Multi-period mean variance asset allocation: Is it bad to win the lottery?

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The Basic Problem

Many financial problems have unhedgeable risk

- Optimal trade execution (sell a large block of shares)
  - $\rightarrow$ Maximize average price received, minimize risk, taking into account price impact
- Long term asset liability management (insurance)
  - $\rightarrow$ Match liabilities with minimal risk
- Minimum variance hedging of contingent claims (with real market constraints)
  - $\rightarrow$ Liquidity effects, different rates for borrowing/lending
- Pension plan investments.
- Wealth management products

Risk-reward tradeoff

All these problems (and many others) involve a tradeoff between risk and reward.

- A classic approach is to use some sort of utility function
- But this has all sorts of practical limitations
  - → What is the utility function of an investment bank?
  - → What *risk aversion* parameter should be selected by the Pension Investment Committee?

Alternative: mean-variance optimization

- When risk is specified by variance, and reward by expected value
  - → Non-technical managers can understand the tradeoffs and make informed decisions

Multi-period Mean Variance

Some issues:

- Standard formulation not amenable to use of dynamic programming
- Variance as risk measure penalizes upside as well as downside
- Pre-commitment mean variance strategies are not time consistent

I hope to convince you that multi-period mean variance optimization is

- Intuitive
- Can be modified slightly to be (effectively) a downside risk measure

Motivating example: Wealth Management (target date fund)

Example: Target Date (Lifecycle) Fund with two assets

Risk free bond B

$$dB = rB\ dt$$
$$r = \text{ risk-free rate}$$

Amount in risky stock index S

$$dS = (\mu - \lambda\kappa)S\ dt + \sigma S\ dZ + (J-1)S\ dq$$

$$\mu = \mathbb{P} \text{ measure drift} \quad ; \quad \sigma = \text{ volatility}$$
$$dZ = \text{ increment of a Wiener process}$$

$$dq = \begin{cases} 0 & \text{with probability } 1 - \lambda dt \\ 1 & \text{with probability } \lambda dt, \end{cases}$$
$$\log J \sim \mathcal{N}(\mu_J, \sigma_J^2). \quad ; \quad \kappa = E[J-1]$$

Optimal Control

Define:

$$
\begin{aligned}
X &= (S(t), B(t)) = \text{Process} \\
x &= (S(t) = s, B(t) = b) = (s, b) = \text{State} \\
(s + b) &= \text{total wealth}
\end{aligned}
$$

Let $(s, b) = (S(t^-), B(t^-))$ be the state of the portfolio the instant before applying a control

The control $c(s, b) = (d, B^+)$ generates a new state

$$
\begin{aligned}
b &\rightarrow B^+ \\
s &\rightarrow S^+ \\
& \quad S^+ = \underbrace{(s + b)}_{wealth\ \ at\ \ t^-} - B^+ - \underbrace{d}_{withdrawal}
\end{aligned}
$$

Note: we allow cash withdrawals of an amount $d \geq 0$ at a rebalancing time

Semi-self financing policy

Since we allow cash withdrawals

$\rightarrow$ The portfolio may not be self-financing

$\rightarrow$ The portfolio may generate a free cash flow

Let $W_a = S(t) + B(t)$ be the **allocated wealth**

- W_a is the wealth available for allocation into $(S(t), B(t))$.

The non-allocated wealth $W_n(t)$ consists of cash withdrawals and accumulated interest

Constraints on the strategy

The investor can continue trading only if solvent

$$\underbrace{W_a(s,b) = s + b > 0}_{Solvency\ \ condition}. \tag{1}$$

In the event of bankruptcy, the investor must liquidate

$$S^+ = 0 \quad ; \quad B^+ = W_a(s,b) \quad ; \quad \text{if } \underbrace{W_a(s,b) \leq 0}_{bankruptcy} \; .$$

