

Decumulation of Retirement Savings:
The Nastiest, Hardest Problem in Finance
Part I: Introduction and Results

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Woudschoten
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Thursday 10:00

Motivation

Defined Benefit Plans (DB) are disappearing

- Corporations/governments no longer willing to take risk of DB plans

Recent survey¹ P7 countries²

- Defined Contribution (DC)³ plan assets: 55% of all pension assets
- Some examples
 - Australia 87% DC
 - US 65% DC
 - Canada 43% DC
 - ...
 - Japan 5% DC

Netherlands → *Collective* DC plan (2027)

¹Thinking Ahead Institute (2023)

²Australia, Canada, Japan, Netherlands, Switzerland, UK, US

³DC plan: retiree takes on all investment risk

The retiree dilemma (Defined Contribution (DC))

A retiree with savings in a DC plan^{4 5} has to decide on

- An investment strategy (stocks vs. bonds)
- A decumulation schedule

The retiree now has two major sources of risk

- Investment risk
- Longevity risk (running out of cash before death)

William Sharpe (Nobel Laureate in Economics) calls this

"The nastiest hardest problem in finance"

⁴In a DC plan, the retiree is responsible for investment/decumulation

⁵RRSP (Canada), SIPP (UK), 401(k)(US), Super Fund (Australia)

The Four per Cent Rule

Based on rolling 30-year historical periods, Bengen (1994) showed:

A retiree who

- Invested in a portfolio of 50% bonds, 50% stocks (US), rebalanced annually
- Withdrew 4% of initial capital (adjusted for inflation) annually
 - Would never have run out of cash, over any rolling 30-year period (from 1926)

Criticism

- Simplistic asset allocation strategy
- Simplistic withdrawal strategy
- Rolling 30 year periods contain large overlaps
 - Underestimates risk of portfolio depletion

Bengen rule

“Play the long game. A retirement income plan should be based on planning to live, not planning to die. A long life will be expensive to support, and it should take precedence over death planning.” Pfau (2018)

Note that Bengen rule is based on assumption that 65-year old will live to be 95

- Should we mortality weight the cash flows (as in an annuity)?
- Example: median life expectancy of 65-year old male $\simeq 87$.
 - Effectively, mortality weighting will weight minimum cash flow of 87-year old by $1/2$
 - If I am 87, and alive, I need 100% of my minimum cash flows
 - If I am dead, I need zero dollars
- We will consider an individual investor, not averaging over a population
 - 30 year retirement, no mortality weighting
 - Consistent with Bengen approach

Fear of running out of cash

Recent survey⁶

- Majority of pre-retirees fear exhausting their savings in retirement more than death

In Canada, a 65-year old male

- Probability of 0.13 of living to be 95
- Probability of 0.02 of living to be 100

Conservative strategy:

→ Assume 30 year retirement (as in Bengen (1994)).

Other assets can be used to hedge extreme longevity⁷

⁶2017 Allianz Generations Ahead Study - Quick Facts #1. (2017), Allianz

⁷Real estate

Objective of this talk

Determine a decumulation strategy which has

- Variable withdrawals (minimum and maximum constraints)
- Minimizes risk of portfolio depletion
- Maximizes total expected withdrawals
- Allows for dynamic, non-deterministic asset allocation

We will treat this as a problem in optimal stochastic control

Formulation

Investor has access to two funds

- A broad stock market index fund
 - *Amount* in stock index S_t
- A constant maturity bond index fund
 - *Amount* in bond index B_t

$$\text{Total Wealth } W_t = S_t + B_t \quad (1)$$

Model the returns of both indexes

- Parametric, jump diffusion
- Non-zero stock-bond correlation
- Fit parameters to market data 1926:1-2019:12
↪ All returns adjusted for inflation

Notation

Withdraw/rebalance at discrete times $t_i \in [0, T]$

The investor has two controls at each rebalancing time

q_i = Amount of withdrawal

p_i = Fraction in stocks after withdrawal (2)

At t_i , the investor withdraws q_i

$$\begin{aligned} W_i^- &= \overbrace{S_i^- + B_i^-}^{\text{wealth before withdrawal}} \\ W_i^+ &= W_i^- - q_i \end{aligned} \quad (3)$$

Then, the investor rebalances the portfolio

$$\begin{aligned} S_i^+ &= p_i W_i^+ \\ B_i^+ &= (1 - p_i) W_i^+ \end{aligned} \quad (4)$$

Can show that

$$q_i = q_i(W_i^-) \quad ; \quad p_i = p_i(W_i^+)$$

Controls

Constraints on controls

$$\begin{aligned} q_i &\in [q_{\min}, q_{\max}] && ; \quad \text{withdrawal amount} \\ p_i &\in [0, 1] && ; \quad \text{fraction in stocks} \\ &\Rightarrow \text{no shorting, no leverage} \end{aligned}$$

Set of controls

$$\mathcal{P} = \{(q_i(\cdot), p_i(\cdot)) : i = 0, \dots, M\} \quad (5)$$

Reward and Risk

Reward: Expected total (real) withdrawals (EW)

$$\text{EW} = E \left[\overbrace{\sum_i q_i}^{\text{total withdrawals}} \right]$$

$E[\cdot] = \text{Expectation}$

Risk measure: Expected Shortfall ES

$$ES(5\%) \equiv \left\{ \text{Mean of worst 5\% of } W_T \right\}$$

$W_T = \text{terminal wealth at } t = T$

ES defined in terms of final wealth, *not losses*⁸

→ Larger is better

⁸ES is basically the negative of CVAR

Objective Function

Multi-objective problem $\rightarrow$ scalarization approach for Pareto points

Find controls $\mathcal{P}$ which maximize (scalarization parameter $\kappa > 0$)⁹

$$\begin{aligned} & \sup_{\mathcal{P}} \left\{ EW + \kappa ES \right\} \\ & \sup_{\mathcal{P}} \left\{ \overbrace{E_{\mathcal{P}} \left[\sum_i q_i \right]}^{\text{total withdrawals}} + \kappa \overbrace{\left(\frac{E_{\mathcal{P}} [W_T \mathbf{1}_{W_T \leq W^*}]}{.05} \right)}^{\text{mean worst 5\% outcomes}} \right\} \\ & \text{s.t.} \quad \text{Prob}[W_T \leq W^*] = .05 \end{aligned}$$

Varying κ traces out the efficient frontier in the (EW, ES) plane

⁹ $E_{\mathcal{P}}[\cdot] \equiv$ expectation under control $\mathcal{P}$.

