Real Options and Finance

Optimal Stochastic Control: Formulation and Solution Techniques

Peter Forsyth[1] Margaret Insley[2]

[1]Cheriton School of Computer Science
University of Waterloo
[2]Department of Economics
University of Waterloo

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Overview

1. Long Term Asset Allocation for the Patient Investor (Forsyth, 1 hour)
2. On the timing of non-renewable resource extraction with regime switching prices: an optimal stochastic control approach (Insley, 1.5 hours)
3. Long Term Asset Allocation: HJB Formulation and Solution (Forsyth, 1 hour)
4. An Options Pricing Approach to Ramping Rate Restrictions at Hydro Power Plants (Insley, 1.5 hours)
5. Monotone Schemes for Two Factor HJB Equations with Nonzero Correlation (Forsyth, 1.5 hours)
6. Methods for Pricing American Options Under Regime Switching (Forsyth, 1 hour)

Long Term Asset Allocation for the Patient Investor

Peter Forsyth[1]

[1]Cheriton School of Computer Science
University of Waterloo

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The Basic Problem

Suppose you are saving for retirement (i.e. 20 years away)

- What is your portfolio allocation strategy?
  - i.e. how much should you allocate to bonds, and how much to equities (i.e. an index ETF)
- How should this allocation change through time?
  - Typical rule of thumb: fraction of portfolio in stocks $= 110$ *minus your age*.
- Target Date (Lifecycle) funds
  - Automatically adjust the fraction in stocks (risky assets) as time goes on
  - Use a specified *"glide path"* to determine the risky asset proportion as a function of time to go
  - At the end of 2014, over \$700 billion invested in US[1]

[1] Morningstar

Risk-reward tradeoff

This problem (and many others) involve a tradeoff between risk and reward.

Intuitive approach: multi-period mean-variance optimization

- When risk is specified by variance, and reward by expected value
  - → Even non-technical managers can understand the tradeoffs and make informed decisions[2]

In this talk, I will determine the optimal asset allocation strategy

- Objective: minimize risk for specified expected gain
- Use tools of *optimal stochastic control*

[2] I am now a member of the University of Waterloo Pension Committee. I have seen this problem first-hand

Multi-period Mean Variance

Criticism: variance as risk measure penalizes upside as well as downside

I hope to convince you that multi-period mean variance optimization

- Can be modified slightly to be (effectively) a downside risk measure
- Has other good properties: small probability of shortfall

Outcome: optimal strategy for a Target Date Fund

- I will show you that most Target Date Funds being sold in the marketplace use a sub-optimal strategy

"All models are wrong: some are useful" [4]

Let S be the price of an underlying asset (i.e. EuroStoxx index).

- A standard model for the evolution of S through time is Geometric Brownian Motion (GBM)
- Basic assumption: price process is stochastic, i.e. unpredictable[3]

$$
\begin{aligned}
\frac{dS}{S} &= \mu\, dt + \sigma\phi\sqrt{dt} \\
& \mu = \text{ drift rate,} \\
& \sigma = \text{ volatility,} \\
& \phi = \text{ random draw from a} \\
& \qquad \text{standard normal distribution}
\end{aligned}
$$

[3] If this were not true, then I (and many others) would be rich

[4] G. Box, of Box-Jenkins and Box-Muller fame.

Monte Carlo Paths: GBM

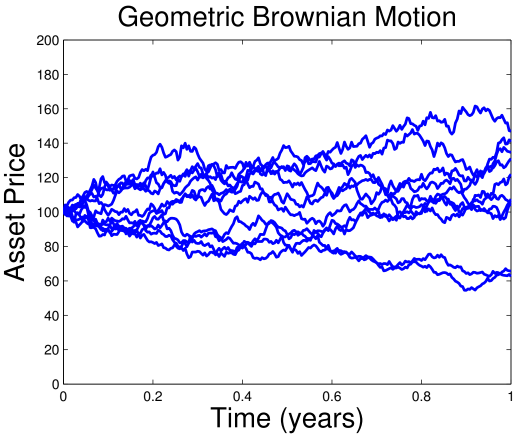


Figure: Ten realizations of possible random paths. Assumption: price processes are stochastic, i.e. unpredictable. $\mu = .10, \sigma = .25$.

What's Wrong with GBM?

- Equity return data suggests market has *jumps* in addition to GBM
  - Sudden discontinuous changes in price
- Most asset allocation strategies ignore the jumps, i.e. market crashes
- But, it seems that we get a financial crisis occurring about once every ten years
- Does it make sense to ignore these events?
- Jumps are also known as:
  - **Black Swans** (see the book with the same title by Nassim Taleb)
  - Fat tail events

EuroStoxx 50 weekly log returns 1986-2014

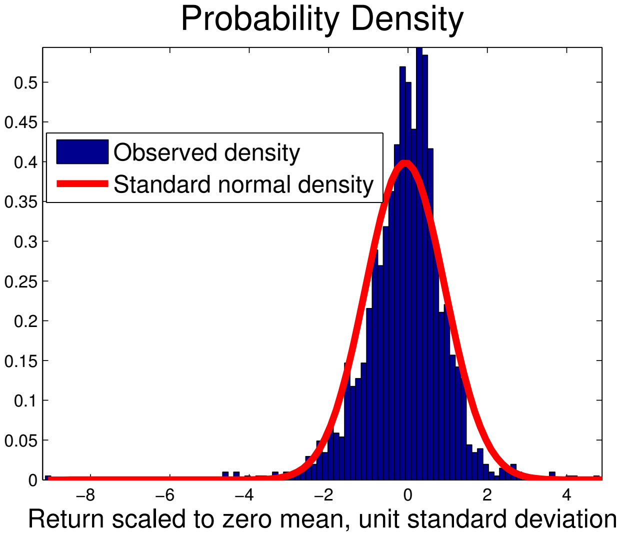


Figure: Higher peaks, heavier tails than normal distribution

A Better Model: Jump Diffusion

$$
\begin{aligned}
\frac{dS}{S} &= \overbrace{(\mu - \lambda\kappa)\; dt + \sigma\phi\sqrt{dt}}^{GBM} + \overbrace{(J-1)dq}^{Jumps} \\
& dq = \begin{cases} 0 & \text{with probability } 1 - \lambda dt \\ 1 & \text{with probability } \lambda dt, \end{cases} \\
& \lambda = \text{ mean arrival rate of Poisson jumps; } S \rightarrow JS \\
& J = \text{ Random jump size } \; ; \;\; \kappa = E[J-1].
\end{aligned}
$$

- GBM plus jumps (jump diffusion)
- When a jump occurs, $S \rightarrow JS$, where J is also random
- This simulates a sudden market crash

Monte Carlo Paths

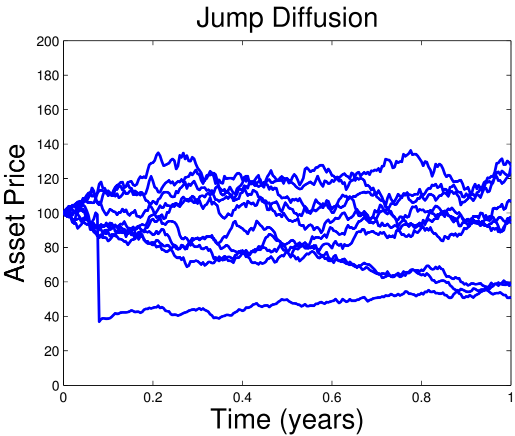


Figure: The arrival rate of the Poisson jump process is .1 per year. Most of the time, the asset follows GBM. In only one of ten stochastic paths, in any given year, can we expect a crash. $\mu = .10, \sigma = .25$.

Example: Target Date (Lifecycle) Fund with two assets

Risk free bond B

$$dB = rB \ dt$$
$$r = \text{ risk-free rate}$$

Amount in risky stock index S

$$dS = \textit{jump diffusion process}$$

Total wealth W

$$W = S + B \tag{1}$$

Objective:

- Optimal allocation of amounts $(S(t), B(t))$, which is multi-period mean-variance optimal
- Optimal strategy is in general a function of (W, t)

Optimal Control

Let:

$$
\begin{aligned}
X &= (S(t), B(t)) = \text{ Process} \\
x &= (S(t) = s, B(t) = b) = (s, b) = \text{ State} \\
(s + b) &= \text{ total wealth}
\end{aligned}
$$

Let $(s, b) = (S(t^-), B(t^-))$ be the state of the portfolio the instant before applying a control

The control $c(s, b) = (d, B^+)$ generates a new state

$$
\begin{aligned}
b &\rightarrow B^+ \\
s &\rightarrow S^+ \\
& S^+ = \underbrace{(s + b)}_{\text{wealth at } t^-} - B^+ - \underbrace{d}_{\text{withdrawal}}
\end{aligned}
$$

Note: we allow cash withdrawals of an amount $d \geq 0$ at a rebalancing time

Semi-self financing policy

Since we allow cash withdrawals

→ The portfolio may not be self-financing

→ The portfolio may generate a free cash flow

Let $W_a = S(t) + B(t)$ be the **allocated wealth**

- W_a is the wealth available for allocation into $(S(t), B(t))$.

The non-allocated wealth $W_n(t)$ consists of cash withdrawals and accumulated interest

Constraints on the strategy

The investor can continue trading only if solvent

$$\underbrace{W_a(s,b) = s + b > 0}_{Solvency\ \ condition}. \quad (2)$$

In the event of bankruptcy, the investor must liquidate

$$S^+ = 0 \quad ; \quad B^+ = W_a(s,b) \quad ; \quad \text{if } \underbrace{W_a(s,b) \leq 0}_{bankruptcy} \ .$$

Leverage is also constrained

$$\frac{S^+}{W^+} \leq q_{\max}$$

$$W^+ = S^+ + B^+ = \text{ Total Wealth}$$

Mean and Variance under control $c(X(t), t)$

Let:

$$\underbrace{E_{t,x}^{c(\cdot)}[W_a(T)]}_{Reward} = \text{Expectation conditional on } (x, t) \text{ under control } c(\cdot)$$

$$\underbrace{\mathrm{Var}_{t,x}^{c(\cdot)}[W_a(T)]}_{Risk} = \text{Variance conditional on } (x, t) \text{ under control } c(\cdot)$$

Important:

- mean and variance of $W_a(T)$ are as observed at time t, initial state x.

