonlinear functions of inventory I, e.g.

$$c_{\max}(I) \quad = \quad k_1\sqrt{I} \quad ; \quad k_1 = \text{ const.} \quad \text{(ideal gas law)}$$

Revenue obtained from selling gas

$$\begin{aligned}
\text{Revenue} \quad &= \quad (c - b(c))\, P \\
& \qquad b(c) = \text{ gas loss during transportation} \\
& \qquad\qquad \text{Revenue } > 0 \rightarrow \text{ gas released and sold} \\
& \qquad\qquad \text{Revenue } < 0 \rightarrow \text{ gas purchased and stored}
\end{aligned}$$

HJB PDE

Value of leasing storage for T years: risk neutral discounted cash flows in $[0, T]$, $(\tau = T - t)$

$$
\begin{aligned}
V(P, I, \tau) \;\; = \;\; \sup_{c(s) \in C(I(s))} E^{c(\cdot)} \Bigg[\int_t^T e^{-r(s-t)} \; \overbrace{[c(s) - b(c(s))]P(s)}^{cash \;\; flows} \; ds \\
+ e^{-r(T-t)} \; \underbrace{V(P(T), I(T), T)}_{lease \;\; termination \;\; penalty} \Bigg]
\end{aligned}
$$

Usual steps: HJB PDE for $V(P, I, \tau)$

$$
\begin{aligned}
V_\tau \;\; = \;\; \mathcal{L}V + \max_{c \in C(I)} \Big\{ (c - b(c))P - (c + a(c))V_I \Big\} \\
\mathcal{L}V \equiv \frac{1}{2}(\sigma P)^2 V_{PP} + \alpha(K(t) - P)V_P - rV
\end{aligned}
$$

Semi-Lagrangian form

Let

$$\frac{DV}{D\tau} = \frac{\partial V}{\partial \tau} + \frac{\partial V}{\partial I}(c + a(c))$$

be the Lagrangian derivative along the trajectory

$$\frac{dI}{d\tau} = c + a(c) \ .$$

We can then write the HJB PDE as

$$\min_{c \in C(I)} \left\{ \frac{DV}{D\tau} - (c - b(c))P - \mathcal{L}V \right\} = 0.$$

Define:

$$\begin{array}{ll} \{P_i, I_j\}_{i=1,\dots,N_1 \ ; \ j=1,\dots,N_2}, & \text{Grid} \\ V_{i,j}^n = V(P_i, I_j, \tau^n) & \text{Discrete Solution} \end{array}$$

Discretization

The final form for the discretization is then

$$V_{i,j}^{n+1} = \max_{c \in C_{i,j}^{n+1}} \left\{ \underbrace{\overbrace{[\Phi^{n+1}(c) V^n]_{i,j}}^{interpolated\ \ value}}_{at\ \ foot\ \ of\ \ characterstic} + \Delta\tau^n \big(c - b(c)\big) P_i \right\} + \Delta\tau^n \underbrace{(\mathcal{L}_h V)_{i,j}^{n+1}}_{standard\ \ FD}, \qquad (1)$$

Define:

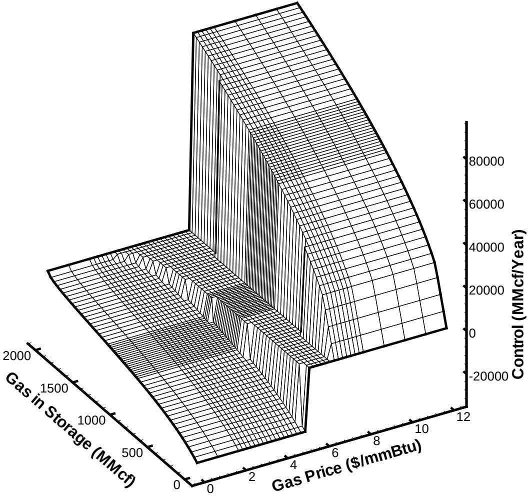
$$\Gamma_{i,j}^n = \max_{c \in C_{i,j}^{n+1}} \left\{ [\Phi^{n+1}(c) V^n]_{i,j} + \Delta\tau^n \big(c - b(c)\big) P_i \right\} \qquad (2)$$

so that (1) becomes[2]

$$V_{i,j}^{n+1} = \Gamma_{i,j}^n + \Delta\tau^n (\mathcal{L}_h V)_{i,j}^{n+1} \qquad (3)$$