Leverage is also constrained

$$\frac{S^+}{W^+} \quad \leq \quad q_{\text{max}}$$

$$W^+ = S^+ + B^+ = \text{ Total Wealth}$$

Mean and Variance under control $c(X(t), t)$

$$\underbrace{E_{t,x}^{c(\cdot)}[\cdot]}_{Reward} = \text{Expectation conditional on } (x, t) \text{ under control } c(\cdot)$$

$$\underbrace{\text{Var}_{t,x}^{c(\cdot)}[\cdot]}_{Risk} = \text{Variance} \quad " \quad " \quad " \quad " \quad "$$

Mean Variance (MV) problem: for fixed λ find control $c(\cdot)$ which solves:

$$\sup_{c(\cdot)\in\mathbb{Z}} \left\{ \underbrace{E_{t,x}^{c(\cdot)}[W_a(T)]}_{Reward\ as\ seen\ at\ time\ t} - \lambda \underbrace{Var_{t,x}^{c(\cdot)}[W_a(T)]}_{Risk\ as\ seen\ at\ time\ t} \right\},$$

$$\mathbb{Z} = \text{ set of admissible controls } ; \ T = \text{ target date}$$

- Varying $\lambda \in [0, \infty)$ traces out the efficient frontier

Embedding(Zhou and Li (2000), Li and Ng (2000))

Equivalent formulation:[1] [2] for fixed λ, if $c^*(\cdot)$ solves the standard MV problem,

$\rightarrow$ $\exists \gamma$ such that $c^*(\cdot)$ minimizes

$$\inf_{c(\cdot) \in \mathbb{Z}} E_{t,x}^{c(\cdot)} \left[\left(W_a(T) - \frac{\gamma}{2} \right)^2 \right] . \tag{2}$$

Once $c^*(\cdot)$ is known

- Easy to determine $E_{t,x}^{c^*(\cdot)}[W_a(T)]$, $Var_{t,x}^{c^*(\cdot)}[W_a(T)]$
- Repeat for different γ, traces out efficient frontier

[1] We are determining the optimal pre-commitment strategy (Basak,Chabakauri: 2010; Bjork et al: 2010). See (Wang and Forsyth (2012)) for a comparison of pre-commitment and time consistent strategies.

[2] We do not require convex constraints.

Equivalence of MV optimization and target problem

MV optimization is equivalent[3] to investing strategy which[4]

- Attempts to hit a target final wealth of $\gamma/2$
- There is a quadratic penalty for not hitting this wealth target
- From (Li and Ng(2000))

$$\underbrace{\frac{\gamma}{2}}_{wealth\ \ target} = \underbrace{\frac{1}{2\lambda}}_{risk\ \ aversion} + \underbrace{E^{c(\cdot)}_{t=0,x_0}[W_a(T)]}_{expected\ \ wealth}$$

Intuition: if you want to achieve $E[W_a(T)]$, you must aim higher

[3] Vigna, Quantitative Finance, to appear, 2014

[4] Strictly speaking, since some values of γ may not represent points on the original frontier, we need to construct the upper left convex hull of these points (Tse, Forsyth, Li (2014), SIAM J. Control Optimization)

HJB PIDE

Determination of the optimal control $c(\cdot)$ is equivalent to determining the value function

$$V(x,t) = \inf_{c \in \mathcal{Z}} \left\{ E^{c}_{t,x}[(W_a(T) - \gamma/2)^2] \right\} ,$$

Define:

$$\begin{aligned} \mathcal{L}V &\equiv \frac{\sigma^2 s^2}{2} V_{ss} + (\mu - \lambda\kappa) s V_s + rbV_b - \lambda V \ , \\ \mathcal{J}V &\equiv \int_0^\infty p(\xi) V(\xi s, b, \tau) \ d\xi \\ & \quad\ p(\xi) = \text{ jump size density} \end{aligned}$$

and the intervention operator $\mathcal{M}(c) \ V(s,b,t)$

$$\mathcal{M}(c) \ V(s,b,t) \ = \ V(S^+(s,b,c), B^+(s,b,c), t)$$

HJB PIDE II

The optimal control $c(\cdot)$ is given by solving the impulse control HJB equation:

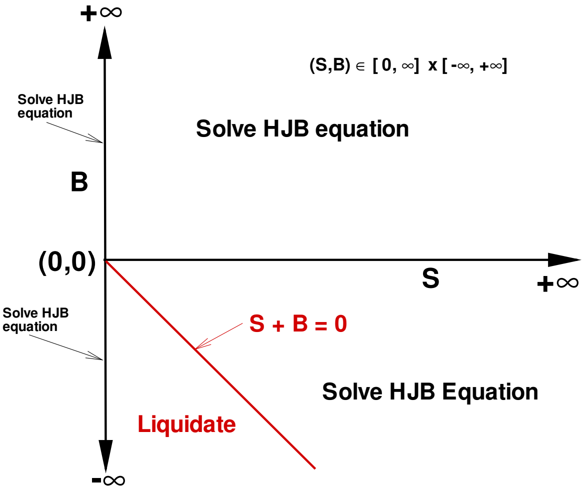
$$\max\left[V_t + \mathcal{L}V + \mathcal{J}V, V - \inf_{c \in \mathcal{Z}} (\mathcal{M}(c)\ V)\right] = 0 \qquad \text{if } (s + b > 0) \qquad (3)$$

Along with liquidation constraint if insolvent

$$V(s, b, t) = V(0, (s + b), t) \qquad \text{if } (s + b) \leq 0 \text{ and } s \neq 0 \qquad (4)$$

Easy to generalize the above equation to handle the discrete rebalancing case.

Computational Domain[5]



[5]If $\mu > r$ it is never optimal to short S

Well behaved utility function

Definition (Well-behaved utility functions)

A utility function $Y(W)$ is a well-behaved function of wealth W if it is an increasing function of W.

Proposition

Pre-commitment MV portfolio optimization is equivalent to maximizing the expectation of a well-behaved quadratic utility function if

$$W_a(T) \leq \frac{\gamma}{2}. \tag{5}$$

Obvious, since value function $V(x, t)$ is

$$V(x,t) = \sup_{c \in \mathcal{Z}} \left\{ E_c^{x,t}[Y(W_a(T))] \right\}$$

$$Y(W) = -(W - \gamma/2)^2$$

Dynamic MV Optimal Strategy

Theorem (Vigna (2014))

Assuming that (i) the risky asset follows a pure diffusion (no jumps), (ii) continuous re-balancing, (iii) infinite leverage permitted, (iv) trading continues even if bankrupt: then the optimal self-financing MV wealth satisfies

$$\begin{aligned} W_a(t) &\leq F(t) \ ; \ \forall t \\ & F(t) = \frac{\gamma}{2} e^{-r(T-t)} = \textit{ discounted wealth target} \end{aligned}$$

$\hookrightarrow$ MV optimization maximizes a well behaved quadratic utility
Result can be generalized[6] to the case of

- Realistic constraints: finite leverage and no trading if insolvent
- But, we must have continuous rebalancing and no jumps

[6] Dang and Forsyth (2013)

Global Optimal Point

Examination of the HJB equation allows us to prove the following result

Lemma (Dang and Forsyth (2013))

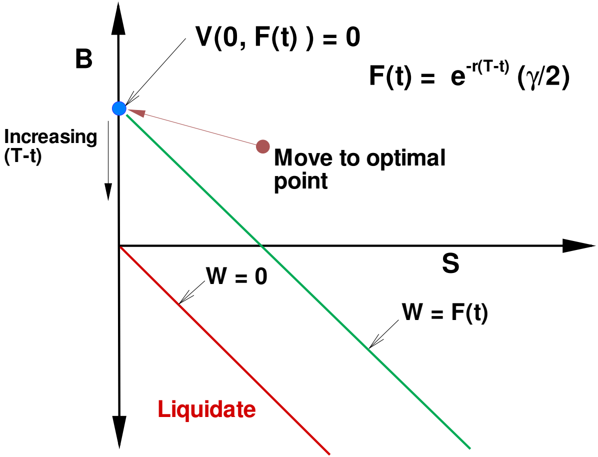
The value function $V(s, b, t)$ is identically zero at

$$V(0, F(t), t) \equiv 0 \; ; \; F(t) = \frac{\gamma}{2} e^{-r(T-t)} \; , \; \forall t$$