Basic problem: find Efficient frontier

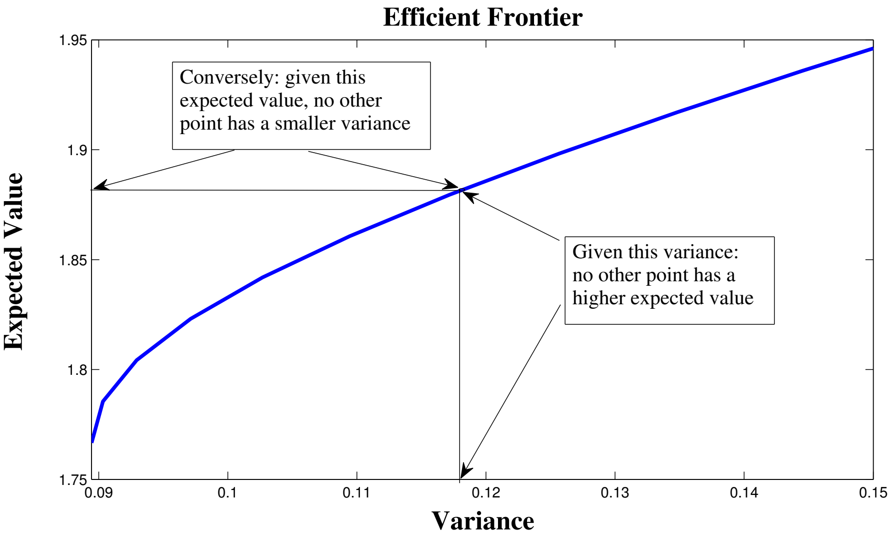
We construct the *efficient frontier* by finding the optimal control $c(\cdot)$ which solves (for fixed λ) [5]

$$\max_{c}\left\{\underbrace{E_{t,x}^{c(\cdot)}[W_a(T)]}_{Reward}-\lambda\underbrace{Var_{t,x}^{c(\cdot)}[W_a(T)]}_{Risk}\right\} \tag{3}$$

• Varying $\lambda \in [0, \infty)$ traces out the efficient frontier

• $\lambda = 0; \rightarrow$ we seek only maximize cash received, we don't care about risk.
• $\lambda = \infty \rightarrow$ we seek only to minimize risk, we don't care about the expected reward.

[5] All investors should pick one of the strategies on the efficient frontier.



Each point on the efficient frontier represents a different strategy $c(\cdot)$.

Mean Variance: Standard Formulation

$$\max_{c(X(u),u\geq t)} \left\{ \underbrace{E_{t,x}^{c(\cdot)}[W_a(T)]}_{Reward\ \ as\ \ seen\ \ at\ \ t} - \lambda \underbrace{Var_{t,x}^{c(\cdot)}[W_a(T)]}_{Risk\ \ as\ \ seen\ \ at\ \ t} \right\} , \qquad \lambda \in [0, \infty) \tag{4}$$

- Let $c_t^*(x,u), u \geq t$ be the optimal policy for (4).

Then $c_{t+\Delta t}^*(x,u), u \geq t + \Delta t$ is the optimal policy which maximizes

$$\max_{c(X(u),u\geq t+\Delta t))} \left\{ \underbrace{E_{t+\Delta t, X(t+\Delta t)}^{c(\cdot)}[W_a(T)]}_{Reward\ \ as\ \ seen\ \ at\ \ t+\Delta t} - \lambda \underbrace{Var_{t+\Delta t, X(t+\Delta t)}^{c(\cdot)}[W_a(T)]}_{Risk\ \ as\ \ seen\ \ at\ \ t+\Delta t} \right\} .$$

Pre-commitment Policy

However, in general

$$\underbrace{c_t^*(X(u),u)}_{optimal\ \ policy\ \ as\ \ seen\ \ at\ \ t} \neq \underbrace{c_{t+\Delta t}^*(X(u),u)}_{optimal\ \ policy\ \ as\ \ seen\ \ at\ \ t+\Delta t} \quad ; \quad \underbrace{u \geq t+\Delta t}_{any\ \ time>t+\Delta t} \ , \tag{5}$$

$\hookrightarrow$ Optimal policy is not *time-consistent*.

The strategy which solves problem (4) has been called the *pre-commitment* policy

Your future self may not agree with your current self!

Ulysses and the Sirens: A pre-commitment strategy

Ulysses had himself tied to the mast of his ship (and put wax in his sailor's ears) so that he could hear the sirens song, but not jump to his death.

Re-formulate MV Problem → Dynamic Programming[6]

For fixed λ, if $c^*(\cdot)$ maximizes

$$\max_{c(X(u),u\geq t)} \left\{ \underbrace{E^c_{t,x}[W_a(T)]}_{Reward} - \lambda \underbrace{Var^c_{t,x}[W_a(T)]}_{Risk} \right\}, \tag{6}$$

→ There exists γ such that $c^*(\cdot)$ minimizes

$$\min_{c(\cdot)} E^{c(\cdot)}_{t,x}\left[\left(W_a(T) - \frac{\gamma}{2}\right)^2\right]. \tag{7}$$

Once $c^*(\cdot)$ is known

- Easy to determine $E^{c^*(\cdot)}_{t,x}[W_a(T)]$, $Var^{c^*(\cdot)}_{t,x}[W_a(T)]$
- Repeat for different γ, traces out efficient frontier

[6] Li and Ng (2000), Zhou and Li (2000)

Equivalence of MV optimization and target problem

MV optimization is equivalent[7] to investing strategy which

- Attempts to hit a target final wealth of $\gamma/2$
- There is a quadratic penalty for not hitting this wealth target
- From (Li and Ng(2000))

$$\underbrace{\frac{\gamma}{2}}_{wealth\ \ target} = \underbrace{\frac{1}{2\lambda}}_{risk\ \ aversion} + \underbrace{E^{c(\cdot)}_{t=0,x_0}[W_a(T)]}_{expected\ \ wealth}$$

Intuition: if you want to achieve $E[W_a(T)]$, you must aim higher

[7] Vigna, Quantitative Finance, 2014

HJB PIDE

Determination of the optimal control $c(\cdot) \Rightarrow$ find the value function

$$V(x,t) = \min_{c(\cdot)} \left\{ E_{x,t}^{c(\cdot)}[(W_a(T) - \gamma/2)^2] \right\} ,$$

Value function

- Given from numerical solution of a Hamilton-Jacobi-Bellman (HJB) partial integro-differential equation (PIDE)
- This also generates the optimal control $c(\cdot)$.

Optimal semi-self-financing strategy

Let

$$\begin{aligned} F(t) &= \frac{\gamma}{2} e^{-r(T-t)} \\ &= \text{discounted target wealth} \end{aligned}$$

Theorem (Dang and Forsyth (2014))

If $W_a(t) > F(t)$, $t \in [0, T]$, an optimal MV strategy is

- *Withdraw cash $W_a(t) - F(t)$ from the portfolio*
- *Invest the remaining amount $F(t)$ in the risk-free asset.*

What should you do with the cash you withdraw (the free cash)?

- Anything you like (e.g. buy an expensive car).
- You are better off withdrawing the cash!

Intuition: Multi-period mean-variance

Optimal target strategy: try to hit $W_a(T) = \gamma/2 = F(T)$.

If $W_a(t) > F(t) = F(T)e^{-r(T-t)}$, then the target can be hit exactly by

- Withdrawing[8] $W_a(t) - F(t)$ from the portfolio
- Investing $F(t)$ in the risk free account

This strategy dominates any other MV strategy

- We never exceed the target
- No *"upside penalization"*

$\rightarrow$ And the investor receives a bonus in terms of a free cash flow

[8] Idea that withdrawing cash may be mean variance optimal was also suggested in (Ehrbar, J. Econ. Theory (1990))

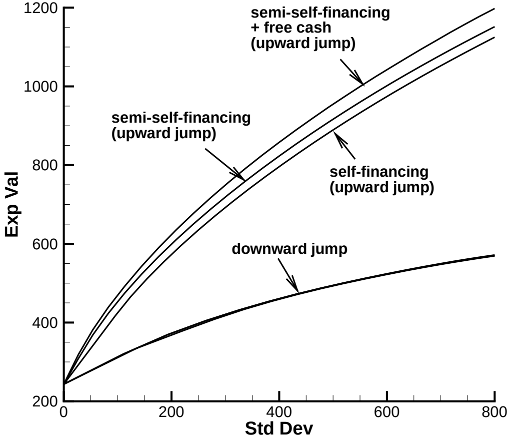
Numerical Examples

initial allocated wealth ($W_a(0)$)	100
r (risk-free interest rate)	0.04450
T (investment horizon)	20 (years)
q_{max} (leverage constraint)	1.5
$t_{i+1} - t_i$ (discrete re-balancing time period)	1.0 (years)

	mean downward jumps	mean upward jumps
μ (drift)	0.07955	0.12168
λ (jump intensity)	0.05851	0.05851
σ (volatility)	0.17650	0.17650
mean log jump size	-0.78832	0.10000

Objective: verify that removing cash when wealth exceeds target is optimal.

Efficient Frontier: sometimes its optimal to spend money[9]



[9] $T = 20$ years, $W_a(0) = 100$. Strictly speaking: upper curve is not an efficient frontier.

Example II

Two assets: risk-free bond, index

- Risky asset follows GBM (no jumps)

According to Benjamin Graham[10], most investors should

- Pick a fraction p of wealth to invest in an index fund (e.g. $p = 1/2$).
- Invest $(1 - p)$ in bonds
- Rebalance to maintain this asset mix

How much better is the optimal asset allocation vs. simple rebalancing rules?

[10] Benjamin Graham, *The Intelligent Investor*

Long term investment asset allocation

Investment horizon (years)	30
Drift rate risky asset μ	.10
Volatility σ	.15
Risk free rate r	.04
Initial investment W_0	100

Benjamin Graham strategy

Constant proportion	Expected Value	Standard Deviation	Quantile
$p = 0.0$	332.01	NA	NA
$p = 0.5$	816.62	350.12	$Prob(W(T) < 800) = 0.56$
$p = 1.0$	2008.55	1972.10	$Prob(W(T) < 2000) = 0.66$

Table: Constant fixed proportion strategy. p = fraction of wealth in risky asset. Continuous rebalancing.

