Decumulation of Retirement Savings: *The Nastiest, Hardest Problem in Finance*

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Motivation

Defined Benefit Plans (DB) are disappearing

→ Corporations/governments no longer willing to take risk of DB plans

Recent survey[1] P7 countries[2]

- Defined Contribution (DC) plan assets: 55% of all pension assets
- Some examples
  - → Australia 87% DC
  - → US 65% DC
  - → Canada 43% DC
  - → ...
  - → Japan 5% DC

[1] Thinking Ahead Institute (2023)

[2] Australia, Canada,Japan, Netherlands, Switzerland, UK, US

The retiree dilemma (Defined Contribution (DC))

A retiree with savings in a DC plan[3] (i.e. an RRSP) has to decide on

- An investment strategy (stocks vs. bonds)
- A decumulation schedule

The retiree now has two major sources of risk

- Investment risk
- Longevity risk (running out of cash before death)

William Sharpe (Nobel Laureate in Economics) calls this

> *"The nastiest hardest problem in finance"*

[3] In a DC plan, the retiree is responsible for investment/decumulation

The Four per Cent Rule

Based on rolling 30-year historical periods, Bengen (1994) showed:

A retiree who

- Invested in a portfolio of 50% bonds, 50% stocks (US), rebalanced annually
- Withdrew 4% of initial capital (adjusted for inflation) annually

→ Would never have run out of cash, over any rolling 30-year period (from 1926)

Criticism

- Simplistic asset allocation strategy
- Simplistic withdrawal strategy
- Rolling 30 year periods contain large overlaps
  → Underestimates risk of portfolio depletion

Bengen rule

> *"Play the long game. A retirement income plan should be based on planning to live, not planning to die. A long life will be expensive to support, and it should take precedence over death planning." Pfau (2018)*

Note that Bengen rule is based on assumption that 65-year old will live to be 95

- Should we mortality weight the cash flows and probability of running out of cash?
- Example: median life expectancy of 65-year old male $\simeq$ 87.
  - → Effectively, mortality weighting will weight minimum cash flow of 87-year old by 1/2
  - → If I am 87, and alive, I need 100% of my minimum cash flows
  - → If I am dead, I need zero dollars
- We will consider an individual investor, not averaging over a population
  - → 30 year retirement, no mortality weighting
  - → Consistent with Bengen approach

Fear of running out of cash

Recent survey[4]

- Majority of pre-retirees fear exhausting their savings in retirement more than death

In Canada, a 65-year old male

- Probability of 0.13 of living to be 95
- Probability of 0.02 of living to be 100

Conservative strategy:

→ Assume 30 year retirement (as in Bengen (1994)).

Other assets can be used to hedge extreme longevity[5]

[4] 2017 Allianz Generations Ahead Study - Quick Facts #1. (2017), Allianz
[5] Real estate

Objective of this talk

Determine a decumulation strategy which has

- Variable withdrawals (minimum and maximum constraints)
- Minimizes risk of portfolio depletion
- Maximizes total expected withdrawals
- Allows for dynamic, non-deterministic asset allocation

We will treat this as a problem in optimal stochastic control

Formulation

Investor has access to two funds

- A broad stock market index fund
  - *Amount* in stock index S_t
- A constant maturity bond index fund
  - *Amount* in bond index B_t

$$\text{Total Wealth } W_t = S_t + B_t \quad (1)$$

Model the returns of both indexes

- Parametric, jump diffusion
- Non-zero stock-bond correlation

• Fit parameters to market data 1926:1-2019:12
↪ All returns adjusted for inflation

Notation

Withdraw/rebalance at discrete times $t_i \in [0, T]$

The investor has two controls at each rebalancing time

$$
\begin{aligned}
q_i &= \text{Amount of withdrawal} \\
p_i &= \text{Fraction in stocks after withdrawal}
\end{aligned} \tag{2}
$$

At t_i, the investor withdraws q_i

$$
\begin{aligned}
W_i^- &= \overbrace{S_i^- + B_i^-}^{\textit{wealth before withdrawal}} \\
W_i^+ &= W_i^- - q_i
\end{aligned} \tag{3}
$$

Then, the investor rebalances the portfolio

$$
\begin{aligned}
S_i^+ &= p_i W_i^+ \\
B_i^+ &= (1 - p_i) W_i^+
\end{aligned} \tag{4}
$$

Can show that

$$
q_i = q_i(W_i^-) \quad ; \quad p_i = p_i(W_i^+)
$$

Controls

Constraints on controls

$$
\begin{aligned}
q_i &\in [q_{\text{min}}, q_{\text{max}}] \quad ; \quad \text{withdrawal amount} \\
p_i &\in [0, 1] \quad ; \quad \text{fraction in stocks} \\
&\Rightarrow \text{ no shorting, no leverage}
\end{aligned}
$$

