Hedging Under Jump Diffusions with Transaction Costs

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Overview

- Single factor diffusion models for equities not adequate for risk management
- Alternatives:

  Stochastic Volatility/Regime Switching: can hedge with underlying plus small number of options (sometimes one)

  Jump processes: hedge with underlying plus infinite number of options!
- Obviously, hedging jumps is hard

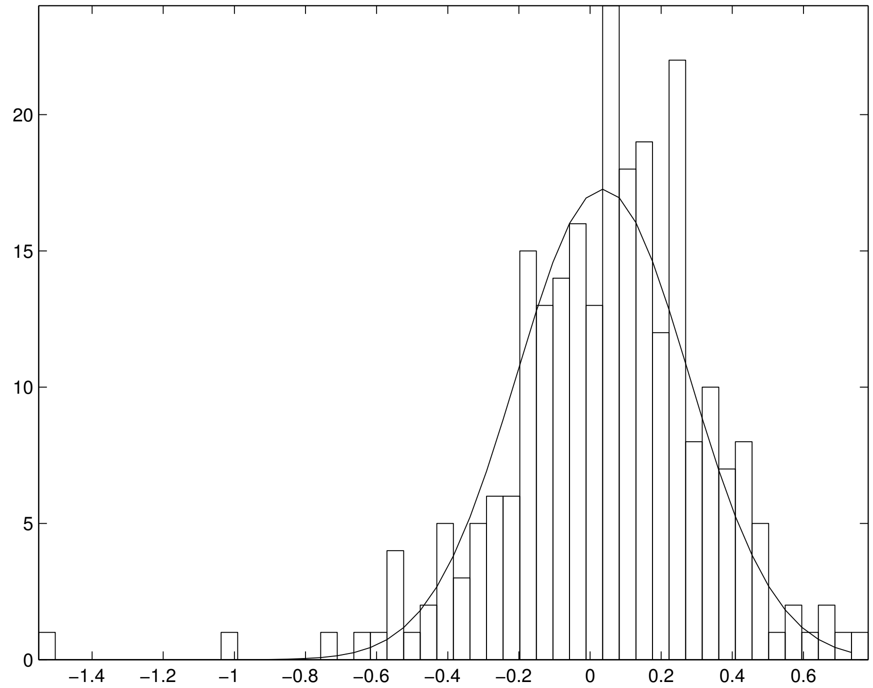
Why Do We Need Jump Models?

- Equity return data suggests jumps.
- Typical local volatility surfaces
  - Heavy skew for short dated options
  - Consistent with jumps
- Large asset price changes more frequent than suggested by Geometric Brownian Motion
- Risk management: if we don't hedge the jumps
  - We are exposed to sudden, large losses

Example: A Drug Company

- This is not Geometric Brownian Motion!
- 80% and 50% drops in one day!

S&P 500 monthly $\log$ returns since 1982



- Scaled to zero mean and unit standard deviation
- Standard normal distribution also shown

$\hookrightarrow$ Extreme events more likely than simple GBM

$\hookrightarrow$ Higher peak, fatter tail than normal distribution

Hedging the Jumps

- If we believe that the underlying process has jumps, hedging portfolio must contain underlying plus options
- Hedging the jumps: previous work (Carr, He *et al*), good results for semi-static hedging (European options)
- We need a dynamic strategy for path dependent options
- Questions:
  - How many options do we need to reduce jump risk?
  - Will the bid-ask spread of the options in our hedging portfolio make a dynamic strategy too expensive?

Overview

- Assume price process is a jump diffusion
- Force delta neutrality (diffusion risk hedged)
- Isolate jump risk and transaction cost (bid/ask spread) terms
  - Model bid-ask spread as a function of moneyness
- At each hedge rebalance time
  - Minimize jump risk and transaction costs
- Test strategy by Monte Carlo simulation

Assumption: Stochastic Process for Underlying Asset S

$$
\begin{aligned}
\frac{dS}{S} &= \mu dt + \sigma dZ + (J-1)dq \\
&\mu = \text{ drift rate,} \\
&\sigma = \text{ volatility,} \\
&dZ = \text{ increment of a Wiener process} \\
&dq = \begin{cases} 0 & \text{with probability } 1 - \lambda dt \\ 1 & \text{with probability } \lambda dt, \end{cases} \\
&\lambda = \text{ mean arrival rate of Poisson jumps; } S \rightarrow JS.
\end{aligned}
$$