Optimal semi-self-financing asset allocation

Fix expected value to be the same as for constant proportion $p = 0.5$.

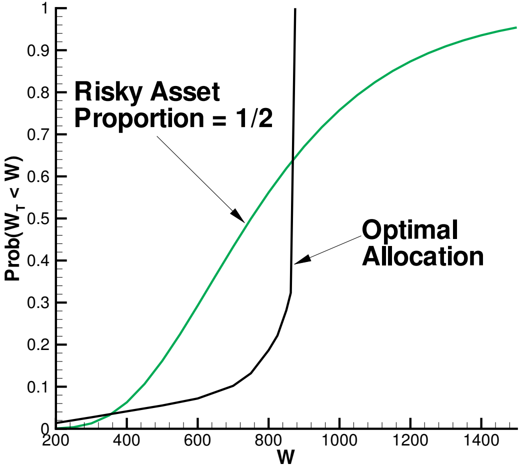
Determine optimal strategy which minimizes the variance for this expected value.

Strategy	Expected Value	Standard Deviation	Quantile
Graham $p = 0.5$	816.62	**350.12**	$Prob(W(T) < 800) =$ **0.56**
Optimal	816.62	**142.85**	$Prob(W(T) < 800) =$ **0.19**

Table: $T = 30$ years. $W(0) = 100$. Semi-self-financing: no trading if insolvent; maximum leverage $= 1.5$, rebalancing once/year.

Standard deviation reduced by 250 %, shortfall probability reduced by $3\times$

Cumulative Distribution Functions



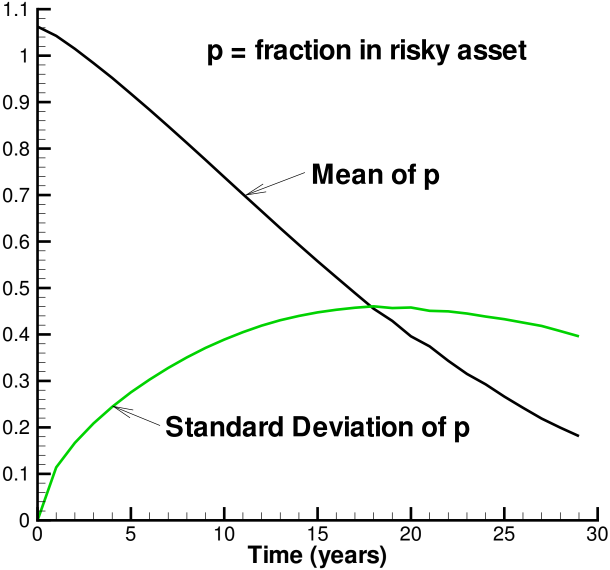
$E[W_T] = 816.62$ for both strategies

Optimal policy: Contrarian: when market goes down $\rightarrow$ increase stock allocation; when market goes up $\rightarrow$ decrease stock allocation

Optimal allocation gives up gains $\gg$ target in order to reduce variance and probability of shortfall.

Investor must pre-commit to target wealth

Mean and standard deviation of the control



Typical Strategy for Target Date Fund: Linear Glide Path

Let p be fraction in risky asset

$$p(t) \quad = \quad p_{start} + \frac{t}{T}(p_{end} - p_{start})$$

Choose parameters so that we get the same expected value as the optimal strategy

$$p_{start} = 1.0 \quad ; \quad p_{end} = 0.0$$

Strategy	Expected Value	Stndrd Dev	$Pr(W(T) < 800)$	Expected Free Cash
$p = 0.5$	817	350	0.56	0.0
Linear[12] Glide Path	817	410	0.58	0.0
Optimal	817	143	0.19	6.3

[12]We can prove that for any deterministic glide path, there exists a superior constant mix strategy

Sensitivity to Market Parameter Estimates

Assume we only know the mean values for the market parameters

- Compute control using mean values
- But: assume parameters (in real world) are uniformly distributed in a range centered on mean
- Compute investment result using Monte Carlo simulations

Interest rate range	Drift rate range	Volatility range
[.02, .06]	[.06, .14]	[.10, .20]

	Strategy: computed using fixed parameters			
Market Parameters	Expected Value	Stndrd Dev	$Pr(W(T) < 800)$	Expected Free Cash
Fixed at Mean	817	143	0.19	6.3
Random	807	145	0.19	30.5

Example III: jump diffusion

Investment horizon (years)	30	Drift rate risky asset μ	0.10
λ (jump intensity)	0.10	Volatility σ	0.10
$E[J]$[13]	0.62	Effective vol (with jumps)	0.16 [14]
Risk free rate r	0.04	Initial Investment W_0	100

Strategy	Expected Value	Standard Deviation	$Pr(W(T)) < 800$
Graham $p = 0.5$[15]	826	**399**	**0.55**
Optimal	826	**213**	**0.23**

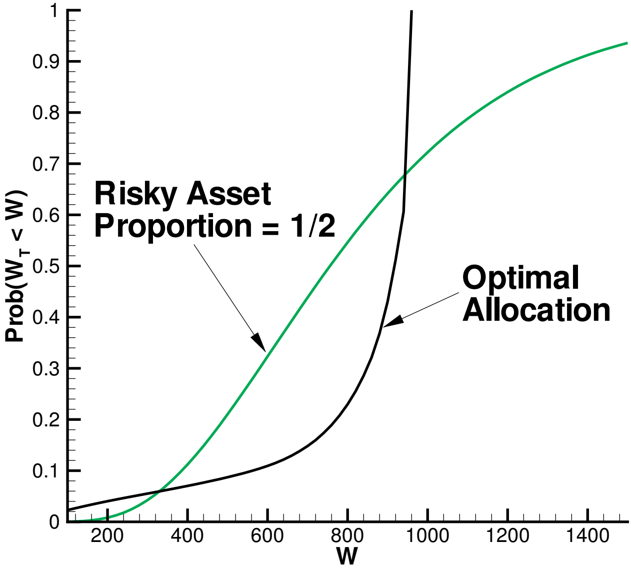
Table: $T = 30$ years. $W(0) = 100$. Optimal: semi-self-financing; no trading if insolvent; maximum leverage $= 1.5$, rebalancing once/year.

[13] When a jump occurs $S \rightarrow JS$.

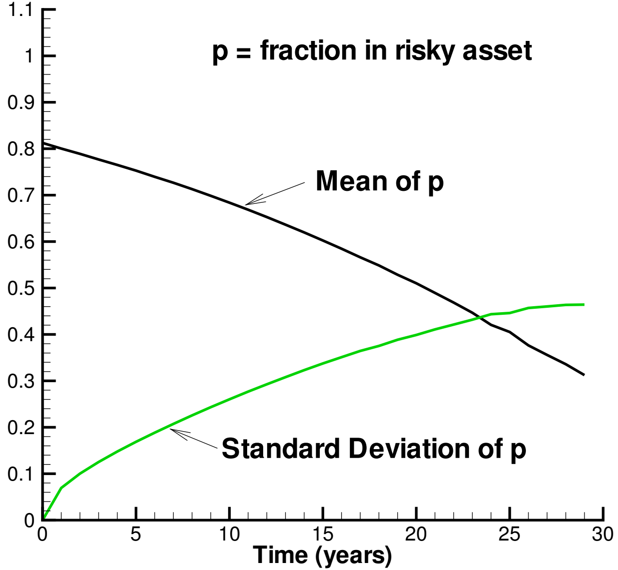
[14] stndrd dev $[S(T)/S(0)]/T$

[15] Yearly rebalancing

Cumulative Distribution Function: Jump diffusion



Mean and standard deviation of the control



Back Testing

Back test problem: only a few non-overlapping 30 year paths
$\hookrightarrow$ Backtesting is dubious in this case

Assume GBM

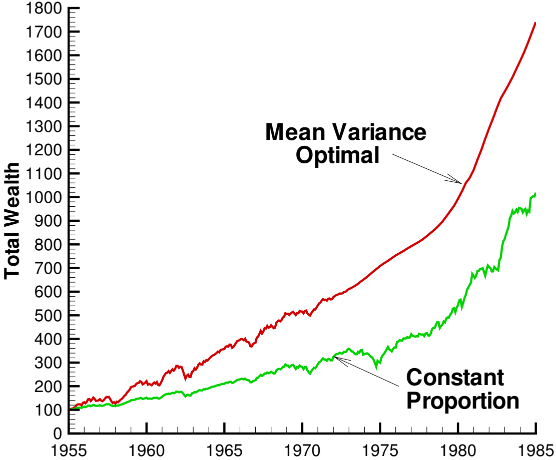
- Estimate μ, σ, r [16] from real data Jan 1, 1934 - Dec 31, 1954
- With these parameters, estimate $E[W(1985)]$ for an equally weighted portfolio ($p = 1/2$) for Jan 1, 1955 - Dec 31, 1984.
- Determine the MV optimal strategy which has same expected value
- Now, run both strategies on observed $1955 - 1984$ data

Second test: repeat: estimate parameters from Jan 1, 1934 - Dec 31, 1984 data

- Compare strategies using real returns from Jan 1, 1985 - Dec 31, 2014

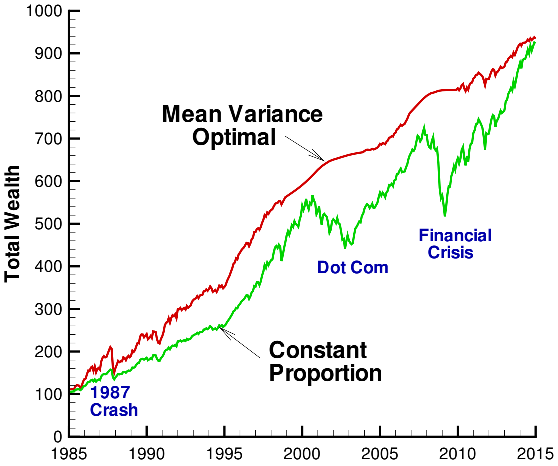
[16] 3 month US treasuries. CRSP value weighted total return.