EW-ES Objective Function

Given an expectation under control $E_{\mathcal{P}}[\cdot]$ (Rockafellar and Uryasev, 2000)

$$\begin{aligned}\text{ES}_{5\%} &= \sup_{W^*} E_{\mathcal{P}} \left[G(W_T, W^*) \right] \\ G(W_T, W^*) &= \left(W^* + \frac{1}{.05} [\min(W_T - W^*, 0)] \right)\end{aligned}$$

Reformulate objective function:

$$\sup_{\mathcal{P}} \sup_{W^*} E_{\mathcal{P}} \left\{ \overbrace{\sum_i q_i}^{\text{total withdrawals}} + \kappa \overbrace{G(W_T, W^*)}^{\text{mean worst 5\% } W_T} + \overbrace{\epsilon W_T}^{\text{Stabilization}} \right\}$$

Why do we need the stabilization term?

↪ More later

Time Consistency

The EW-ES objective function is not formally *time consistent*

Time inconsistency

⇒ Investor has incentive to deviate from initial optimal policy at later times

EW-ES policy computed at time zero

↪ Pre-commitment policy

Induced time consistent policy

At t_0 we compute the pre-commitment EW-ES control

- For $t > t_0$ we assume that the investor follows the *induced time consistent* control (Strub et al (2019))
- This control is identical to the pre-commitment control at t_0
- No incentive to deviate from this control at $t > t_0$

Induced time consistent control determined from (fixed W^*)

$$\sup_{\mathcal{P}} E_{\mathcal{P}} \left\{ \sum_i q_i + \kappa G(W_T, W^*) + \epsilon W_T \right\}$$

W^* from pre-commitment solution at time zero

Alternative: equilibrium mean-ES control

$\hookrightarrow$ Does not actually control tail risk! (Forsyth(2020)) ¹⁰

¹⁰For more discussion of time consistency, induced time consistency, pre-commitment, see Bjork et al (2021), Vigna (2020, 2022), Strub et al (2019), Forsyth (2020)

Withdrawal Control: limiting case

Theorem 1 (Bang-bang withdrawal control: continuous limit)

Assume that

- *the stock and bond indexes follow a parametric jump-diffusion*
- *the portfolio is continuously rebalanced, and withdrawals occur at the continuous (finite) rate $\hat{q} \in [\hat{q}_{\min}, \hat{q}_{\max}]$*

then the optimal control is bang-bang, i.e. the optimal withdrawal $\hat{q}^$ is either $\hat{q}^* = \hat{q}_{\min}$ or $\hat{q}^* = \hat{q}_{\max}$.*

Proof.

See Forsyth (North American Actuarial Journal (2022))



But of course, in real life, we do not withdraw/rebalance continuously.

Scenario: all amounts indexed to inflation

- DC account at $t = 0$ (age 65) \$1,000K (one million)
- Minimum withdrawal from DC account \$35K per year¹¹
- Maximum withdrawal from DC \$60K per year
- No shorting, no leverage ($p \in [0, 1]$)
- Annual rebalancing/withdrawals
- Retiree owns mortgage-free real estate worth \$400K

Investment Horizon

- $T = 30$ years, i.e. from age 65 to 95
 $\Rightarrow$ Plan to live long and prosper



¹¹Assume gov't benefits of 22K/year. Minimum income $\simeq 22K + 35K = 57K$ /year.

Scenario II

Why do we include real estate in the scenario?

Since $q_{\min} = 35K$ per year, W_t can become negative

- When $W_t < 0$, assume retiree is borrowing, using a reverse mortgage¹²
 - Reverse mortgages allow borrowing of 50% of home value
 - In our case: \$200K
- Once $W_t < 0$
 - All stocks are liquidated
 - Debt accumulates at borrowing rate
- If $W_T > 0$, then real-estate is a bequest
- Real estate is a hedge of last resort: not fungible with other wealth
 - This mental bucketing of real estate is a well-known behavioral finance result.¹³

¹²See Pfeiffer et al, Journal of Financial Planning (2013)

¹³I also observe this with my fellow retirees: real-estate is a separate bucket

Numerical Method I

Pre-commitment control at t_0 (same as induced time consistent control)

Interchange $\sup \sup(\dots)$

$$\sup_{W^*} \overbrace{\sup_{\mathcal{P}} E_{\mathcal{P}} \left\{ \underbrace{\sum_i q_i + \kappa G(W_T, W^*) + \epsilon W_T}_{\text{maximize over } W^*} \right\}}^{\text{Solve using Dynamic Programming (fixed } W^*)}$$

Solve inner DP problem using PIDE methods

Numerical Method II

Inner maximization: dynamic programming

- Conditional expectations at t_i^+
 - Solve linear 2-d PIDE
 - Use δ -monotone Fourier method (Forsyth and Labahn (2019))
- Optimal controls at each rebalancing time
 - Discretize controls
 - Find maximum by exhaustive search
- Guaranteed to converge to the solution as discretization parameters $\rightarrow 0$