Option Price $V = V(S,t)$ Given by PIDE/LCP

$$\min(V_\tau - \mathcal{L}V - \lambda\mathcal{I}V, V - V^*) = 0 \qquad \text{American}$$

$$V_\tau = \mathcal{L}V - \lambda\mathcal{I}V \qquad \text{European}$$

$$\mathcal{L}V \equiv \frac{\sigma^2}{2}S^2V_{SS} + (r - \lambda\kappa)SV_S - (r+\lambda)V$$

$$\mathcal{I}V \equiv \int_0^\infty V(SJ)g^{\mathbb{Q}}(J)\,dJ$$

T = maturity date, $\kappa = E^{\mathbb{Q}}[J-1]$, $V^* =$ payoff ,

r = risk free rate , $\tau = T - t$,

$g^{\mathbb{Q}}(J)$ = probability density function of the jump amplitude J

Hedging Strategy

Hedging Portfolio Π

$$\Pi \;=\; -V + eS + \vec{\phi} \cdot \vec{I} + B$$

- Short option worth V
- Long e units underlying worth S
- Long N additional instruments worth $\vec{I} = [I_1, I_2, \ldots, I_N]^T$, with weights $\vec{\phi} = [\phi_1, \phi_2, \ldots, \phi_N]^T$
- Cash worth B

Jump Risk

- In $t \rightarrow t + dt$, $\Pi \rightarrow \Pi + d\Pi$.
- Use Ito's formula for finite activity jump diffusions, force delta neutrality
- Assume mid-point option prices given by linear pricing PIDE
- Recall: $\mathbb{Q} =$ pricing measure; $\mathbb{P} =$ real world measure
  - In practice, $\mathbb{Q}$ measure parameters obtained by calibration
- $\mathbb{P}$ measure parameters unknown to hedger

Change in Delta Neutral Portfolio

$$d\Pi = \text{ Jump Risk } = \lambda^{\mathbb{Q}} \, dt \, \mathbb{E}^{\mathbb{Q}}\Big[\Delta V - (\vec{\phi} \cdot \Delta \vec{I} + e \Delta S)\Big] + dq^{\mathbb{P}} \Big[-\Delta V + (\vec{\phi} \cdot \Delta \vec{I} + e \Delta S)\Big]$$

$$\Delta S = JS - S \;\; ; \;\; \Delta V = V(JS) - V(S)$$

$$\Delta \vec{I} = \vec{I}(JS) - \vec{I}(S)$$

Note: if $\mathbb{E}^{\mathbb{Q}} = \mathbb{E}^{\mathbb{P}}$, deterministic drift term exactly compensates random term. But in general $\mathbb{E}^{\mathbb{Q}} \neq \mathbb{E}^{\mathbb{P}}$, i.e. usually $\mathbb{Q}$ is more pessimistic than $\mathbb{P}$

Minimizing Jump Risk

When a jump occurs $dq^{\mathbb{P}} \neq 0$, the random change in Π is

$$\Delta H(J) \;\; = \;\; -\Delta V + \vec{\phi} \cdot \Delta \vec{I} + e \Delta S$$

Let $W(J)$ be any positive weighting function.
Consider:

$$F(\vec{\phi}, e)_{jump} \;\; = \;\; \int_0^\infty [\Delta H(J)]^2 \, W(J) dJ$$

Minimizing Jump Risk

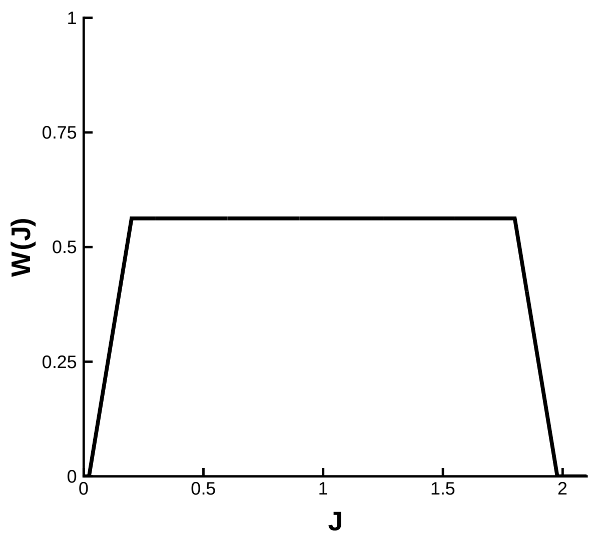
If $F(\vec{\phi}, e)_{jump} = 0$, then both the deterministic and random component of jump risk is zero.