Since $V(s, b, t) \geq 0$

- $V(0, F(t), t) = 0$ is a global minimum
- Any admissible policy which allows moving to this point is an optimal policy
- Once this point is attained, it is optimal to remain at this point

Movement of Globally Optimal Point[7]



[7] This is only admissible if $\gamma > 0$

Optimal semi-self-financing strategy

Theorem (Dang and Forsyth (2013))

If $W_a(t) > F(t)$,[8] $t \in [0, T]$*, an optimal MV strategy is*[9]

- *Withdraw cash $W_a(t) - F(t)$ from the portfolio*
- *Invest the remaining amount $F(t)$ in the risk-free asset.*

Corollary (Well behaved utility function)

In the case of discrete rebalancing, and/or jumps, the optimal semi-self-financing MV strategy is

- *Equivalent to maximizing a well behaved quadratic utility function*[10]

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[8] $F(t)$ is the discounted wealth target

[9] A similar semi-self-financing strategy for the discrete rebalancing case was first suggested in (Cui, Li, Wang, Zhu (2012) *Mathematical Finance*).

[10] A similar idea is termed *time consistency in efficiency* (Li, Cui, Zhu (2011))

Intuition: Multi-period mean-variance

Optimal target strategy: try to hit $W_a(T) = \gamma/2 = F(T)$.

If $W_a(t) > F(t) = F(T)e^{-r(T-t)}$, then the target can be hit exactly by

- Withdrawing $W_a(t) - F(t)$ from the portfolio
- Investing $F(t)$ in the risk free account

Optimal control for the target problem $\equiv$ optimal control for the Mean Variance problem

This strategy dominates any other MV strategy

$\rightarrow$ And the investor receives a bonus in terms of a free cash flow

What happens if we win the lottery?

Classic Mean Variance

- If you win the lottery, and exceed your wealth target
  - Since gains $>$ target are penalized.

  $\rightarrow$ Optimal strategy: lose money!

Precommitment, semi-self-financing optimal strategy

- If you win the lottery, and exceed your wealth target

  $\rightarrow$ Invest $F(t)$[11] in a risk-free account

  $\rightarrow$ Withdraw any remaining cash from the portfolio

  $\rightarrow$ No incentive to act irrationally

[11] $F(t)$ is the discounted target wealth

Numerical Method

We solve the HJB impulse control problem numerically using a finite difference method

- We use a semi-Lagrangian timestepping method
- Can impose realistic constraints on the strategy
  - Maximum leverage, no trading if insolvent
  - Arbitrarily shaped solvency boundaries
- Continuous or discrete rebalancing
- Nonlinearities
  - Different interest rates for borrowing/lending
  - Transaction costs
- Regime switching (i.e. stochastic volatility and interest rates)

We can prove[12] that the method is monotone, consistent, ℓ_∞ stable

→ Guarantees convergence to the viscosity solution

[12] Dang and Forsyth (2014) Numerical Methods for PDEs

Numerical Examples

initial allocated wealth ($W_a(0)$)	100
r (risk-free interest rate)	0.04450
T (investment horizon)	20 (years)
q_{max} (leverage constraint)	1.5
discrete re-balancing time period	1.0 (years)

	mean downward jumps	mean upward jumps
μ (drift)	0.07955	0.12168
λ (jump intensity)	0.05851	0.05851
σ (volatility)	0.17650	0.17650
mean log jump size	-0.78832	0.10000
compensated drift	0.10862	0.10862

Efficient Frontier: discrete rebalancing

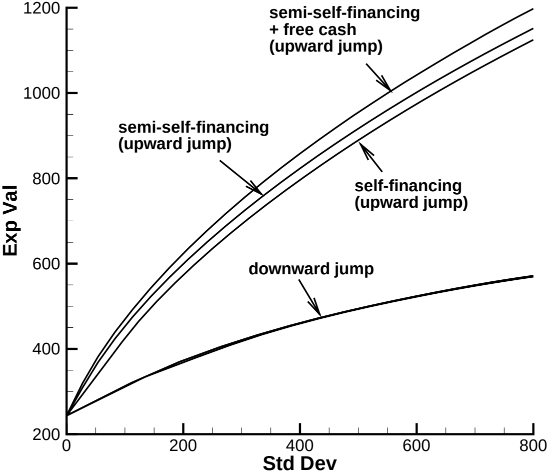


Figure: $T = 20$ years, $W_a(0) = 100$.