Outer maximization over W^*

- Discretize W^* , use coarse PIDE grid
 - Find optimal W^* by exhaustive search
- Use coarse grid W^* as starting point for 1-d optimization on finer grids

Data

Center for Research in Security Prices (CRSP) US

- Cap weighted index, all stocks on all major US exchanges 1926:1-2019:12
- US 10 year Treasury index
- Monthly data, inflation adjusted by CPI

Synthetic Market

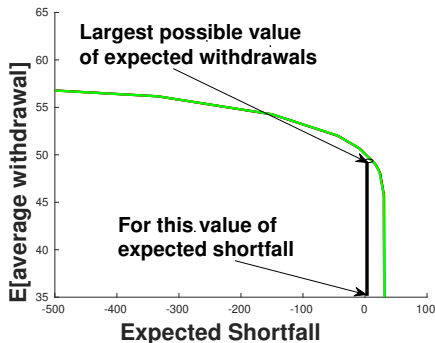
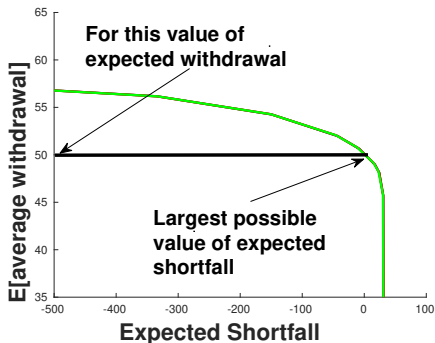
- Stock/bond returns driven by parametric jump-diffusion model, calibrated to data
- Optimal controls computed in the *synthetic* market

Historical market

- Stock/bond returns from stationary block bootstrap resampling of actual data¹⁴
- No assumptions about stock/bond processes
- Used to test control robustness computed in the synthetic market

¹⁴Dichtl et al (2016, Appl. Econ.), Anarkulova et al (JFE,2022)

Pareto optimal points (Units: Thousands)

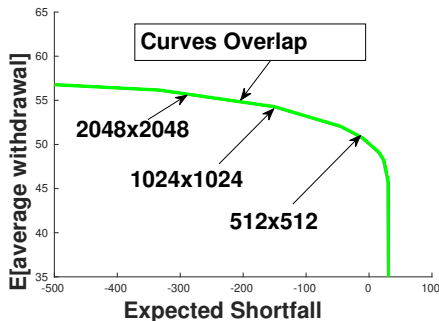


Varying scalarization parameter κ

→ Traces out efficient frontier

- y-axis is annual average expected withdrawals
- E.g.: 50K ($W_0 = 1000K$) corresponds to 5% withdrawal rate
- Recall ES is mean of worst 5% $W_T \Rightarrow$ larger is better

EW-ES efficient frontier (Units: thousands)



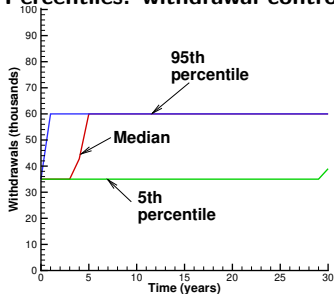
- Solutions with different PIDE grids
- ES is the mean of the worst 5% of outcomes
- Each pt on curve, different κ
- Reverse mortgage hedge
 - Any point $ES > -200K$ is acceptable

Note Efficient Frontier almost vertical at right hand end

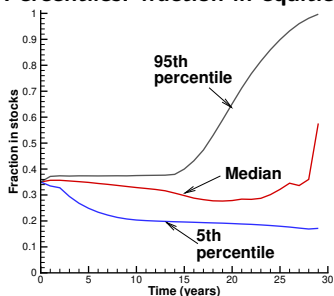
- Base case: constant withdrawal 35K/year
- Tiny increase in risk (smaller ES)
 - ⇒ Average withdrawal 50K per year (never less than 35K)

Point on Frontier: (EW,ES) = (52K/year, -42K)

Percentiles: withdrawal control

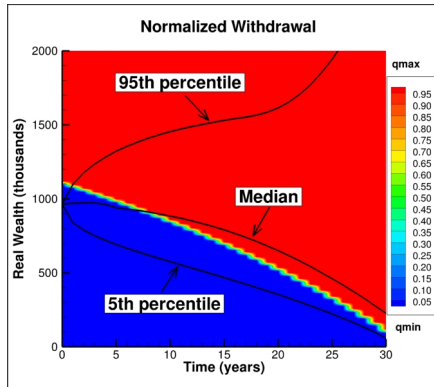
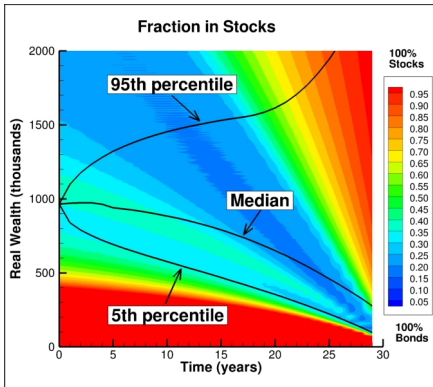


Percentiles: fraction in equities



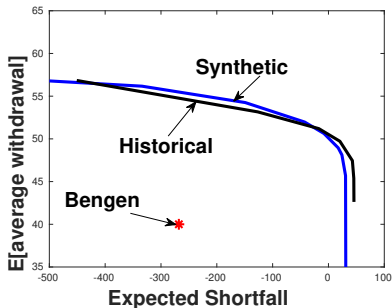
- $ES \simeq -42K$
- 5th percentile wealth at $t = 30 \simeq 58K$
- Average withdrawal $\simeq 52K/\text{year}$

Point on Frontier: (EW,ES) = (52K/year, -42K)



- Withdrawal controls $\simeq$ *bang-bang*, i.e. only withdraw either $q_{\min}$ or $q_{\max}$.
- Median $W_t \simeq 1000K \rightarrow 300K$

Robustness Check: Efficient Frontier (Units: thousands)



Bengen 4% rule: bootstrapped historical market^{a b}

⇒ **very inefficient**

⇒ **More risky than advertised**, ES
 $\simeq -270K$

^aBengen suggests 50% in stocks.