Objective: make $F(\vec{\phi}, e)_{jump}$ (weighted jump risk) as small as possible

- Problem: What weighting function to use?
- Ideally, $W(J) = \mathbb{P}$ measure jump distribution, but this is unobservable
- If you guess wrong, results can be be very bad

Weighting Function

Practical Solution: set $W(J)$ to be nonzero for *likely* jump sizes $S \rightarrow JS$ (triangular tails avoid numerical problems)



$$F(\vec{\phi}, e)_{jump}$$
$$= \int_0^{\infty} [\Delta H(J)]^2 W(J) dJ$$

Bid-Ask Spreads

- Assume that hedger buys/sells at PIDE midpoint price $\pm$ one half spread
- This represents a lost transaction cost at each hedge rebalance time

$$F(\vec{\phi}, e)_{spread} \;=\; \sum_{portfolio} \Big(\text{Money lost due to spreads} \Big)^2$$

Objective Function

At each hedge rebalance time, choose $(e, \vec{\phi})$ (weights in underlying and hedging options), so that

- Portfolio is Delta neutral
- Minimize

$$
\begin{aligned}
\text{Objective Function} \;&=\; \xi F(\vec{\phi}, e)_{jump} + (1-\xi) F(\vec{\phi}, e)_{spread} \\
& \xi = 1 \rightarrow \text{ Minimize jump risk only} \\
& \xi = 0 \rightarrow \text{ Minimize trans. cost only}
\end{aligned}
$$

Review Assumptions: Synthetic Market

- Price process is Merton type jump diffusion
- All options in market can be bought/sold for the fair price plus/minus one half spread
- Mid-point option prices determined by linear pricing PIDE
- $\mathbb{Q}$ measure parameters: Andersen and Andreasen (2000)
- $\mathbb{P}$ measure market parameters: utility equilibrium model
- Hedger knows the $\mathbb{Q}$ measure market parameters
- Hedger does not know $\mathbb{P}$ measure market parameters

Basic Testing Method

- Choose target option, set of hedging instruments, hedging horizon
- Carry out MC simulations of hedging strategy, assume underlying follows a jump diffusion, with specified $\mathbb{P}$ measure parameters, option prices given by solution of PIDE
- Record discounted relative $P\&L$ at end of hedging horizon (or exercise) $t = T^*$ for each MC simulation

$$\text{Relative P \& L} = \frac{\exp\{-rT^*\}\Pi(T^*)}{V(S_0, 0)}$$

$$V(S_0, 0) = \text{ Initial Target Option Price}$$

Base Case Example

- Target option: one year European straddle
- Hedging horizon: 1.0 years, rebalance 40 times
- Initial $S_0 = 100$
- Hedging portfolio: underlying plus five $.25$ year puts/calls with strikes near S_0 (liquidate portfolio at $t = .25, .50., .75$, buy new .25 year options)
- Case 1: no bid-ask spreads
- Case 2: flat relative bid-ask spreads

$$\text{Relative Spread: underlying} = .002$$

$$\text{Relative Spread: options} = .10$$

Optimization Weights

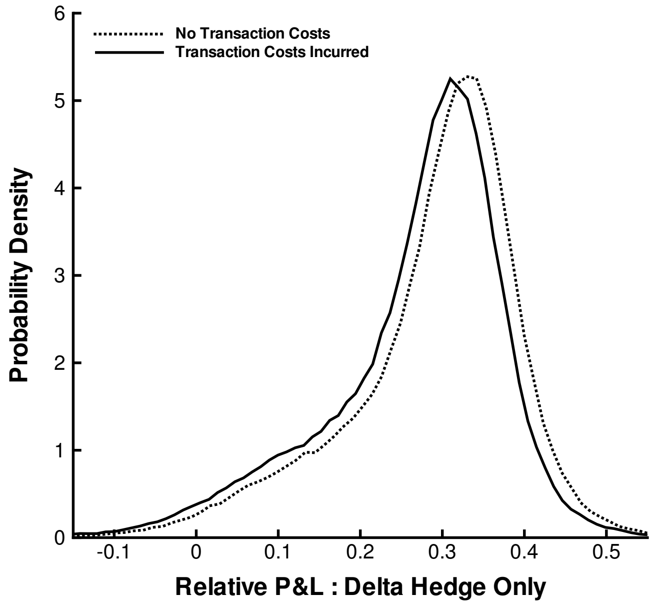
Recall that, at each rebalance date, we minimize:

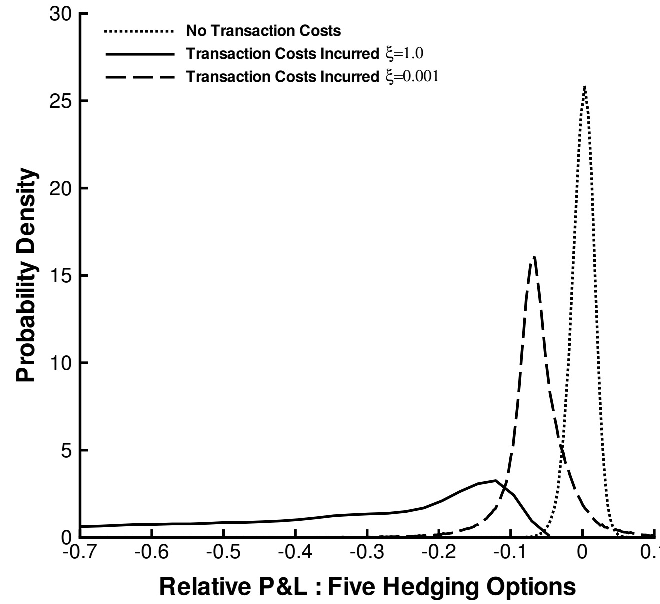
$$\begin{aligned}\text{Objective Function} &= \xi \times (\text{ Jump Risk}) \\ &\quad +(1-\xi) \times (\text{ Transaction Cost})\end{aligned}$$

How to pick ξ?

- No *right* answer
- Tradeoff between risk and cost
- We simply compute the density of the $P\&L$ for a range of ξ values, report results which give smallest standard deviation.

Base Case Results





Dotted - no transaction costs

Solid - transaction cost in market; not in objective function

Dashed - transaction cost in market; transaction cost in objective function

Base Case Summary

- Delta hedging alone not very good
- If there are bid-ask spreads, and you don't take them into account when determining portfolio weights

  → Hedging with options worse then delta hedging!
- Minimizing both jump risk and transaction costs

  → Small standard deviation

  → Cumulative transaction cost comparable with relative spreads assumed for hedging options.

A More Realistic Example

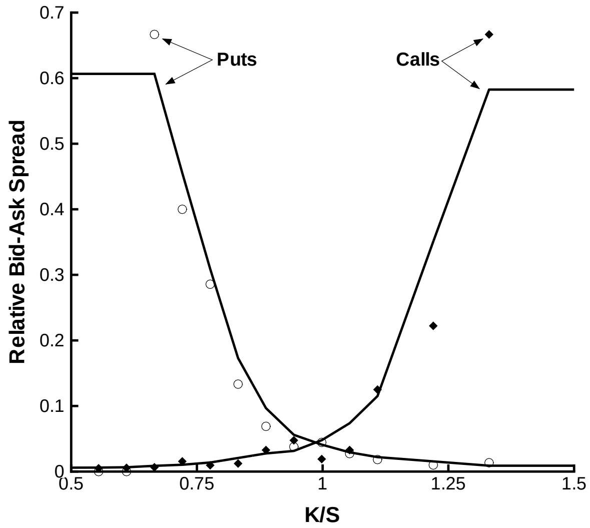
- Use better model for bid-ask spreads
- Allow a larger number of possible options for use in hedge portfolio ($10 - 14$ possible hedging options)
  - Consider $.25$ year puts/calls with strikes at $\$10$ intervals, centered near $S = 100$
- Realistic bid-ask spread model should make deep out of the money options too expensive to use

Bid-Ask Spreads

Relative bid-ask spreads, Amazon, 22Oct2005 puts/calls, as of August 10, 2005 vs. K/S.

Model relative spread as a function of moneyness (K/S)

Flat top data to avoid unrealistically large relative spread.