Set of controls

$$
\mathcal{P} = \{(q_i(\cdot), p_i(\cdot))) : i = 0, \dots, M\} \tag{5}
$$

Reward and Risk

Reward: Expected total (real) withdrawals (EW)

$$\text{EW} = E\Big[\overbrace{\sum_i q_i}^{\textit{total withdrawals}} \Big]$$

$E[\cdot] =$ Expectation

Risk measure: Expected Shortfall *ES*

$$ES(5\%) \equiv \Big\{ \text{Mean of worst 5\% of } W_T \Big\}$$

$W_T =$ terminal wealth at $t = T$

ES defined in terms of final wealth, *not losses*[6]

→ Larger is better

[6] ES is basically the negative of CVAR

Objective Function

Multi-objective problem $\rightarrow$ scalarization approach for Pareto points

Find controls $\mathcal{P}$ which maximize (scalarization parameter $\kappa > 0$)

$$\sup_{\mathcal{P}} \left\{ EW + \kappa \ ES \right\}$$

$$\sup_{\mathcal{P}} \left\{ \overbrace{E_{\mathcal{P}}[\sum_i q_i]}^{total \ \ withdrawals} + \kappa \overbrace{\left(\frac{E_{\mathcal{P}}[W_T \ 1_{W_T \leq W^*}]}{.05} \right)}^{mean \ \ worst \ \ 5\% \ \ outcomes} \right\}$$

$$\text{s.t.} \qquad Prob[W_T \leq W^*] = .05$$

Varying κ traces out the efficient frontier in the (EW, ES) plane

EW-ES Objective Function

Given an expectation under control $E_{\mathcal{P}}[\cdot]$ (Rockafellar and Uryasev, 2000)

$$\text{ES}_{5\%} = \sup_{W^*} E_{\mathcal{P}}\Big[G(W_T, W^*)\Big]$$

$$G(W_T, W^*) = \left(W^* + \frac{1}{.05}\left[\min(W_T - W^*, 0)\right]\right)$$

Objective function:

$$\sup_{\mathcal{P}} \sup_{W^*} E_{\mathcal{P}}\Bigg\{ \overbrace{\sum_i q_i}^{total\ \ withdrawals} + \kappa \overbrace{G(W_T, W^*)}^{mean\ \ worst\ \ 5\%\ \ W_T} + \overbrace{\epsilon W_T}^{Stabilization} \Bigg\}$$

Why do we need the stabilization term?

$\hookrightarrow$ More later

Time Consistency

The EW-ES objective function is not formally *time consistent*

Time inconsistency

$\Rightarrow$ Investor has incentive to deviate from initial optimal policy at later times

EW-ES policy computed at time zero
$\hookrightarrow$ Pre-commitment policy

Induced time consistent policy

At t_0 we compute the pre-commitment EW-ES control

- For $t > t_0$ we assume that the investor follows the *induced time consistent* control (Strub et al (2019))
- This control is identical to the pre-commitment control at t_0
- No incentive to deviate from this control at $t > t_0$

Induced time consistent control determined from (fixed W^*)

$$\sup_{\mathcal{P}} E_{\mathcal{P}} \left\{ \sum_i q_i + \kappa G(W_T, W^*) + \epsilon W_T \right\}$$

W^* from pre-commitment solution at time zero

Alternative: equilibrium mean-ES control
$\hookrightarrow$ Does not actually control tail risk! (Forsyth(2020)) [7]

[7] For more discussion of time consistency, induced time consistency, pre-commitment, see Bjork et al (2021), Vigna (2020, 2022), Strub et al (2019), Forsyth (2020)

Withdrawal Control: limiting case

Theorem 1 (Bang-bang withdrawal control: continuous limit)

Assume that

- *the stock and bond indexes follow a parametric jump-diffusion*
- *the portfolio is continuously rebalanced, and withdrawals occur at the continuous (finite) rate* $\hat{q} \in [\hat{q}_{\text{min}}, \hat{q}_{\text{max}}]$

then the optimal control is bang-bang, i.e. the optimal withdrawal $\hat{q}^*$ *is either* $\hat{q}^* = \hat{q}_{\text{min}}$ *or* $\hat{q}^* = \hat{q}_{\text{max}}$.

Proof.

See Forsyth (North American Actuarial Journal (2022)) □

But of course, in real life, we do not withdraw/rebalance continuously.