^bExperimentally, 40% in stocks maximized ES.

Controls computed and stored in the *synthetic* market

- Parametric model calibrated to historical data

Controls tested¹⁵ in the bootstrapped historical market

→ Controls are robust to parametric model misspecification

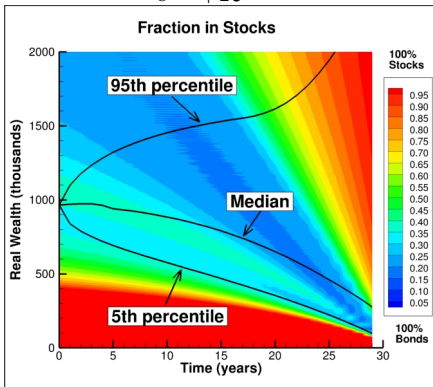
¹⁵ “Out-of-sample” test.

Stabilization term (EW,ES) = (52K/year, -42K)

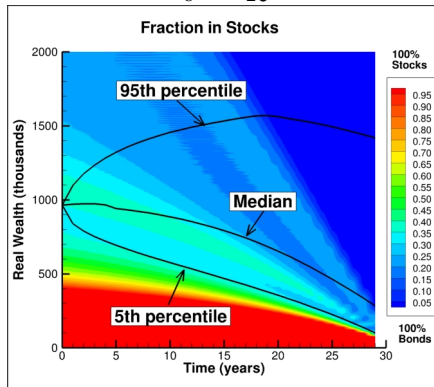
Recall objective function:

$$\sup_{\mathcal{P}} \sup_{W^*} \left\{ \overbrace{EW}^{\text{total withdrawals}} + \overbrace{\kappa G(W_T, W^*)}^{\text{mean worst 5\% outcomes}} + \overbrace{\epsilon W_T}^{\text{Stabilization}} \right\}$$

$$\epsilon = +10^{-6}$$



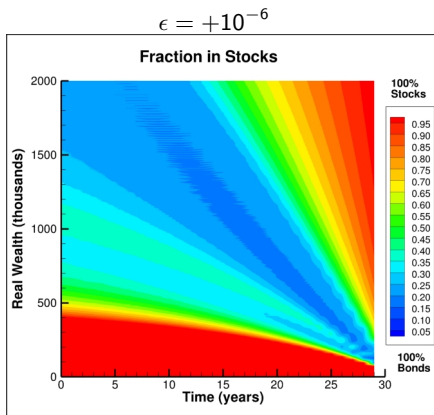
$$\epsilon = -10^{-6}$$



Stabilization term

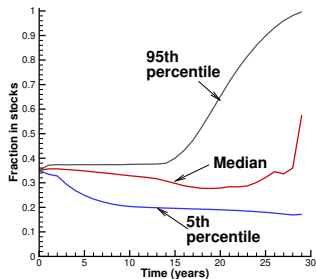
Plots of efficient EW-ES frontiers overlap for $\epsilon = \pm 10^{-6}$

Recall that we are assuming the investor follows the induced time consistent strategy

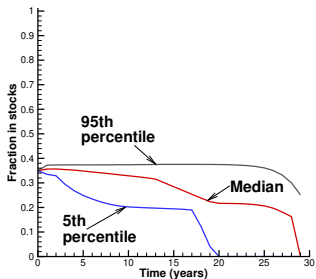


- $W^* = 58K$
- Suppose that $t = 25$, i.e. 90 years old
- $W = 2000K$, you will never run out of cash with $q_{max} = 60K/\text{year}$
- It does not matter whether you invest 100% in stocks or bonds

If you are Warren Buffet, this problem is ill-posed



$$\epsilon = +10^{-6}$$



$$\epsilon = -10^{-6}$$

Fraction Stocks < 0.4 at 95th percentile

If you are rich and old, then it does not matter what you do

- $\epsilon = +10^{-6}$ invest 100% in stocks
- $\epsilon = -10^{-6}$ invest 100% in bonds

But these lucky large wealth outcomes $\Rightarrow$ no effect on (EW,ES) frontier

Peter Ponzo: Canasta Strategy

Peter Ponzo (retired Applied Math Professor from Waterloo)

- Retired: 1993; passed away: 2020
- In 1993, took commuted value of his pension
 - One-half → annuity (interest rate: 9.8%)
 - One-half → self-directed investments
 - Wrote a blog about his attempts to *“beat the market”*
- It turned out that beating the market was not easy!

But: he summarized his withdrawal strategy: **“Canasta Strategy”**
“If we have a good year, we take a trip to China,...,if we have a bad year, we stay home and play canasta.”

This is a **bang-bang** control!