[2] Each timestep: interpolation/optimization step (2) $\rightarrow$ time advance step (3). Easy to show: monotone, consistent, stable

Control Surface: $t = 0$



- Injection $c < 0$
- Production $c > 0$
- Note region where optimal to do nothing
- Option value of waiting

Example II: control appearing in the 2nd derivative terms

Continuous time mean variance portfolio allocation: stochastic volatility

Long term investor can can allocate wealth into two assets:

Amount in Risk-free bond B

$$
\begin{aligned}
dB &= r\ B\ dt \\
& \quad r = \text{ risk-free rate}
\end{aligned}
$$

Amount in risky-asset S

$$
\begin{aligned}
\frac{dS}{S} &= (r + \xi V)\ dt + \sqrt{V}\ dZ_1 \\
& \quad \xi V = \text{ market price of volatility risk}
\end{aligned}
$$

Variance process V

$$
\begin{aligned}
dV(t) &= \kappa(\theta - V(t))\ dt + \sigma \sqrt{V(t)}\ dZ_2 \\
& \quad \sigma = \text{ vol of vol ; } \kappa = \text{ Mean-reversion speed} \\
& \quad \theta = \text{ mean variance ; } \rho\ dt = dZ_1\ dZ_2
\end{aligned}
$$

SDE for Total Wealth

The investor's total wealth $W = S + B$ follows the process

$$dW(t) = (r + p\xi V(t))\, W(t)\ dt + p\sqrt{V}\, W(t)\ dZ_1 .$$

$$p = \left(\frac{S}{W}\right) = \text{ fraction invested in risky asset}$$

Constraints on control p

- Trading must stop if $W = 0$
- Leverage is constrained: $p \leq p_{\text{max}}$

Objective: determine optimal control $p(W, t)$ which generates points on the efficient frontier

$$\sup_p \Big\{ \underbrace{E^{p(\cdot)}[W(T)]}_{ExpectedValue} - \lambda \underbrace{Var^{p(\cdot)}[W(T)]}_{Variance} \Big\}$$

- Varying $\lambda \in [0, \infty)$ traces out the efficient frontier

Reformulate MV Problem $\Rightarrow$ Dynamic Programming

Embedding technique[3] for fixed λ, if $p^*(\cdot)$ maximizes

$$\sup_{p(\cdot)\in\mathbb{Z}} \left\{ \underbrace{E^p[W(T)]}_{ExpectedValue} - \lambda \underbrace{Var^p[W(T)]}_{Variance} \right\} ,$$

$\mathbb{Z}$ is the set of admissible controls

$\rightarrow \exists\ \gamma$ such that $p^*(\cdot)$ minimizes

$$\inf_{p(\cdot)\in\mathbb{Z}} E^{p(\cdot)}\left[\left(W(T) - \frac{\gamma}{2} \right)^2 \right] .$$

[3]Zhou and Li (2000), Li and Ng (2000)