Back Test: Jan 1, 1955 - Dec 31, 1984[17]



[17] $W(Jan\ 1,\ 1954) = 100$. GBM parameters estimated from $1934 - 1954$ data. Estimated $E[W(Dec\ 31\ 1984) \mid t = Jan\ 1,\ 1955] = 625$ same for both strategies. Estimated parameters: $\mu = .12, \sigma = .18, r = .0063$. MV optimal target 641.4. Historical returns used for $1955 - 1984$. Maximum leverage 1.5.

Back Test: Jan 1, 1985 - Dec 31, 2014[18]



[18] $W(1985) = 100$. GBM parameters estimated from $1934 - 1984$ data. Estimated $E[W(Dec\ 31\ 2014) \mid t = Jan\ 1,\ 1985] = 967$ same for both strategies. Estimated parameters: $\mu = .11, \sigma = .16, r = .037$. MV optimal target 1010.5. Historical returns used for $1985 - 2014$. Maximum leverage 1.5

Bootstrap Resampling Backtest: 1926-2014

- CRSP data January 1 1926 - December 31 2014
- Three month US T-bills January 1 1926 - December 31 2014[19]

Estimate GBM parameters:

CRSP		T-Bill (3-month)
Drift (μ)	Volatility (σ)	Mean rate (r)
.112	.189	.0352

Strategy	Expected Value	Standard Deviation	$Pr(W(T)) < 800$
Graham $p = 0.5$[20]	915	**506**	**0.50**
Optimal	915	**200**	**.13**

Table: $T = 30$ years. $W(0) = 100$. Optimal: semi-self-financing; no trading if insolvent; maximum leverage $= 1.5$, rebalancing once/year.

[19] From Federal Reserve website. Starts in 1934. 1926-1934 data from NBER

[20] Continuous rebalancing

Bootstrap Resampling: 1926-2014

Now, use real historical data, quarterly returns

- For each MC simulation, draw 30 years of returns (with replacement) from historical returns (blocksize 10 years)
- 10,000 simulations, each block starts at random quarter

Strategy	Expected Value	Standard Deviation	$Pr(W(T)) < 800$	Expected Free Cash
Graham $p = 0.5$[21]	953	**514**	**0.47**	0.0
Optimal	922	**164**	**0.09**	129

Table: $T = 30$ years. $W(0) = 100$. Optimal: semi-self-financing; no trading if insolvent; maximum leverage $= 1.5$, rebalancing once/year.

[21] yearly rebalancing

Conclusions

- Optimal allocation strategy dominates simple constant proportion strategy by a large margin
  - → Probability of shortfall $\simeq$ 3 times smaller!
- But
  - → Investors must pre-commit to a wealth target
  - → Investors must commit to a long term strategy ($>$ 20 years)
  - → Investors buy-in when market crashes, de-risk when near target
- Standard *"glide path"* strategies of Target Date funds
  - → Inferior to constant mix strategy[22]
  - → Constant mix strategy inferior to optimal control strategy
- Optimal mean-variance policy
  - Seems to be insensitive to parameter estimates
  - Good performance even if jump processes modelled
  - Historical backtests: works as expected

[22] See also *"The false promise of Target Date funds"*, Esch and Michaud (2014); *"Life-cycle funds: much ado about nothing?"*, Graf (2013)

Long Term Asset Allocation: HJB Equation Formulation and Solution

Peter Forsyth[1]

[1]Cheriton School of Computer Science
University of Waterloo

September, 2015
A Coruna

Outline

1. Dynamic mean variance
   - Embedding result
   - Equivalence to quadratic target
   - Removal of spurious points
2. HJB PDE
   - Intuitive discretization
   - Semi-Lagrangian timestepping and explicit control
   - Unconditionally stable, monotone and consistent

Dynamic Mean Variance: Abstract Formulation

Define:

$$
\begin{aligned}
X &= \text{Process} \\
&\frac{dX}{dt} = \text{SDE} \\
x &= (X(t) = x) = \text{State} \\
W(X(t)) &= \text{total wealth}
\end{aligned}
$$

Control $c(X(t), t)$ is applied to $X(t)$
Define admissible set $\mathcal{Z}$, i.e.

$$c(x, t) \in \mathcal{Z}(x, t)$$

Mean and Variance under control $c(X(t), t)$

Let:

$$\underbrace{E_{t,x}^{c(\cdot)}[W(T)]}_{Reward} = \text{Expectation conditional on } (x, t) \text{ under control } c(\cdot)$$

$$\underbrace{\mathrm{Var}_{t,x}^{c(\cdot)}[W(T)]}_{Risk} = \text{Variance conditional on } (x, t) \text{ under control } c(\cdot)$$

Important:

- mean and variance of $W(T)$ are as observed at time t, initial state x.

Basic Problem: Find Pareto Optimal Strategy

We desire to find the investment strategy $c^*(\cdot)$ such that, there exists no other other strategy $c(\cdot)$ such that

$$\underbrace{E_{t,x}^{c(\cdot)}[W_T]}_{Reward\ \ under\ \ strategy\ \ c(\cdot)} \geq \underbrace{E_{t,x}^{c(*\cdot)}[W_T]}_{Reward\ \ under\ \ strategy\ \ c^*(\cdot)}$$

$$\underbrace{\text{Var}_{t,x}^{c(\cdot)}[W_T]}_{Risk\ \ under\ \ strategy\ \ c(\cdot)} \leq \underbrace{\text{Var}_{t,x}^{c^*(\cdot)}[W_T]}_{Risk\ \ under\ \ strategy\ \ c^*(\cdot)}$$

and at least one of the inequalities is strict.

In other words

- There exists no other strategy which simultaneously has higher expected value and smaller variance
- This is a Pareto optimal strategy
- There is a family of such strategies

Pareto optimal points

Let

$$\mathcal{E} = E_{t,x}^{c(\cdot)}[W_T] \quad ; \quad \mathcal{V} = \mathrm{Var}_{t,x}^{c(\cdot)}[W_T]$$

The *achievable set* $\mathcal{V}$ is

$$\mathcal{Y} = \{(\mathcal{V}, \mathcal{E}) \,:\, c(\cdot) \in \mathcal{Z}\},$$

Given $\lambda > 0$, define[1]

$$\mathcal{Y}_{P(\lambda)} = \{(\mathcal{V}, \mathcal{E}) \in \bar{\mathcal{Y}} : \lambda\mathcal{V} - \mathcal{E} = \inf_{(\mathcal{V}_*, \mathcal{E}_*) \in \mathcal{Y}} (\lambda\mathcal{V}_* - \mathcal{E}_*\}$$

The efficient frontier $\mathcal{Y}_P$ is

$$\mathcal{Y}_P = \bigcup_{\lambda > 0} \mathcal{Y}_{P(\lambda)}$$

The efficient frontier is a collection of Pareto points

[1] $\bar{\mathcal{Y}}$ is the closure of $\mathcal{Y}$.

Efficient Frontier[2]

Consider a straight line in the $(\mathcal{V}, \mathcal{E})$ plane (for fixed λ)

$$\lambda \mathcal{V} - \mathcal{E} = C_1 \tag{1}$$

From

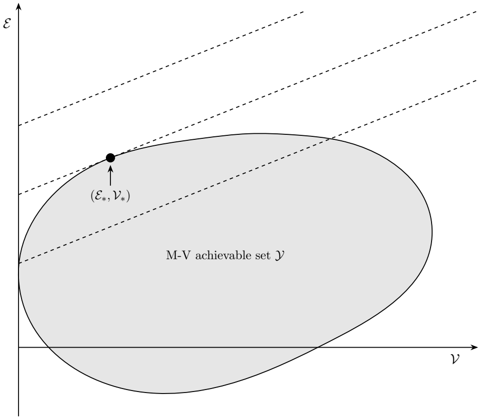
$$\mathcal{Y}_{P(\lambda)} = \{(\mathcal{V}, \mathcal{E}) \in \bar{\mathcal{Y}} : \lambda \mathcal{V} - \mathcal{E} = \inf_{(\mathcal{V}_*, \mathcal{E}_*) \in \mathcal{Y}} (\lambda \mathcal{V}_* - \mathcal{E}_*\}$$

we can find points on the efficient frontier by choosing C_1 as small as possible so that

- Intersection of $\mathcal{Y}$ and straight line (1) has at least one point

[2]We may not get all the Pareto points here if $\mathcal{Y}$ is not convex

Intuition



Move dotted lines line $\lambda\mathcal{V} - \mathcal{E} = C_1$ to the left as much as possible (decrease C_1)

Line will touch $\mathcal{Y}$ at Pareto point

Problem

Pareto point

$$\lambda \mathcal{V} - \mathcal{E} = \inf_{(\mathcal{V}_*, \mathcal{E}_*) \in \mathcal{Y}} (\lambda \mathcal{V}_* - \mathcal{E}_* \} \quad (2)$$

Problem arises from variance

$$\mathcal{V} = E^c[W(T)^2] - (E^c[W(T)])^2$$

$(E^c[W(T)])^2$ cannot be handled with standard dynamic programming

- Cannot directly formulate (2) as an HJB equation

Consider an objective function of form (for fixed γ)

$$\inf_{(\mathcal{V}, \mathcal{E}) \in \mathcal{Y}} \mathcal{V} + \mathcal{E}^2 - \gamma \mathcal{E} \quad (3)$$

Note that

$$\mathcal{V} + \mathcal{E}^2 = E^c[W(T)^2]$$

Minimizing (3) can be done using dynamic programming

Embedded Objective Function Intuition

Examine points $(\mathcal{V}, \mathcal{E}) \in \mathcal{Y}$ such that (for fixed γ)

$$\mathcal{V} + \mathcal{E}^2 - \gamma\mathcal{E} = \inf_{(\mathcal{V}_*, \mathcal{E}_*) \in \mathcal{Y}} \mathcal{V}_* + \mathcal{E}_*^2 - \gamma\mathcal{E}_*$$

Consider the parabola

$$\mathcal{V} + \mathcal{E}^2 - \gamma\mathcal{E} = C_2 \tag{4}$$

Choose C_2 as small as possible, so that

- Intersection of parabola and $\mathcal{Y}$ has at least one point

Rewriting equation (4)