Scenario: all amounts indexed to inflation

- DC account at $t = 0$ (age 65) \$1,000K (one million)
- Minimum withdrawal from DC account \$35K per year[8]
- Maximum withdrawal from DC \$60K per year
- Annual rebalancing/withdrawals
- Retiree owns mortgage-free real estate worth \$400K

Investment Horizon

- $T = 30$ years, i.e. from age 65 to 95

  $\Rightarrow$ Plan to live long and prosper

[8] Assume gov't benefits of 22K/year. Minimum income $\simeq 22K + 35K = 57K$/year.

Scenario II

Why do we include real estate in the scenario?

Since $q_{\min} = 35K$ per year, W_t can become negative

- When $W_t < 0$, assume retiree is borrowing, using a reverse mortgage[9]
  - Reverse mortgages allow borrowing of 50% of home value
  - In our case: $200K
- Once $W_t < 0$
  - All stocks are liquidated
  - Debt accumulates at borrowing rate
- If $W_T > 0$, then real-estate is a bequest
- Real estate is a hedge of last resort: not fungible with other wealth
  - This mental bucketing of real estate is a well-known behavioral finance result.[10]

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[9] See Pfeiffer et al, Journal of Financial Planning (2013)

[10] I also observe this with my fellow retirees: real-estate is a separate bucket

Numerical Method I

Pre-commitment control at t_0 (same as induced time consistent control)

Interchange $\sup \sup (\ldots)$

$$\underbrace{\sup_{W^*} \overbrace{\sup_{\mathcal{P}} E_{\mathcal{P}} \left\{ \sum_i q_i + \kappa G(W_T, W^*) + \epsilon W_T \right\}}^{Solve\ \ using\ \ Dynamic\ \ Programming\ \ (fixed\ \ W^*)}}_{maximize\ \ over\ \ W^*}$$

Solve inner DP problem using PIDE methods

Numerical Method II

Inner maximization: dynamic programming

- Conditional expectations at t_i^+
  - Solve linear 2-d PIDE
  - Use ϵ-monotone Fourier method (Forsyth and Labahn (2019))
- Optimal controls at each rebalancing time
  - Discretize controls
  - Find maximum by exhaustive search
- Guaranteed to converge to the solution as discretization parameters $\to 0$

Outer maximization over W^*

- Discretize W^*, use coarse PIDE grid
  - $\to$ Find optimal W^* by exhaustive search
- Use coarse grid W^* as starting point for 1-d optimization on finer grids

Data

Center for Research in Security Prices (CRSP) US

- Cap weighted index, all stocks on all major US exchanges 1926:1-2019:12
- US 10 year Treasury index
- Monthly data, inflation adjusted by CPI

Synthetic Market

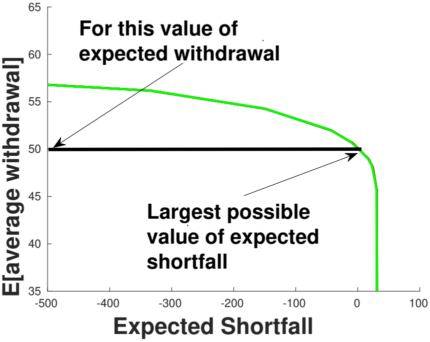
- Stock/bond returns driven by parametric jump-diffusion model, calibrated to data
- Optimal controls computed in the *synthetic* market

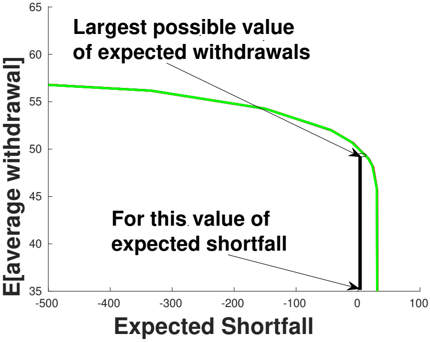
Historical market

- Stock/bond returns from stationary block bootstrap resampling of actual data[11]
- No assumptions about stock/bond processes
- Used to test control robustness computed in the synthetic market

[11] Dichtl et al (2016, Appl. Econ.), Anarkulova et al (JFE,2022)

Pareto optimal points (Units: Thousands)