Value Function $\mathcal{U}(w, v, \tau)$[4]

$$\mathcal{U}(w, v, \tau) \quad = \quad \inf_{p(\cdot) \in \mathbb{Z}} E^{p(\cdot)}_{\tau, w, v}\left[\left(W(T) - \frac{\gamma}{2}\right)^2\right]$$

$$w = \text{ wealth } ; \; v = \text{ local variance } ; \; \tau = T - t$$

HJB PDE for optimal allocation strategy $p(\cdot)$:

$$\begin{aligned} \mathcal{U}_\tau \quad = \quad & \inf_{p \in \mathcal{Z}} \Bigg\{ (r + p\xi v) w \, \mathcal{U}_w + \kappa(\theta - v) \, \mathcal{U}_v \\ & + \left(\frac{(p\sqrt{v}w)^2}{2} \right) \mathcal{U}_{ww} + \overbrace{(p\rho\sigma\sqrt{v}w) \, \mathcal{U}_{wv}}^{x-derivative \;\; term} + \left(\frac{\sigma^2 v}{2} \right) \mathcal{U}_{vv} \Bigg\}, \end{aligned}$$

$$\mathcal{U}(w, v, 0) = \left(w - \frac{\gamma}{2} \right)^2 .$$

Given $p(\cdot)$, compute $E^{p(\cdot)}[W_T]$, $Var^{p(\cdot)}[W_T]$

[4] For a fixed γ, this gives one point on the efficient frontier.

Main Problem: cross derivative term

Construct finite difference grid

$$\{w_i, v_j\}_{i=1,\dots,N_1\ ;\ j=1,\dots,N_2}$$

We need to construct a monotone scheme

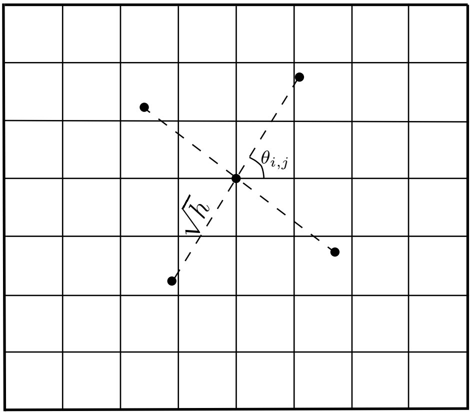
- Control appears in 2nd derivative terms

Solution: Wide stencil[5]

- Grid spacing $O(h)$
- At each node, do virtual rotation, eliminate x-derivative term, finite difference on rotated grid
- Values are interpolated from *real grid*
- Size of virtual stencil $O(\sqrt{h})$

[5] Debrebant and Jakobsen (2013) factor the diffusion tensor

Local Rotation



Note: local rotation angle $\theta_{i,j}$ depends on

- Node location, i.e. (w_i, v_j)
- Control p at this node

Discretization: fully implicit timestepping

$$
\begin{aligned}
\mathcal{U}_{i,j}^{n} &= \mathcal{U}(w_i, v_j, \tau^n) \\
\mathbf{U}^n &= (\mathcal{U}_{1,1}^{n}, \mathcal{U}_{2,1}^{n}, \ldots, \mathcal{U}_{N_1,1}^{n}, \ldots, \mathcal{U}_{1,N_2}^{n}, \ldots, \mathcal{U}_{N_1,N_2}^{n}) \\
& \quad \mathcal{U}_{\ell}^{n} = \mathcal{U}_{i,j}^{n}, \quad \ell = i + (j-1)N_1.
\end{aligned}
$$

Similarly the vector of optimal controls is

$$
\mathcal{P} = (p_{1,1}, \ldots, p_{N_1 N_2})
$$

The nonlinear algebraic equations are then[6]

$$
\inf_{\mathcal{P} \in Z} \left\{ -\mathbf{A}(\mathcal{P})\mathbf{U}^{n+1} + \mathbf{C}(\mathcal{P}) \right\} = 0,
$$

$$
\mathbf{A} = \text{ matrix of discretized equations} \tag{4}
$$

[6] Row ℓ of $\mathbf{A}, \mathbf{C}$ depends only on p_ℓ

Policy Iteration[7]