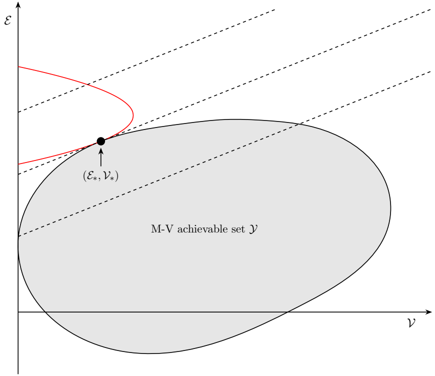
$$\begin{aligned} \mathcal{V} &= -\left(\mathcal{E}^2 - \gamma\mathcal{E}\right) + C_2 \\ &= -\left(\mathcal{E} - \gamma/2\right)^2 + \gamma^2/4 + C_2 \\ &= -\left(\mathcal{E} - \gamma/2\right)^2 + C_3. \end{aligned} \tag{5}$$

Parabola faces left, symmetric about line $\mathcal{E} = \gamma/2$

Embedded Pareto Points

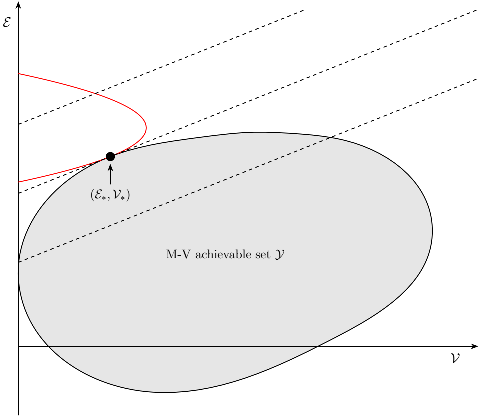
Suppose $(\mathcal{V}_*, \mathcal{E}_*)$ is a Pareto point $\rightarrow \exists \lambda > 0$, C_1, s.t.

$$\lambda \mathcal{V}_* - \mathcal{E}_* = C_1$$



Move parabola to left as much as possible, and intersect line $\lambda \mathcal{V}_* - \mathcal{E}_* = C_1$ at a single point.

Tangency Condition



Parabola $\mathcal{V} = -(\mathcal{E} - \gamma/2)^2 + C_3$ tangent to line $\lambda\mathcal{V} - \mathcal{E} = C_1$ at $(\mathcal{V}_*, \mathcal{E}_*)$

$$\left(\frac{\partial \mathcal{E}}{\partial \mathcal{V}}\right)_{parabola} = \lambda \quad ; \quad \lambda = \text{ slope of dotted lines}$$

$$\rightarrow \qquad \gamma/2 = 1/(2\lambda) + \mathcal{E}_*$$

Embedding Result

Theorem 1 ((Li and Ng (2000); Zhou and Li (2000))

If

$$\lambda \mathcal{V}_0 - \mathcal{E}_0 = \inf_{(\mathcal{V},\mathcal{E})\in\mathcal{Y}} (\lambda \mathcal{V} - \mathcal{E}), \quad (6)$$

then

$$\mathcal{V}_0 + \mathcal{E}_0^2 - \gamma \mathcal{E}_0 = \inf_{(\mathcal{V},\mathcal{E})\in\mathcal{Y}} (\mathcal{V} + \mathcal{E}^2 - \gamma \mathcal{E}), \quad (7)$$

$$\gamma = \frac{1}{\lambda} + 2\mathcal{E}_0$$

Implication

- We can determine all the Pareto points from (6) by solving problem (7)

Value function

Note:

$$\mathcal{V} + \mathcal{E}^2 - \gamma\mathcal{E} = E^c_{t,x}[W^2(T)] - (E^c_{t,x}[W(T)])^2 + (E^c_{t,x}[W(T)])^2 - \gamma E^c_{t,x}[W(T)]$$
$$= E^c_{t,x}[(W(T) - \frac{\gamma}{2})^2] + \frac{\gamma^2}{4},$$

Define value function (ignore $\gamma^2/4$ term when minimizing)

$$\mathcal{U}(x,t) = \inf_{c(\cdot)\in\mathcal{Z}} E^{c(\cdot)}_{t,x}[(W(T) - \gamma/2)^2] \quad (8)$$

Implication: Given point $(\mathcal{V}^*, \mathcal{E}^*)$ on the efficient frontier, generated by control $c^*(\cdot)$, then $\exists\gamma$ s.t.

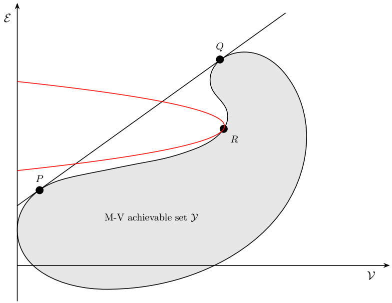
$\rightarrow$ $c^*(\cdot)$ is the optimal control for (8)

Spurious points

But, converse not necessarily true: i.e. there may be some $\gamma \in (-\infty, +\infty)$ s.t. $c^*(\cdot)$ which solves

$$\mathcal{U}(x,t) = \inf_{c(\cdot)\in\mathcal{Z}} E_{t,x}^{c(\cdot)}[(W(T) - \gamma/2)^2] \tag{9}$$

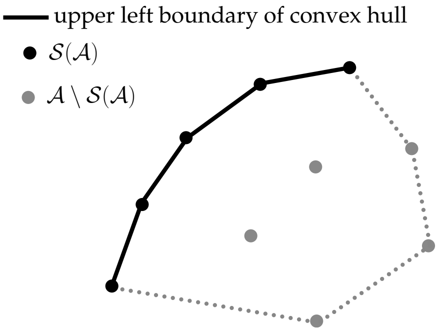
which does not correspond to a point on the efficient frontier



Remove Spurious Points[3]

Suppose we solve value function $\forall \gamma \in (-\infty, +\infty)$

- Generate set of candidate points on the efficient frontier $\mathcal{A}$
- Determine upper left convex hull $\mathcal{S}(\mathcal{A})$
- Valid points on efficient frontier: $\mathcal{A} \cap \mathcal{S}(\mathcal{A})$



[3] Tse, Forsyth, Li (2014, SIAM Cont. Opt.); Dang,Forsyth, Li (2015, Num. Math.)

Review: asset allocation, bond and stock

Risk free bond B

$$
\begin{aligned}
dB &= rB\ dt \\
& \quad r = \text{ risk-free rate}
\end{aligned}
$$

Amount in risky stock index S

$$
dS = (\mu - \lambda\kappa)S\ dt + \sigma S\ dZ + (J-1)S\ dq
$$

$$
\mu = \mathbb{P} \text{ measure drift} \quad ; \quad \sigma = \text{ volatility}
$$

$$
dZ = \text{ increment of a Wiener process}
$$

$$
dq = \begin{cases} 0 & \text{with probability } 1 - \rho dt \\ 1 & \text{with probability } \rho dt, \end{cases}
$$

$$
\log J \sim \mathcal{N}(\mu_J, \sigma_J^2). \quad ; \quad \kappa = E[J-1]
$$

Optimal Control

Define:

$$
\begin{aligned}
X &= (S(t), B(t)) = \text{Process} \\
x &= (S(t) = s, B(t) = b) = (s, b) = \text{State} \\
(s+b) &= \text{total wealth}
\end{aligned}
$$

Let $(s, b) = (S(t^-), B(t^-))$ be the state of the portfolio the instant before applying a control

The control $c(s, b) = (d, B^+)$ generates a new state

$$
\begin{aligned}
b &\rightarrow B^+ \\
s &\rightarrow S^+ \\
& S^+ = \underbrace{(s+b)}_{wealth\ \ at\ \ t^-} - B^+ - \underbrace{d}_{withdrawal}
\end{aligned}
$$

Note: we allow cash withdrawals of an amount $d \geq 0$ at a rebalancing time

Constraints on the strategy

The investor can continue trading only if solvent

$$\underbrace{W(s,b) = s + b > 0}_{Solvency\ \ condition}. \tag{10}$$

In the event of bankruptcy, the investor must liquidate

$$S^+ = 0 \quad ; \quad B^+ = W(s,b) \quad ; \quad \text{if } \underbrace{W(s,b) \leq 0}_{bankruptcy} \ .$$

Leverage is also constrained

$$\frac{S^+}{W^+} \quad \leq \quad q_{\max}$$

$$W^+ = S^+ + B^+ = \text{ Total Wealth}$$

HJB PIDE

Determination of the optimal control $c(\cdot)$ is equivalent to determining the value function

$$V(x,t) = \inf_{c \in \mathcal{Z}} \left\{ E_c^{x,t}[(W(T) - \gamma/2)^2] \right\} ,$$

Define:

$$\begin{aligned} \mathcal{L}V &\equiv \frac{\sigma^2 s^2}{2} V_{ss} + (\mu - \rho\kappa) s V_s + r b V_b - \lambda V \ , \\ \mathcal{J}V &\equiv \int_0^\infty p(\xi) V(\xi s, b, \tau) \ d\xi \\ & \quad\ p(\xi) = \text{ jump size density } ; \ \rho = \text{ jump intensity} \end{aligned}$$

and the intervention operator $\mathcal{M}(c) \ V(s,b,t)$

$$\mathcal{M}(c) \ V(s,b,t) \ = \ V(S^+(s,b,c), B^+(s,b,c), t)$$

HJB PIDE II

The value function (and the control $c(\cdot)$) is given by solving the impulse control HJB equation

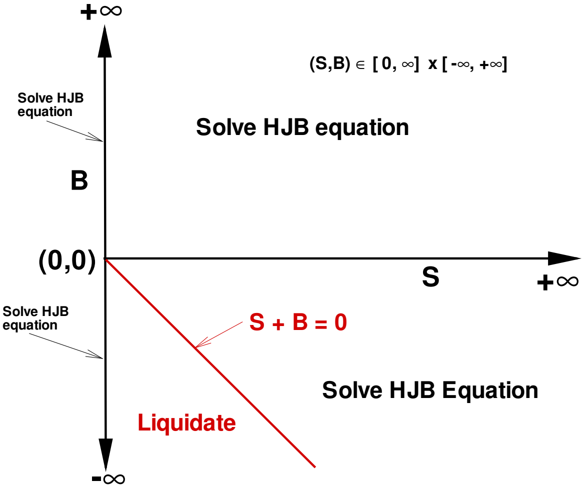
$$\max\left[V_t + \mathcal{L}V + \mathcal{J}V, V - \inf_{c \in \mathcal{Z}} (\mathcal{M}(c)\ V) \right] = 0 \qquad \text{if } (s + b > 0) \qquad (11)$$

Along with liquidation constraint if insolvent

$$V(s, b, t) = V(0, s + b, t) \qquad \text{if } (s + b) \leq 0 \text{ and } s \neq 0 \qquad (12)$$

We can easily generalize the above equation to handle the discrete rebalancing case.