Example II

Two assets: risk-free bond, index

- Risky asset follows GBM (no jumps)

According to Benjamin Graham[13], most investors should

- Pick a fraction p of wealth to invest in an index fund (i.e. $p = 1/2$).
- Invest $(1 - p)$ in bonds
- Rebalance to maintain this asset mix

How much better is the optimal asset allocation vs. simple rebalancing rules?

[13] Benjamin Graham, *The Intelligent Investor*

Long term investment asset allocation

Investment horizon (years)	30
Drift rate risky asset μ	.10
Volatility σ	.15
Risk free rate r	.04
Initial investment W_0	100

Benjamin Graham strategy			
Constant proportion	Expected Value	Standard Deviation	Quantile
$p = 0.0$	332.01	NA	NA
$p = 0.5$	816.62	350.12	$Prob(W(T) < 800) = 0.56$
$p = 1.0$	2008.55	1972.10	$Prob(W(T) < 2000) = 0.66$

Table: Constant fixed proportion strategy. $p =$ fraction of wealth in risky asset. Continuous rebalancing.

Optimal semi-self-financing asset allocation

Fix expected value to be the same as for constant proportion $p = 0.5$.

Determine optimal strategy which minimizes the variance for this expected value.

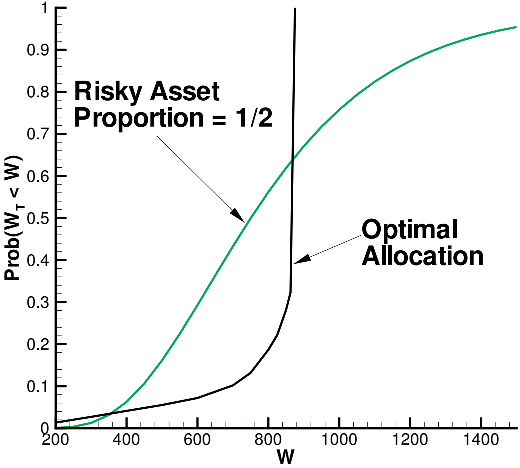
- We do this by determining the value of $\gamma/2$ (the wealth target) by Newton iteration

Strategy	Expected Value	Standard Deviation	Quantile
Graham $p = 0.5$	816.62	350.12	$Prob(W(T) < 800) = 0.56$
Semi-self-financing	816.62	142.85	$Prob(W(T) < 800) = 0.19$

Table: $T = 30$ years. $W(0) = 100$. Semi-self-financing: no trading if insolvent; maximum leverage $= 1.5$, rebalancing once/year.

Standard deviation reduced by 250 %, shortfall probability reduced by $3\times$

Cumulative Distribution Functions



$E[W_T] = 816.62$ for both strategies

Optimal policy: $\uparrow W$ risk off; $\downarrow W(t)$ risk on

Optimal allocation gives up gains $\gg$ target in order to reduce variance and probability of shortfall.

Investor must pre-commit to target wealth

Conclusions

- Pre-commitment mean variance strategy
  - Equivalent to quadratic target strategy
- Semi-self-financing, pre-commitment mean variance strategy
  - Minimizes quadratic loss w.r.t. a target
  - Dominates self-financing strategy
  - Extra bonus of free cash-flow
- Example: target date fund
  - Optimal strategy dominates simple constant proportion strategy by a large margin
    - $\rightarrow$ Probability of shortfall $\simeq$ 3 times smaller!
  - But
    - $\rightarrow$ Investors must pre-commit to a wealth target
- Optimal stochastic control: teaches us an important life lesson
  - Decide on a life target ahead of time and stick with it
  - If you achieve your target, do not be greedy and want more