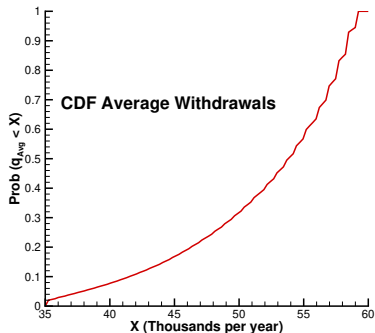
Conclusions

- Optimal strategy: flexible withdrawals, dynamic stock-bond allocation
 - Less risk, higher average withdrawals¹⁶ compared to 4% rule
 - Bootstrap resampling $\Rightarrow$ controls are robust
- In the continuous withdrawal limit
 - Optimal withdrawals are *bang-bang*, i.e. only withdraw at either maximum or minimum rate
- Discrete rebalancing: withdrawal controls are very close to bang-bang
- Intuition: if you are lucky, and make money in stocks, take money off the table and go on a cruise
 - Otherwise: sit tight

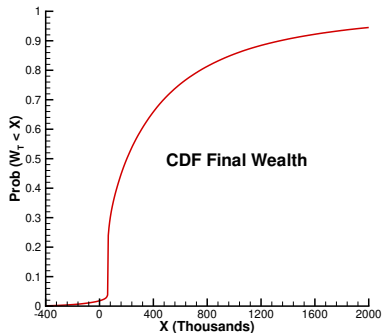
¹⁶Optimal: 5% EW, with $ES \simeq 0$; Bengen: 4% EW, with $ES \simeq -270K$.

Cumulative Distribution Functions: (EW,ES) = (52K/year, -42K)

Average withdrawal



Wealth at $T = 30$ years



Bootstrap resampled historical data (blksize = 3 months)

- $> 94\%$ probability: average withdrawals $> 40K$ per year
- $> 98\%$ probability: $W_T > 0$

Decumulation of Retirement Savings:
The Nastiest, Hardest Problem in Finance
Part II: Numerical Algorithms

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Woudschoten
September 27-29, 2023
Friday 9:00

Decumulation of Retirement Savings

Recall from the first talk

- Retiree wants to maximize total withdrawals
- Minimize risk of running out of cash (30 year retirement)
- Can invest in a mix of stocks and bonds
- At each (yearly) rebalancing time
 - Choose amount to withdraw q
 - Fraction in stocks p
- No shorting/leverage for investments
- $q \in [q_{\min}, q_{\max}]$

Stochastic Process: Stock Index

Let S_t be the real (inflation adjusted) *amount* in a stock index
 S_t follows a jump diffusion process

$$\frac{dS_t}{S_{t-}} = (\mu - \lambda\gamma) dt + \sigma dZ + d \left(\sum_{i=1}^{\pi_t^s} (\xi_i - 1) \right),$$

σ^s = volatility

dZ = increment of Wiener process

π_t^s = Poisson process with intensity λ

$S_{t-} \rightarrow \xi_i S_t$ at jump times

$\gamma = E[\xi - 1]$

$\xi \simeq$ double exponential distribution (1)

Stochastic Process: Bond Index

Let B_t be the real (inflation adjusted) *amount* in a **constant maturity** bond index

Model real returns of the bond index directly as a stochastic process

- Common practitioner approach (Lin et al, IME (2015))
- Avoids modelling interest rates, inflation
- Easy to calibrate to historical data

B_t follows a jump diffusion process

$$\frac{dB_t}{B_{t-}} = \dots \text{ similar to stock process} \quad (2)$$

Parameters for both processes calibrated to historical data

Recall

Withdraw/rebalance at discrete times $t_i \in [0, T]$

The investor has two controls at each rebalancing time

q_i = Amount of withdrawal

p_i = Fraction in stocks after withdrawal (3)

At t_i , the investor withdraws q_i

$$\begin{aligned} W_i^- &= \overbrace{S_i^- + B_i^-}^{\text{wealth before withdrawal}} \\ W_i^+ &= W_i^- - q_i \end{aligned} \quad (4)$$

Then, the investor rebalances the portfolio

$$\begin{aligned} S_i^+ &= p_i W_i^+ \\ B_i^+ &= (1 - p_i) W_i^+ \end{aligned} \quad (5)$$

Can show that

$$q_i = q_i(W_i^-) \quad ; \quad p_i = p_i(W_i^+)$$

Controls

Constraints on controls

$$\begin{aligned} q_i &\in [q_{\min}, q_{\max}] && \text{ ; withdrawal amount} \\ p_i &\in [0, 1] && \text{ ; fraction in stocks} \\ &&& \text{no shorting, no leverage} \end{aligned}$$

Set of controls

$$\mathcal{P} = \{(q_i(\cdot), p_i(\cdot)) : i = 0, \dots, M\} \quad (6)$$

$$\begin{aligned} \mathcal{P}_n &= \{(q_i(\cdot), p_i(\cdot)) : i = n, \dots, M\} \\ &\text{tail of the controls} \end{aligned} \quad (7)$$

EW-ES Objective Function

Objective function:

$$\sup_{\mathcal{P}} \sup_{W^*} E_{\mathcal{P}} \left\{ \overbrace{\sum_i q_i}^{\text{total withdrawals}} + \overbrace{\kappa G(W_T, W^*)}^{\text{mean worst 5\% outcomes}} + \overbrace{\epsilon W_T}^{\text{Stabilization}} \right\}$$

Numerical Method I

Interchange $\sup \sup(\dots)$

$$\sup_{W^*} \overbrace{\sup_{\mathcal{P}} E_{\mathcal{P}} \left\{ \sum_i q_i + \kappa G(W_T, W^*) + \epsilon W_T \right\}}^{\text{Solve using Dynamic Programming (fixed } W^*)}$$

$\underbrace{\hspace{10em}}_{\text{maximize over } W^*}$

Solve inner DP problem using PIDE methods

Inner problem: value function

$$V(s, b, W^*, t_n^-) = \sup_{\mathcal{P}_n} \left\{ E_{\mathcal{P}_n}^{(S_n^-, B_n^-), t_n^-} \left[\sum_{i=n}^M q_i \right. \right. \\ \left. \left. + \kappa \left(W^* + \frac{1}{\alpha} \min((W_T - W^*), 0) \right) \middle| (S_n^-, B_n^-) = (s, b) \right] \right\} .$$