Computational Domain[4]



[4] If $\mu > r$ it is never optimal to short S

Discretization

Define nodes in the s and b direction

$$\{s_1, s_2, \ldots, s_{i_{\max}}\} \quad ; \quad \{b_1, \ldots, b_{j_{\max}}\}$$

Assume constant timesteps

$$\Delta\tau = \tau^{n+1} - \tau^n$$

Assume:

$$\Delta s_{\max} = \max_i (s_{i+1} - s_i) \ ; \ \Delta b_{\max} = \max_j (b_{j+1} - b_j) \ ; \ \Delta\tau_{\max} = \max_n (\tau^{n+1} - \tau^n)$$

Assume control B^+ is discretized

$$\Delta B^+_{\max} = \max_j (B^+_{j+1} - B^+_j) = \Delta b_{\max}$$

Discretization parameter h such that

$$\Delta s_{\max} = C_1 h \ ; \ \Delta b_{\max} = C_2 h \ ; \ \Delta\tau_{\max} = C_3 h$$

Notation

Recall that we want to solve

$$\max\left[V_t + \mathcal{L}V + \mathcal{J}V, V - \inf_{c \in \mathcal{Z}} (\mathcal{M}(c)\ V)\right] = 0 \tag{13}$$

where

$$\mathcal{L}V = \mathcal{P}V + rbV_b$$

and

$$\mathcal{P}V = \frac{\sigma^2 s^2}{2} V_{ss} + (\mu - \rho\kappa) s V_s - \lambda V\ ,$$

Intuitive Derivation of Discretization

Consider a set of discrete rebalancing times $\{t_1, t_2, \ldots\}$
Define

$$t_m^+ = t_m + \epsilon \quad ; \quad t_m^- = t_m - \epsilon \quad ; \quad \epsilon \to 0^+ \tag{14}$$

Portfolio has $s = S(t)$ and $b = B(t)$ stock and bond at $t = t_m^+$

Over $[t_m^+, t_{m+1}^+]$

1. $[t_m^+, t_{m+1}^-]$: stock evolves randomly, bond unchanged (no interest paid)
2. $[t_{m+1}^-, t_{m+1}]$: interest paid $B \to Be^{r\Delta t}$
3. $[t_{m+1}, t_{m+1}^+]$: rebalance portfolio

Details: steps 1 and 2

Step $[t_m^+, t_{m+1}^-]$ (bond amount constant)

- The value function $\bar{V}(s, b, t)$ evolves according to the PIDE

$$V_t + \overbrace{\mathcal{P}V}^{No\ \ rbV_b\ \ term} + \overbrace{\mathcal{J}V}^{Jump\ \ term} = 0,$$

Step $[t_{m+1}^-, t_{m+1}]$ (Stock amount constant)

- Pay interest earned in $[t_m^+, t_{m+1}^-]$

$$V(s, b, t_{m+1}^-) \;=\; V(s, be^{r\Delta t}, t_{m+1}) \quad ; \quad \text{by no-arbitrage}$$

$$\Delta t = t_{m+1} - t_m$$

Detail: Step 3

Step $[t_{m+1}, t_{m+1}^+]$

- Optimal rebalance

$$
\begin{aligned}
V(s, b, t_{m+1}) \;=\; \min \Bigg[& \overbrace{V(s, b, t_{m+1}^+)}^{do\ \ nothing}, \\
& \overbrace{\min_c V(S^+(s, b, c), B^+(s, b, c), t_{m+1}^+)}^{rebalance} \Bigg]
\end{aligned}
$$

Combine Steps 2 and 3 (pay interest and rebalance)

$$
\begin{aligned}
V(s, b, t_{m+1}^-) = \min \Bigg[& V(s, be^{r\Delta t}, t_{m+1}^+), \\
& \min_c V(S^+(s, be^{r\Delta t}, c), B^+(s, be^{r\Delta t}, c), t_{m+1}^+) \Bigg]
\end{aligned}
$$

Backwards time: discrete solution

Let $V_h(s_i, b_j, \tau^n)$ be the discrete approximate solution at (s_i, b_j, τ^n)

Now, we write these steps down in backwards time $\tau = T - t$

$$\tau_-^n = T - t_{m+1}^+ \; ; \; \tau_+^n = T - t_{m+1}^- \; ; \; \tau_-^{n+1} = T - t_m^+ \; ; \; \tau_+^{n+1} = T - t_m^- \; .$$

We proceed from $\tau_-^n \to \tau_-^{n+1}$ (note reverse time order)

Step $\tau_-^n \to \tau_+^n$: (rebalance and pay interest)

$$\begin{aligned} V_h(s_i, b_j, \tau_+^n) &= \min \Big[V_h(s_i, b_j e^{r\Delta\tau}, \tau_-^n), \\ &\qquad \min_{c \in \mathcal{Z}_h} V_h(S^+(s_i, b_j e^{r\Delta\tau}, c), B^+(s_i, b_j e^{r\Delta\tau}, c), \tau_-^n) \Big] \; . \end{aligned}$$

Solve PIDE

Step $\tau_+^n \rightarrow \tau_-^{n+1}$: with $V_h(s_i, b_j, \tau_+^n)$ as the initial condition.

Fully implicit timestepping $P_h, \mathcal{J}_h$ discretized operators)

$$V_h(s_i, b_j, \tau_-^{n+1}) - \Delta\tau P_h V_h(s_i, b_j, \tau_-^{n+1}) - \Delta\tau \mathcal{J}_h V_h(s_i, b_j, \tau_-^{n+1})$$
$$= V_h(s_i, b_j, \tau_+^n)$$

Final Discretization

Let $V_{i,j}^n \equiv V_h(s_i, b_j, \tau^n)$

$$\frac{V_{i,j}^{n+1}}{\Delta\tau} - P_h V_{i,j}^{n+1} - \mathcal{J}_h V_{i,j}^{n+1} = \frac{\widetilde{V}_{i,j}^n}{\Delta\tau}$$
$$\widetilde{V}_{i,j}^n = \left(\min\left[V_h(s_i, b_j e^{r\Delta\tau}, \tau^n),\right.\right.$$
$$\left.\left.\min_{c\in\mathcal{Z}_h} V_h(S^+(s_i, b_j e^{r\Delta\tau}, c), B^+(s_i, b_j e^{r\Delta\tau}, c), \tau^n)\right]\right) \tag{15}$$
.

This is actually

- Semi-Lagrangian timestepping applied to PIDE
- Impulse control is handled explicitly

Discretization Properties

1. Central, forward, backward differencing used to discretize $\mathcal{P}$, positive coefficient condition enforced

$$\begin{aligned} \mathcal{P}_h V_{i,j}^n &= \alpha_{i,j} V_{i-1,j} + \beta_{i,j} V_{i+1,j}^n - (\alpha_{i,j} + \beta_{i,j} + \lambda) V_{i,j}^n \\ & \alpha_{i,j} \geq 0 \; ; \; \beta_{i,j} \geq 0 \; . \end{aligned} \tag{16}$$

2. FFT and interpolation used to discretize jump term, such that

$$\mathcal{J}_h V_{i,j}^n = \sum_k q_k^{i,j} V_{k,j}^n \tag{17}$$

$$0 \leq q_k^{i,j} \leq 1 \; ; \; \sum_k q_k^{i,j} \leq 1 \; . \tag{18}$$

3. Linear interpolation used to approximate V_h at off grid points (needed for optimal control)

Convergence

Lemma 2

If properties (1-3) hold, then discretization (15) is monotone, consistent (in the viscosity sense) and unconditionally ℓ_∞ stable.

Theorem 3 (Convergence)

Provided that the original impulse control problem satisfies the strong comparison property, then discretization (15) converges to the viscosity solution of (13).

Proof.

This follows from Lemma 2 and results in (Barles, Souganidis (1993)). □

Implementation

Note that this discretization method consists of two steps:

1. Determine optimal control at each node (linear search of discretized control $\mathcal{Z}_h$ used)

$$\widetilde{V}^n_{i,j} = \left(\min \left[V_h(s_i, b_j e^{r\Delta\tau}, \tau^n), \right.\right.$$
$$\left.\left. \min_{c \in \mathcal{Z}_h} V_h(S^+(s_i, b_j e^{r\Delta\tau}, c), B^+(s_i, b_j e^{r\Delta\tau}, c), \tau^n) \right] \right)$$

2. Time advance step: solve linear PIDE (use method in d'Halluin et al, 2005)

$$\frac{V^{n+1}_{i,j}}{\Delta\tau} - P_h V^{n+1}_{i,j} - \mathcal{J}_h V^{n+1}_{i,j} = \frac{\widetilde{V}^n_{i,j}}{\Delta\tau}$$

This is very simple to implement

- Easy to alter existing Semi-Lagrangian software to add impulse control

Validation Test

Refinement	Timesteps	S Nodes	B Nodes (also $\mathcal{Z}_h$ nodes)
0	50	58	115
1	100	115	229
2	200	229	457
3	400	457	913

No-jump case, $q_{\max} = \infty$, exact solution known

Investment Horizon	10
Lending rate r_l	.04
Borrowing rate r_b	.04
Trading ceases if insolvent	yes
Volatility σ	0.15
Drift μ	0.15
Initial Wealth	100
Maximum Leverage Ratio $q_{\max}$	∞
Jumps	No

Validation Test

Refine	Mean	Change	Ratio	Standard Deviation	Change	Ratio
0	377.714			62.069		
1	381.379	3.665		56.292	-5.776	
2	383.104	1.724	2.1	53.503	-2.789	2.1
3	383.966	0.862	2.0	52.108	-1.394	2.0
Exact	384.826	N/A	N/A	50.686	N/A	N/A

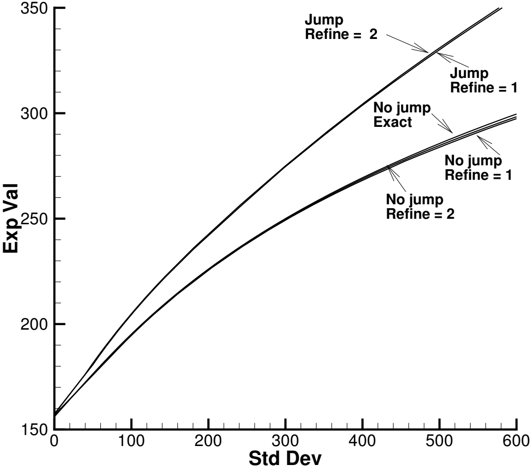
Table: A single point on the efficient frontier $\gamma = 800$. No jumps, $q_{\max} = \infty$.