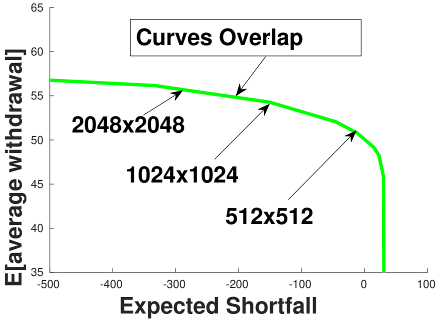


Varying scalarization parameter κ

$\rightarrow$ Traces out efficient frontier

- y-axis is annual average expected withdrawals
- E.g.: 50K ($W_0 = 1000K$) corresponds to 5% withdrawal rate
- Recall ES is mean of worst 5% $W_T \Rightarrow$ larger is better

EW-ES efficient frontier (Units: thousands)

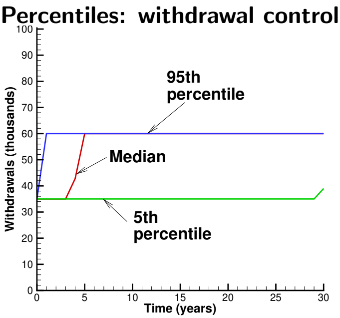


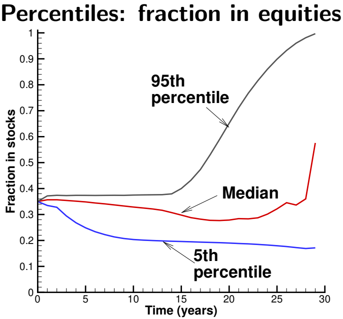
- Solutions with different PIDE grids
- ES is the mean of the worst 5% of outcomes
- Each pt on curve, different κ
- Reverse mortgage hedge
  - $\rightarrow$ Any point $ES > -200K$ is acceptable

Note Efficient Frontier almost vertical at right hand end

- Base case: constant withdrawal 35K/year
- Tiny increase in risk (smaller ES)
  - $\Rightarrow$ Average withdrawal 50K per year (never less than 35K)

Point on Frontier: (EW,ES) = (52K/year, -42K)



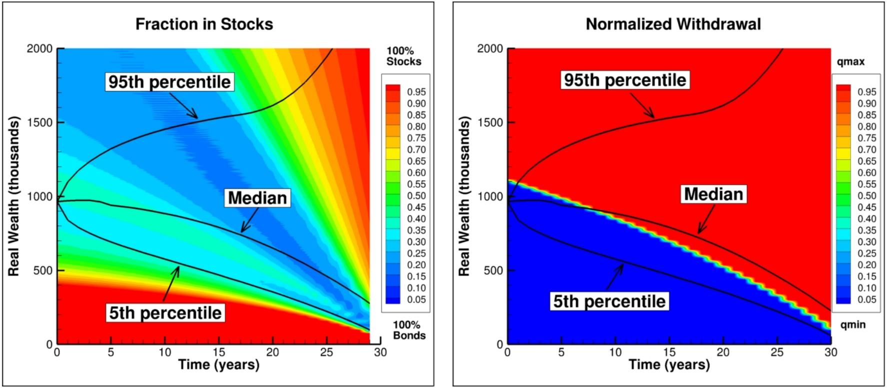


→ $ES \simeq -42K$

→ 5th percentile wealth at $t = 30 \simeq$ 58K

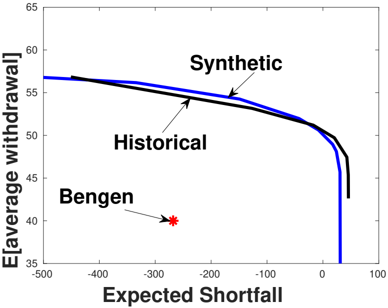
→ Average withdrawal ≃52K/year

Point on Frontier: (EW,ES) = (52K/year, -42K)



- Withdrawal controls $\simeq$ *bang-bang*, i.e. only withdraw either $q_{\min}$ or $q_{\max}$.
- Median $W_t \simeq 1000K \rightarrow 300K$

Robustness Check: Efficient Frontier (Units: thousands)



- Bengen 4% rule: bootstrapped historical market
  - ⇒ very inefficient
  - ⇒ More risky than advertised, ES $\simeq$ -270K