- Same target option (one year straddle)
- Forty rebalances
- Optimization method should pick out cheapest options to minimize jump risk

Realistic Spread Model: Results

	$(P\&L)/$ Initial option price			
Hedging Strategy	Mean	Standard Deviation	Percentiles 0.02%	0.2%
Delta Hedge	0.0565	1.0395	-11.55	-9.01
Ten Hedging Options	-0.0639	0.0230	-0.1493	-0.1250
Fourteen Hedging Options	-0.0667	0.0206	-0.1251	-0.1152

- Note that ten hedging options $\rightarrow$ 99.98% of the time we can lose no more than 15% of the initial option premium
- Note positive mean for simple delta hedging

Some Analysis: No Transaction Costs

Suppose that the weighting function $W(J)$ is such that for any function $f(J)$

$$\int_0^\infty f^2(J) g^{\mathbb{P}}(J)\, dJ \;\leq\; \int_0^\infty f^2(J) W(J)\, dJ \;<\; \infty$$

Recall notation:

$$\begin{aligned} T &= \text{Expiry time of option} \\ \Delta H(J) &= \text{Jump Risk} \\ \Delta t &= \text{Hedge rebalance interval} \end{aligned}$$

Global Bound: Hedging Error

Theorem 1. *In the limit as $\Delta t \rightarrow 0$, and if at each hedge rebalance time*

- *The hedge portfolio Π is delta neutral*
- $\int_0^\infty [\Delta H(J)]^2 W(J)\, dJ \leq \epsilon$

Then

$$E^{\mathbb{P}}[(\textit{Total Hedging Error})_T^2] \leq \mathbb{C}_1 \epsilon$$

where $\mathbb{C}_1$ is a constant.

Adding in Transaction Costs

Theorem 2. *In the limit as* $\Delta t \to 0$, *assuming* Π *is delta neutral and, at each rebalance time*

$$\xi \left\{ \int_0^\infty \left[\Delta H_J(S_t, t) \right]^2 W(J)\, dJ \right\} + (1-\xi) \left\{ \textit{ Transaction Costs } \right\}^2 < \epsilon\, \Delta t^2$$

holds for some $\xi \in (0,1)$. *Then*

$$E^{\mathbb{P}}[(\textit{Total Hedging Error})_T^2] \leq \mathbb{C}_2 \epsilon$$

Adding in Transaction Costs II

- It is always possible to minimize transaction costs by not trading
- It may not be possible to minimize both transaction costs and jump risk as required by the Theorem
- As a practical solution, we attempt to make the objective function as small as possible at each rebalance time

$$\begin{aligned}\text{Objective Function} &= \xi \left\{ \int_0^{\infty} \Big[\Delta H_J(S_t, t) \Big]^2 W(J)\, dJ \right\} \\ &\quad + (1-\xi) \Big\{ \text{ Transaction Costs } \Big\}^2\end{aligned}$$

Adding in Transaction Costs III

It follows from this result that if ξ is fixed for given Δt, and we minimize the local objective function at each rebalance time, then the best choice for ξ is

$$\xi \;\; = \;\; \mathbb{C}_3 (\Delta t)^2$$

This is observed in the numerical experiments.

Note that this means that more weight is put on the transaction cost term in the objective function as $\Delta t \rightarrow 0$.

This is required to avoid infinite transaction costs

Conclusions

- In market with jumps $\rightarrow$ delta hedging is bad
- Need to use additional options in the hedging portfolio
- If hedging portfolio is determined only on basis of minimizing jump risk $\rightarrow$ bid-ask spreads cause poor results when hedging with options
- If both jump risk and transaction costs minimized
  - Standard deviation much reduced compared to delta hedge
  - Relative cumulative transaction costs $\simeq 6 - 7\%$
- Similar results for American options

Let's Start a Hedge Fund ☺

- Recall that hedge fund managers typically receive 20% of the gain in an investment portfolio, but *no penalty if a loss*.
- Hedge fund strategy
  - Select asset which has large, infrequent jumps
  - Sell contingent claims (on this asset) with positive gamma, delta hedge
  - In a market with jumps, recall that this strategy has a positive mean
- This means that we, as hedge fund managers make money most of the time (and collect large bonuses)
- When a jump occurs, the investors are left with large, unhedged losses, hedge fund is bankrupt, but we retire rich!
- Sound familiar?