Numerical Results: Jumps vs. No-jumps

	Jumps	No Jumps
Investment Horizon T	10	10
Lending rate r_ℓ	.0445	.0445
Borrowing rate r_b	.0445	.0445
Trading ceases if insolvent	yes	yes
Volatility σ	0.1765	.281751
Drift μ	.0795487	.0795487
Initial Wealth	100	100
Maximum Leverage Ratio $q_{\max}$	∞	∞
Jump Intensity λ	.0585046	N/A
$\log J \sim \mathcal{N}(\mu_J, \sigma_J^2)$	$\mu_J = -.79, \sigma_J = .45$	NA

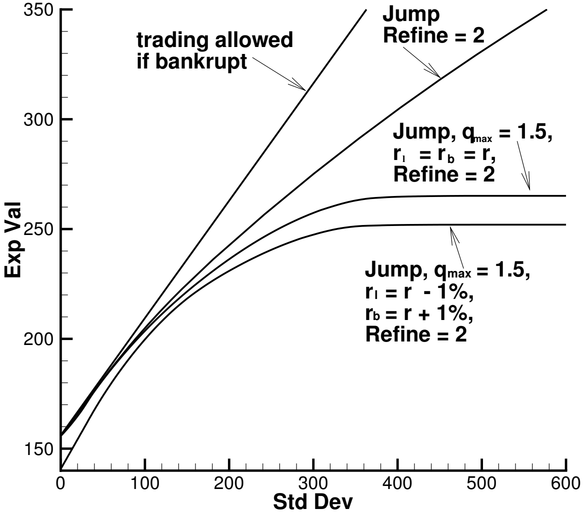
For the no-jump case, the volatility is the *effective volatility* computed based on jump parameters, as in (Navas, 2000).

Jumps vs. No Jumps[5]



[5]Maximum leverage $q_{\max} = \infty$.

Leverage constraint, Jumps[6]



[6] $q_{\max} = \infty$ unless noted. No trading if bankrupt: exact solution. Unequal lending/borrowing rates r_ℓ, r_b if shown.

Conclusions

- Can reformulate continuous time mean-variance using an embedding method
  - Embedded problem can be solved using HJB equation
  - Spurious points on emebedded efficient frontier easily removed.
- HJB Discretization
  - Semi-Lagrangian timestepping, explicit impulse control
  - Unconditionally monotone, consistent, ℓ_∞ stable
  - Guaranteed to converge to viscosity solution
  - Easy to implement
  - Transaction costs, unequal borrowing/lending rates, discrete rebalancing straightforward to model
  - Regular additions/withdrawals of capital can be easily handled.
- Efficient frontiers
  - Exact solution known for simple cases (trading continues if bankrupt, no leverage constraint)
  - Realistic constraints: large influence on efficient frontiers

An Options Pricing Approach to Ramping Rate Restrictions at Hydro Power Plants

Presentation at the University of A Coruña

Shilei Niu[1] Margaret Insley[2]

[1]International Business School Suzhou
Xi'an Jiaotong-Liverpool University

[2]Department of Economics
University of Waterloo

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Table of contents

Outline

Costs and benefits of hydro

- Hydro is considered an environmentally friendly source of power.
- Water release rates can be easily adjusted to meet peak demands.
- Environmental damage to the aquatic ecosystem by frequent changes in water flows.
- Negative environmental impacts from changing water release rates is case specific.

Restrictions on hydro operations

- Regulators may impose restrictions on
  - minimum and maximum water levels in reservoirs
  - release rates
  - the rate of change in the release rate - ramping rate
- How should these restrictions be chosen?

Restrictions on hydro operation

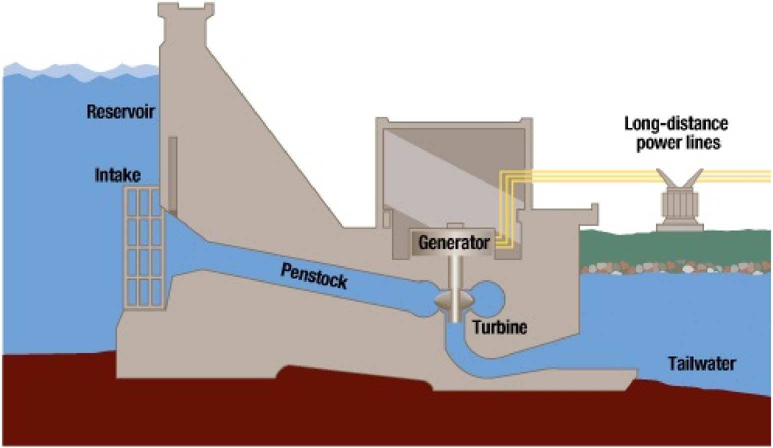
- Regulators need to consider impact of restrictions on hydro profitability as well as ability of electricity grid to meet peak demands.
- If restrictions imply greater reliance on fossil fuels to meet peak electricity demands, there would be other negative environmental consequences.
- Optimal choice balances all of these costs an benefits.
- This research considers only one aspect of the problem: consequences for the hydro operator.

Research Questions

- What is the effect of ramping restrictions on:
  - The value of a hydro operation
  - The optimal operation of a hydro plant
- What factors cause ramping restrictions to have a larger or smaller effect?

The answers to these questions can help inform a regulator's policy decisions about ramping restrictions.

Hydro dam


Reservoir
Intake
Penstock
Generator
Turbine
Long-distance power lines
Tailwater

Optimal operation of a hydro dam

- A complex dynamic optimization problem
- Electricity production depends in a non-linear fashion on the speed of water released through turbines as well as on reservoir head
- Releasing water reduces the head and negatively affects the amount of power produced in the next period.
- Must balance water inflows and outflows while responding to changing electricity demandand prices, and meeting regulatory restrictions.

Optimal operation of a hydro dam

- A complex dynamic optimization problem
- Electricity production depends in a non-linear fashion on the speed of water released through turbines as well as on reservoir head
- Releasing water reduces the head and negatively affects the amount of power produced in the next period.
- Must balance water inflows and outflows while responding to changing electricity demandand prices, and meeting regulatory restrictions.

Optimal operation of a hydro dam

- A complex dynamic optimization problem
- Electricity production depends in a non-linear fashion on the speed of water released through turbines as well as on reservoir head
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- Must balance water inflows and outflows while responding to changing electricity demandand prices, and meeting regulatory restrictions.

Complex nature of electricity prices

- Marked daily patterns and seasonal patterns.
- Limited storage, so price spikes are not uncommon
- Significant changes in electricity markets over the past decade.
- Government incentives to expand solar and wind energy to reduce carbon emissions.
- Intermittent power sources are thought to increase volatility of electricity prices
- Hydro operators can benefit from this volatility..

This research

- Models the optimal decision of a hydro operator under various ramping restrictions as a stochastic control problem.
- Uses a regime switching model of electricity prices - more realistic than some other models used in the literature.
- Results in a Hamilton Jacobi Bellman equation solved numerically using a fully implicit finite difference approach with semi-Lagrangian time stepping
- Analyzes the impact of ramping restrictions for a prototype hydro dam using using a model of German EEX spot electricity prices.

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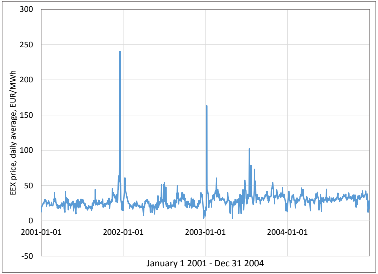
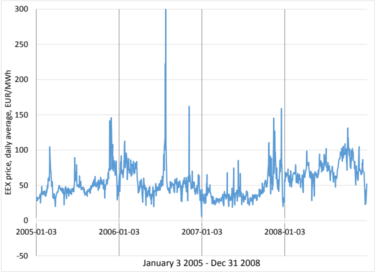
Outline

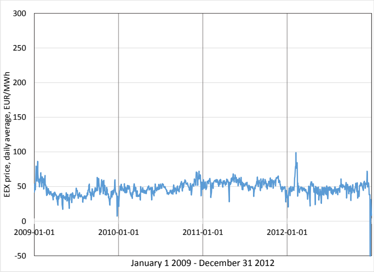


(a) 2001-2004

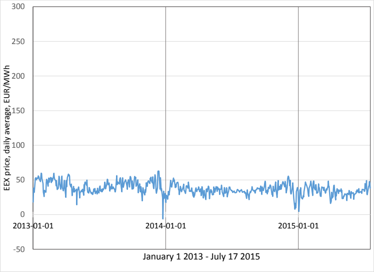


(b) 2005-2008

Figure 1: EEX Daily Average Electricity Prices EUR/ MWh, Market Data Day-ahead Auction, German/Austria Phelix, Source: Datastream.



(a) 2009-2012



(b) 2013-2015

Figure 2: EEX Daily Average Electricity Prices EUR/ MWh, Market Data Day-ahead Auction, German/Austria Phelix, Source: Datastream.