Where:

$$\text{Subject to } \begin{cases} (S_t, B_t) \text{ follow processes (1) and (2);} \\ W_\ell^+ = S_\ell^- + B_\ell^- - q_\ell \\ S_\ell^+ = p_\ell(\cdot) W_\ell^+; \quad B_\ell^+ = (1 - p_\ell(\cdot)) W_\ell^+ \\ t_\ell = \text{rebalancing times} \end{cases} .$$

Dynamic Programming Approach

Terminal condition at $t_M = T$

$$V(s, b, W^*, T^+) = \kappa \left(W^* + \frac{\min((s + b - W^*), 0)}{.05} \right).$$

At any rebalancing time t_n

$\hookrightarrow$ Advance the solution backwards $t_n^+ \rightarrow t_n^-$

$$\begin{aligned} V(s, b, W^*, t_n^-) &= \sup_{(p,q)} \left\{ q + \left[V(w^+ p, w^+(1-p), W^*, t_n^+) \right] \right\} \\ w^- &= s + b \\ w^+ &= w^- - q \end{aligned}$$

$$t_n^+ = t_n + \epsilon, \quad t_n^- = t_n - \epsilon, \quad \epsilon \uparrow 0^+$$

Between rebalancing times

For $t \in (t_{n-1}^+, t_n^-)$

$\hookrightarrow$ No cashflows, no discounting, for $h \rightarrow 0$

$\hookrightarrow$ Tower property

$$V(s, b, W^*, t) = E \left[V(S(t+h), B(t+h), W^*, t+h) \right. \\ \left. | S(t) = s, B(t) = b \right]$$

Apply Ito's Lemma for jump/diffusion processes

$\rightarrow$ 2-D Partial Integro Differential Equation (PIDE)

$\rightarrow$ Independent variables (s, b, t)

Numerical Algorithm: Details

Discretize state space (s, b)

$\hookrightarrow$ 2-D grid, with mesh parameter h

Solve PIDE, using Fourier method

- Standard Fourier methods may not be monotone
- Example: Two possible controls $\mathcal{P}^A, \mathcal{P}^B$ are such that

$$\mathcal{P}^A = \{(q_i(\cdot), p_i(\cdot)) : i = 0, \dots, M\} \in \mathcal{A}$$

$$\mathcal{P}^B = \{(q_i(\cdot), p_i(\cdot)) : i = 0, \dots, M\} \in \mathcal{B}$$

- Assume $\mathcal{A} \subset \mathcal{B}$

Then we should have the monotonicity property (optimal control maximizes V)

$$V^A(s, b, t) \leq V^B(s, b, t) ; \quad \forall (s, b, t)$$

We use a δ -monotone Fourier method $\rightarrow$ guarantees

$$V^A(s, b, t) \leq V^B(s, b, t) + \delta$$

Given fixed h , δ can be made arbitrarily small

Numerical Details II

At rebalancing times:

- Discretize the controls with spacing $O(h)$
- Find optimal (p, q) by exhaustive search
- For off-grid points
 - Use linear interpolation of discretized value function

Actual value function $\hat{V}(s_0, b_0, t_0)$

$$\hat{V}(s_0, b_0, t_0) = \sup_{W^*} \overbrace{V(s_0, b_0, W^*, t_0)}^{\text{Inner PIDE Solve}}$$

Solve problem on sequence of grids

- On coarse grid, discretize W^* , maximize by exhaustive search
- On finer grids, use coarse grid estimate for W^* as starting point
 - Find optimal W^* using 1-d optimization algorithm

Numerical Details III

Solve control problem on grid

- At each rebalancing time, store optimal controls

Determine statistical quantities

- **Synthetic Market:** use stored controls, do Monte Carlo simulations with parametric SDE model of stocks and bonds
- **Historical Market:** use stored controls, do bootstrap resampling of historical stock, bond returns

Bootstrap simulations

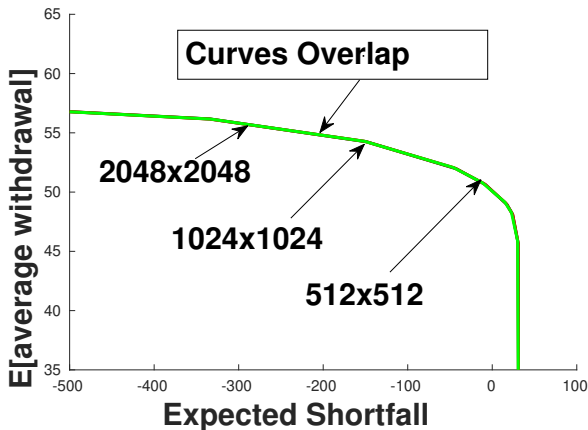
- *Out of sample* test
- No assumptions about market stochastic processes

Numerical Example

- DC account at $t = 0$ (age 65) \$1,000K (one million)
- Minimum withdrawal from DC account \$35K per year²
- Maximum withdrawal from DC \$60K per year
- No shorting, no leverage ($p \in [0, 1]$)
- Annual rebalancing/withdrawals
- Retiree owns mortgage-free real estate worth \$400K
 - Hedge of last resort if account exhausted
- Investment horizon: age 65 to 95

²Assume gov't benefits of 22K/year. Minimum income
 $\simeq 22K + 35K = 57K/\text{year}$.

Convergence Check: Synthetic Market



⇒ Even coarse grid gives good solution

Alternative Approach: Machine Learning

- Does not use dynamic programming
 - Efficient in cases where performance criteria is high dimensional
 - Control is low dimensional (see van Staden, Forsyth, Li, SIFIN (2023))
 - Can be used in cases where no dynamic programming principle exists (e.g. mean semi-variance)
- Does not require a parametric model of stochastic processes for stock and bond
- Can be extended to higher dimensional problems (e.g. more assets)

Basic idea ³

- Go back to original problem formulation
- Approximate control directly using a Neural Network (NN)
- Approximate expectations by sampling paths
- Optimize w.r.t. NN parameters

³See also Han (2016), Andersson, Oosterlee (2023).