Algorithm 1 Policy Iteration

Let $(\hat{\mathbf{U}})^0 =$ Initial estimate for $\mathbf{U}^{n+1}$
for $k = 0, 1, 2, \ldots$ until converge **do**
  $\mathcal{P}_\ell^k = \arg\min\limits_{\mathcal{P}_\ell \in Z} \left\{ -[\mathbf{A}(\mathcal{P})]\hat{\mathbf{U}}^k + \mathbf{C}(\mathcal{P}) \right\}_\ell$
  Solve $[\mathbf{A}(\mathcal{P}^k)]\hat{\mathbf{U}}^{k+1} = \mathbf{C}(\mathcal{P}^k)$
  if converged **then**
    break from the iteration
  end if
end for

Theorem (Convergence of Policy Iteration)

If $[\mathbf{A}(\mathcal{P})]$ is an $\mathcal{M}$ matrix, Policy iteration converges to the unique solution of equation (4).

[7] Use ILU-PCG method to solve matrix, complexity $= O((N_1 N_2)^{5/4})$.

Numerical Example

κ	θ	σ	ρ	ξ
5.04	0.0457	0.48	-0.767	1.605

Table: $\mathbb{P}$ measure Heston parameters[8]

Investment Horizon T	10
The risk free rate r	0.03
Leverage constraint $p_{\max}$	2
Initial wealth w_0	100
Initial variance v_0	0.0457

[8] Aıt-Sahalia, Kimmel, Journal of Financial Economics (2007)

Efficient Frontier (vary speed of mean reversion κ)

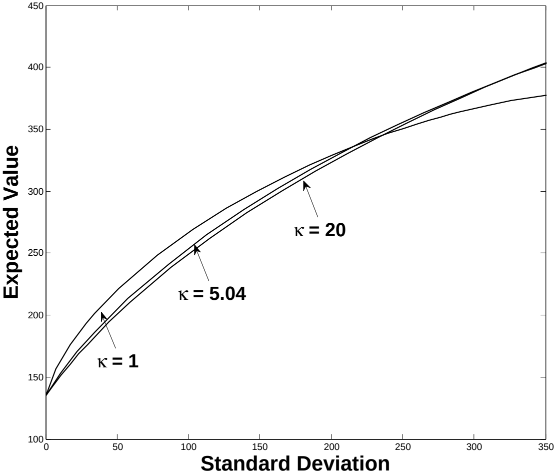


Figure: $T = 10$ years. $w_0 = 100$. $\kappa = 5 \simeq 2.5$ months mean reversion time. Curves for $\kappa = 5, 20$ very close.

Long Term Investment: Stochastic Volatility Unimportant?

Fix γ (parameter that traces out efficient frontier)

- A Assume constant volatility GBM[9], compute and store optimal strategy
- B Assume stochastic volatility[10], compute and store optimal strategy

Assume real world follows stochastic volatility, compute result using MC simulations, for both **A** and **B**

	$\gamma = 540$		$\gamma = 1350$	
	Mean	Stndrd Dev	Mean	Stndrd Dev
GBM Control **A**	212.68	58.42	329.13	207.23
Stoch Vol Control **B**	213.99	58.53	331.28	207.37

Table: $\kappa T > 20$, stochastic vol well approximated by GBM.

[9] Constant volatility = mean value from stochastic vol model

[10] $\kappa \simeq 5$, $T = 10$ years

Conclusions

- If the control appears only in the first derivative term
  - $\rightarrow$ Semi-Lagrangian timestepping simple and effective
- Similar timestepping method can be used for impulse control.
- Control appearing in 2nd derivative terms
  - $\rightarrow$ For non-zero correlation, need monotone discretization (wide stencil)
  - $\rightarrow$ Non-linear algebraic equations easily solved using Policy iteration
- Low accuracy control (e.g. GBM for stoch vol, coarse control set discretization)
  - $\rightarrow$ Accurate value function. Why?
- Challenges:
  - Higher dimensions
  - Wide stencil only 1*st* order
  - Solution of local optimization problem at each node (need global optimum)