Controls computed and stored in the *synthetic* market

- Parametric model calibrated to historical data

Controls tested[12] in the bootstrapped historical market

→ Controls are robust to parametric model misspecification

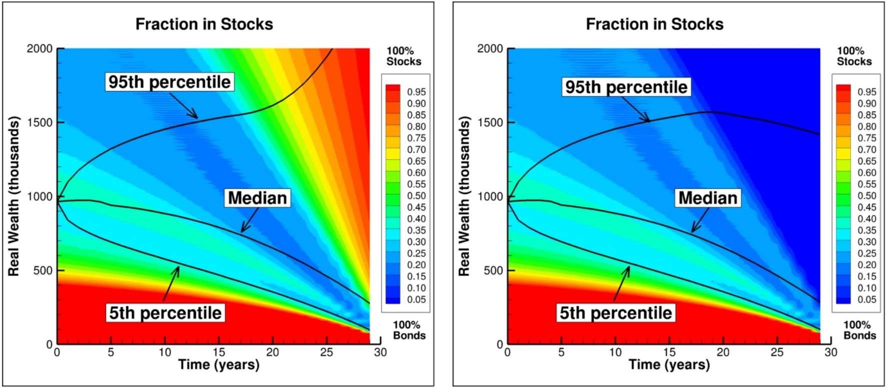
[12] "Out-of-sample" test.

Stabilization term (EW,ES) = (52K/year, -42K)

Recall objective function:

$$\sup_{\mathcal{P}} \sup_{W^*} \left\{ \overbrace{EW}^{\text{total withdrawals}} + \overbrace{\kappa \; G(W_T, W^*)}^{\text{mean worst 5\% outcomes}} + \overbrace{\epsilon W_T}^{\text{Stabilization}} \right\}$$

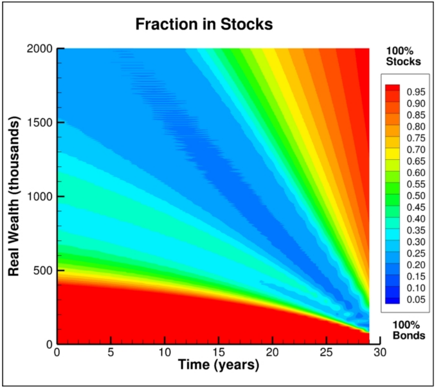
$\epsilon = +10^{-6}$ $\epsilon = -10^{-6}$



Stabilization term

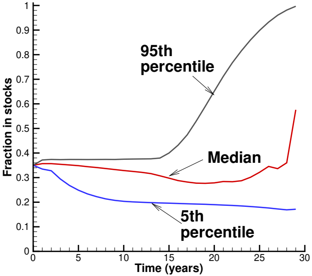
Plots of efficient EW-ES frontiers overlap for $\epsilon = \pm 10^{-6}$
Recall that we are assuming the investor follows the induced time consistent strategy

$\epsilon = +10^{-6}$

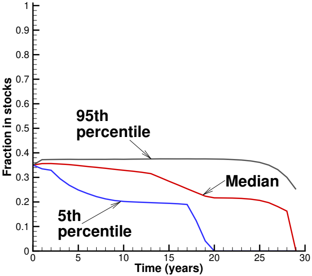


- $W^* = 58K$
- Suppose that $t = 25$, i.e. 90 years old
- W = 2000K, you will never run out of cash with $q_{max} =$ 60K/year
- It does not matter whether you invest 100% in stocks or bonds

If you are Warren Buffet, this problem is ill-posed



$\epsilon = +10^{-6}$



$\epsilon = -10^{-6}$

Fraction Stocks < 0.4 at 95th percentile

If you are rich and old, then it does not matter what you do

- $\epsilon = +10^{-6}$ invest 100% in stocks
- $\epsilon = -10^{-6}$ invest 100% in bonds

But these lucky large wealth outcomes $\Rightarrow$ no effect on (EW,ES) frontier

Peter Ponzo: Canasta Strategy

Peter Ponzo (retired Applied Math Professor from Waterloo)

- Retired: 1993; passed away: 2020
- In 1993, took commuted value of his pension
  - One-half $\rightarrow$ annuity (interest rate: 9.8%)
  - One-half $\rightarrow$ self-directed investments
  - Wrote a blog about his attempts to *"beat the market"*
- It turned out that beating the market was not easy!

But: he summarized his withdrawal strategy: "Canasta Strategy"

> *"If we have a good year, we take a trip to China,...,if we have a bad year, we stay home and play canasta."*

This is a bang-bang control!

Conclusions

- Optimal strategy: flexible withdrawals, dynamic stock-bond allocation
  - → Less risk, higher average withdrawals[13] compared to 4% rule
  - → Bootstrap resampling ⇒ controls are robust
- In the continuous withdrawal limit
  - → Optimal withdrawals are *bang-bang*, i.e. only withdraw at either maximum or minimum rate
- Discrete rebalancing: withdrawal controls are very close to bang-bang
- Intuition: if you are lucky, and make money in stocks, take money off the table and go on a cruise
  - → Otherwise: sit tight

[13] Optimal: 5% EW, with ES $\simeq 0$; Bengen: 4% EW, with ES $\simeq -270K$.