NN Framework

Approximate controls

$$\begin{aligned}q_i(W_i^-, t_i^-) &\simeq \hat{q}(W_i^-, t_i^-; \theta_q) \\p_i(W_i^+, t_i^+) &\simeq \hat{p}(W_i^+, t_i^+; \theta_p) \\ \mathcal{P} &\simeq \hat{\mathcal{P}} = \{\hat{q}(\cdot), \hat{p}(\cdot)\}\end{aligned}$$

$$\{\hat{q}(W_i^-, t_i^-; \theta_q), \hat{p}(W_i^+, t_i^+; \theta_p)\}$$

- fully connected feedforward NNs, parameterized by (θ_q, θ_p)
- Separate NN for $\hat{q}$ and $\hat{p}$.
- Note that using time t as input
 - recurrent network
- Wealth is only state variable needed in this case

Solve for control directly (Policy Function Approximation)

Recall Objective function

$$\sup_{\mathcal{P}} \sup_{W^*} E_{\mathcal{P}} \left\{ \overbrace{\sum_i q_i}^{\text{total withdrawals}} + \overbrace{\kappa G(W_T, W^*)}^{\text{mean worst 5\% outcomes}} + \overbrace{\epsilon W_T}^{\text{Stabilization}} \right\}$$

Generate M sample paths (use stochastic model)

W_T^j = Final wealth along j^{th} path

q_i^j = Withdrawal at time t_i along j^{th} path

Approximate $E[\cdot]$ by mean of samples

$$\sup_{W^*, \theta_q, \theta_p} \frac{1}{M} \sum_{j=1}^M \left\{ \sum_i q_i^j + \kappa G(W_T^j, W^*) + \epsilon W_T^j \right\}$$

Simultaneously maximize over $(W^*, \theta_p, \theta_q)$

NN Method

Each NN has output activation function that encodes constraints

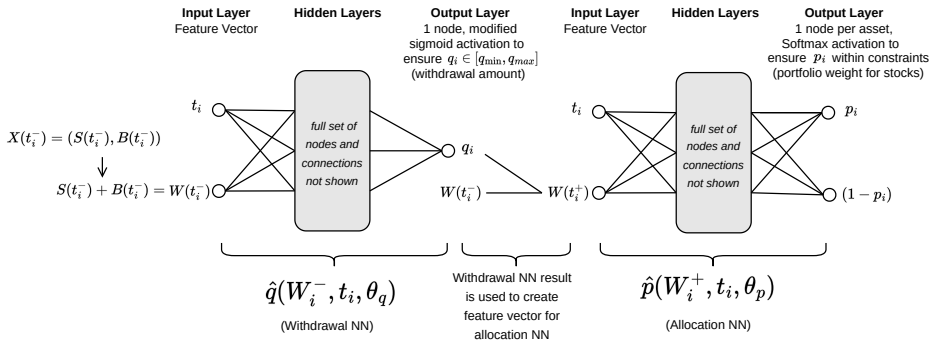
→ Allows unconstrained optimization (i.e. SGD)

No need to have inner/outer optimization

→ W^* maximized along with (θ_q, θ_p)

- A single network $\hat{q}(W^-, t; \theta_q)$ approximates the q control for all t
- Similarly for the p control
 - Contrasts with *stacked NN* approach used previously
- Note: we generate paths using parameterized SDEs
 - We are agnostic to method used to generate paths

NN Framework Diagram



Output of $\hat{q}$ network

$\Rightarrow$ Input to $\hat{p}$ network

Withdrawal Control Heatmaps

Withdrawal control is 'bang-bang': Switches abruptly between q_{min} and q_{max} .

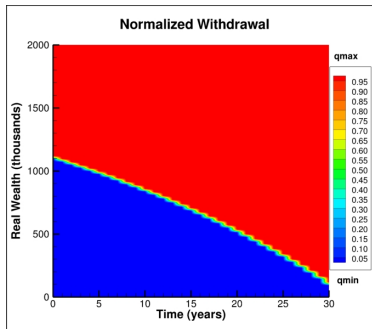


Figure: Withdrawal amount, PDE Control, $\epsilon = 10^{-6}$

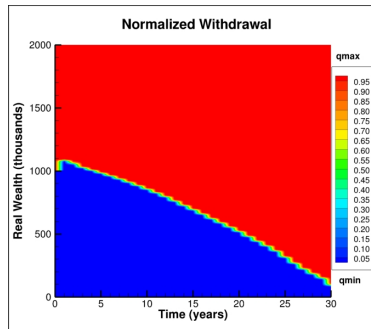


Figure: Withdrawal amount, NN Control, $\epsilon = 10^{-6}$

Units: thousands of dollars

Stock Allocation Control Heatmaps (1)

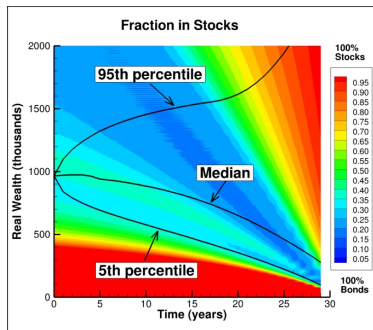


Figure: Fraction in stocks, PDE Control, $\epsilon = 10^{-6}$

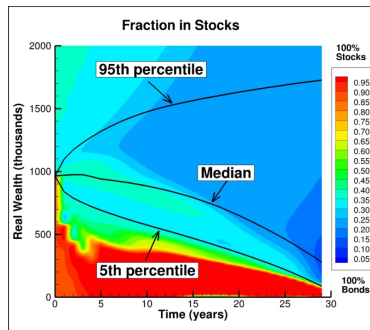


Figure: Fraction in stocks, NN Control, $\epsilon = 10^{-6}$

Effect of stabilization term clearly shown in PDE heatmap, but NN is not sensitive enough (ϵ is tiny). *Units: thousands of dollars*

Stock Allocation Control Heatmaps (2)

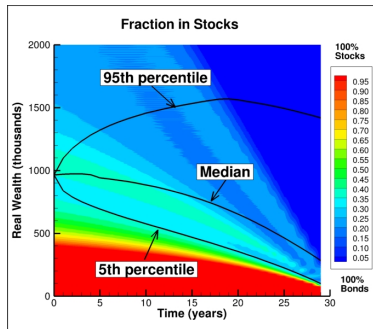


Figure: Fraction in stocks, PDE Control, $\epsilon = -10^{-6}$

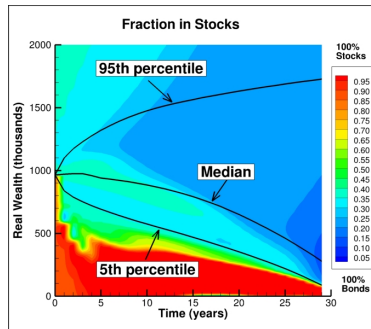


Figure: Fraction in stocks, NN Control, $\epsilon = 10^{-6}$

Making **stabilization term negative** shows that NN control is somewhere in between +/- epsilon versions of PDE control. *Units: thousands of dollars*

Efficient Frontier Comparison: Synthetic Market

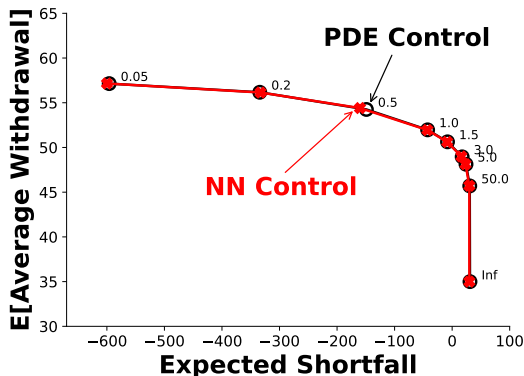


Figure: Comparison of EW-ES frontier for NN and PDE methods. Labels on nodes are the κ values. Units: thousands of dollars

PDE frontier virtually the same, $\epsilon = \pm 10^{-6}$

Bootstrap Resampling

Stationary Block Bootstrap resampling

- Monthly historical data: 1926:1-2020:1
- Draw blocks of data (with replacement) from historical data
 - Simultaneously draw stock and bond returns
 - Sampling in blocks preserves serial correlation
- Blocksizes are drawn from a geometric distribution
 - Random blocksizes reduce edge effects, preserve stationarity
- Concatenate blocks to form a single path of T years
- Dubious algorithm available to determine expected blocksize

Typical parameters

- 10^5 training samples, 10^5 test samples
- Probability of a single identical train, test path $< 10^{-29}$

The universe is 10^{18} seconds old.

Train on Synthetic Data, Test on Historical Data

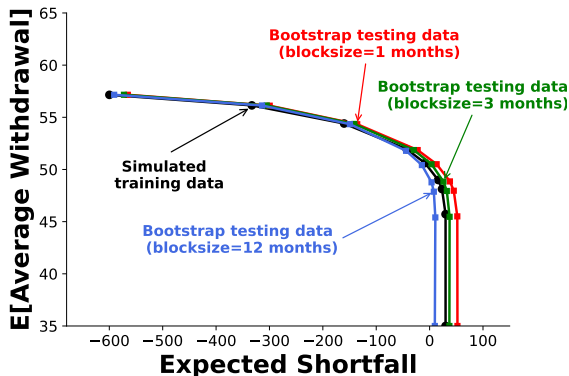


Figure: Comparison of EW-ES frontier for NN training performance vs. tests on resampled historical data. *Units: thousands of dollars*

Train with Historical Data, Test on Synthetic Data

Demonstrates NN framework's ability to use other datasets and still yield good results.

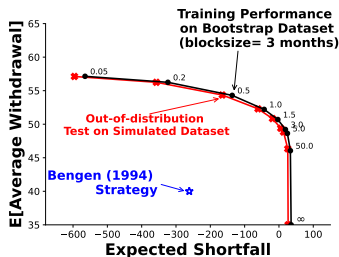


Figure: Historical training data, block size = 3 months

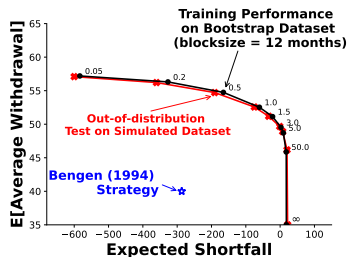


Figure: Historical training data, block size = 12 months

Labels on nodes: κ values. Units: thousands of dollars

Conclusions

- Train/test combinations $\rightarrow$ multi-period optimization is robust
- NN method $\rightarrow$ accurate results compared to ground truth
 - $\rightarrow$ Even for bang-bang controls
- Advantages of NN
 - Does not depend on parametric SDE model (data driven)
 - Can solve high dimensional problems
 - Can be used for problems which do not have DP principle

But

CPU time for computing a single point on the efficient frontier

- PDE: medium grid (C++) $\simeq$ 400 sec (laptop)
- NN: 2 hours (Pytorch + GPUs)

Low dimensional problem, parametric model for stochastic processes

$\rightarrow$ PDEs win