Table 1: Descriptive Statistics for EEX, Market Day-ahead Auction Prices, Selected Years

	2001-2004	2005-2008	2009-2012	2013-2015
Mean, EUR/MWh	29.14	55.56	47.42	37.74
Median	26.96	49.13	47.41	35.97
Maximum	1719.72	2436.63	210	130.27
Minimum	0	-101.52	-221.99	-62.03
Standard Deviation	23.32	40.56	16.77	14.11
Skewness	24.68	17.1959	-1.5367	0.379
Kurtosis.	1350.59	777.88	29.92	5.05

Electricity Price Models

- Weron (2014) provides a good review of the different approaches.
- We desire a parsimonious model which captures important features, but is not so complex as to make computation intractable.
- Structural versus reduced form models - We use a reduced form model
- Regime switching is thought to do a good job of replicating key characteristics - the existence of price spikes and “spike clusters”.

Electricity Price Models

- Spot price models (P-measure) versus risk neutral price models (Q-measure)
- For optimal decisions and valuation, the risk neutral price process is desired as its parameters are adjusted for risk
- Current literature has tended to estimate spot price models
- To use a spot price model for determining optimal decisions, risk adjustment must be made through an estimate of the market price of risk.

Janczura and Weron estimates

- Janczua and Weron (2009) estimate a regime switching electricity price model using German EEX spot prices from 2001-2009.
- Base regime: CIR (Cox-Ingersoll-Ross) process
- Spike regime: shifted lognormal distribution (with higher mean and variance than those in the base regime) which assigns zero probability to prices below the median m.
- $$dP = \eta(\mu_1 - P)dt + \sigma_1\sqrt{P}dZ.$$
  $$\log(P - m) \sim N(\mu_2, \sigma_2^2),\ P > m.$$

Janczura and Weron estimates

- J and W use two time samples to compare estimates under different market conditions, 2001-2005 and 2005-2009. The latter is more volatile, the former gives a better fit of data to model.
- Like much of the literature, Janczura and Weron use mean daily spot prices to estimate their model.
- Ideally we would like a model estimated using hourly data, to incorporate the regular daily cycle in electricity prices.
- We use the JW estimates and then in a sensitivity case we impose a daily cycle to determine the impact on results.

Janczura and Weron estimates

Table 2: Parameter Values Estimated by Janczura and Weron 2009

Parameter	Jan. 1 2001 - Jan. 2 2005	Jan. 3 2005 - Jan 3 2009
μ_1	47.194 EUR/MWh	46.033 EUR/MWh
μ_2	3.44	3.41
η	0.36	0.30
σ_1	0.73485	1.28452
σ_2	0.83066	1.66433
m	46.54 EUR/MWh	45.19 EUR/MWh
λ_{12}	0.0089	0.0116
λ_{21}	0.8402	0.6481

Base regime: $dP = \eta(\mu_1 - P)dt + \sigma_1\sqrt{P}dZ.$
Spike regime: $log(P - m) \sim N(\mu_2, \sigma_2^2),\ P > m.$

P-measure price process

We adapt this model to conform to a standard Ito process and model base and spike regimes as follows.

$$dP = \eta(\mu_1 - P)dt + \sigma_1 \sqrt{P} dZ + P(\xi^{12} - 1)dX_{12}.$$

$$dP = \sigma_2(P - m)dz + P(\xi^{21} - 1)dX_{21}, \; P > m.$$

Q-measure price process

We deduct a market price of risk to derive SDE's to be used for hydro plant valuation.

$$dP = [\eta(\mu_1 - P) - \Lambda_1 \sigma_1 \sqrt{P}]dt + \sigma_1 \sqrt{P} d\hat{Z} + P(\xi^{12} - 1)d\hat{X}_{12}.$$

$$dP = \sigma_2 (P - m) d\hat{z} + P(\xi^{21} - 1)d\hat{X}_{21}, \; P > m.$$

- $d\hat{Z}$ and $d\hat{z}$ are increments of the standard Gauss-Wiener processes under the Q measure
- Λ_1 is the market price of risk which adjusts the drift term in the base regime from the P to the Q measure
- $d\hat{X}_{12}$ and $d\hat{X}_{21}$ indicate the transition of the Markov chain under the Q measure.

Base case parameter values

Table 3: Parameters for the Regime Switching Model, Benchmark

Parameter	Value	Parameter	Value
μ_1	47.194 EUR/MWh	η	0.36
m	46.54 EUR/MWh	c	20 EUR/MWh
σ_1	0.73485	σ_2	0.83066
ξ^{12}	1.6470	ξ^{21}	0.6072
λ_{12}^{Q}	0.0089	λ_{21}^{Q}	0.8402
Λ_1	-0.2481	-	-
$P_1^{\max}$	200 EUR/MWh	$P_2^{\max}$	200 EUR/MWh
$P_1^{\min}$	0 EUR/MWh	$P_2^{\min}$	48 EUR/MWh
T	168h	$\bar{r}$	0.05 annually

Base regime:
$dP = [\eta(\mu_1 - P) - \Lambda_1\sigma_1\sqrt{P}]dt + \sigma_1\sqrt{P}d\hat{Z} + P(\xi^{12} - 1)d\hat{X}_{12}$. Spike regime: $dP = \sigma_2(P - m)d\hat{z} + P(\xi^{21} - 1)d\hat{X}_{21},\ P > m$.

Market price of risk

- Empirical evidence is mixed in terms of magnitude, variability and sign.
- This paper uses an estimated value from Cartea and Figueroa (2005) for England and Wales. Their estimate is -0.2481.
- We undertake sensitivity analysis for a range of values.

Outline

Physical and environmental constraints

- minimum and maximum water release rates: $r^{\min} \leq r \leq r^{\max}$
- minimum and maximum water content: $w^{\min} \leq w \leq w^{\max}$
- Equation of motion for water:

$$dw = a(\ell - r)dt$$

where a is a constant converting water measurement units, r is the release rate, ℓ is the inflow rate.

Ramping rate control

- The ramping rate z is the control variable.

$$dr = zdt$$

- Up-ramping and down-ramping constraints:

$$dr \leq r^u dt$$

$$-dr \leq r^d dt$$

where r^u and r^d represent the maximum allowed up-ramping and down-ramping rates respectively.

- Ramping constraints may be written as:

$$-r^d \leq z \leq r^u$$

Parameterizing the hydro dam

- Empirical analysis is done for a medium sized dam.
- Physical details of the dam are based on the Abitibi Canyon generating station in NE Ontario.
- Constant water inflow is assumed of 6671 CFS
- Other details given in the next table

Parameter Values for the Prototype Hydro Station

Table 4: Base case parameter values for hydro station

Parameter	Value	Parameter	Value
inflow rate, ℓ	6671 CFS	$w^{\max}$	17000 acre-feet
grav constant, g	32.15 feet/sec^2	$w^{\min}$	7000 acre-feet
constant, head/water, b	0.0089	$r^{\max}$	15000 CFS
efficiency factor, e	0.87	$r^{\min}$	2000 CFS
generator capacity	19000 CFS	$q^{\max}$	336 MW
–	–	$q^{\min}$	0 MW
–	–	r^u	3000 CFS-hr
–	–	r^d	3000 CFS-hr

Power production

- General power production relation

$$q(r, w) \propto r \times h(r, w) \times e(r, h).$$

where $q=$ power output, r = water release rate, h = gross head, w = water content, e = the efficiency factor.

- Under some simplifying assumptions we use:

$$\begin{aligned} q(r, h(w)) &= 0.001 \times g \times r \times h(w) \times e \\ &= 0.28 \times r \times h(w). \end{aligned}$$

where g = gravitational constant , $e = 0.87$.

- Linear functional form between the head and water content:

$$h(w) = b \times w. \tag{1}$$

where b is assumed to be 0.0089.

Outline

Present value of net revenue from power generation

$$\int_{t_1}^{T} e^{-\rho t} q(r, h(w))(P - c)dt. \quad (2)$$

where,

- ρ is the discount rate,
- q is the amount power produced which is a function of the water release rate r and the head h,
- c is the unit cost of hydro power production, which is assumed to be a positive constant.

Objective function

$V^{\imath}(P, w, r, t_1)$: value of the hydro plant under the optimal control in regime $\imath$ under the risk neutral measure.

$$V^{\imath}(P, w, r, t_1) = \max_{z} E^{Q}\left[\int_{t_1}^{T} e^{-\rho(t-t_1)} H(r, w) q(r, h(w))(P - c) dt \right.$$

$$\left. | P(t_1) = \tilde{P}, w(t_1) = \tilde{w}, r(t_1) = \tilde{r} \right].$$

subject to

$$Z(r) \subseteq [z^{\min}, z^{\max}].$$

$$dw = H(r, w) a(\ell - r) dt.$$

$$dr = z dt.$$

$$dP = \mu^{\imath}(P, t) dt + \sigma^{\imath}(P, t) dZ + \sum_{\jmath=1}^{N} P(\xi^{\imath\jmath} - 1) dX_{\imath\jmath}.$$

HJB equation

$$\bar{r}V^{\imath} = \sup_{z\in Z(r)}(z\frac{\partial V^{\imath}}{\partial r}) + H(r,w)a(\ell - r)\frac{\partial V^{\imath}}{\partial w} + \frac{1}{2}(\sigma^{\imath})^2(P,t)\frac{\partial^2 V^{\imath}}{\partial P^2}$$
$$+ (\mu^{\imath}(P,t) - \Lambda^{\imath}\sigma^{\imath}(P,t))\frac{\partial V^{\imath}}{\partial P}$$
$$+ H(r,w)q(r,h(w))(P-c) + \frac{\partial V^{\imath}}{\partial t} + \sum_{\substack{\jmath=1\\ \jmath\neq\imath}}^{N}\lambda^{Q}_{\imath\jmath}(V^{\jmath} - V^{\imath}).$$

where $\bar{r}$ is the risk free interest rate, $\Lambda^{\imath}$ is the market price of risk in state $\imath$ and $\lambda^{Q}_{\imath\jmath}$ is the risk-neutral transition intensity from state $\imath$ to $\jmath$ ($\jmath \neq \imath$).

Boundary conditions

- At $t = T$ $(\tau = 0)$, we assume the value of the plant is zero:

$$V^{\imath}(P, w, r, \tau = 0) = 0. \tag{3}$$

- For $P \to 0$ we take the limit of the HJB equation to obtain the boundary condition.
- For $P \to \infty$, we apply the commonly used boundary condition $V^{\imath}_{PP} = 0$, which implies that

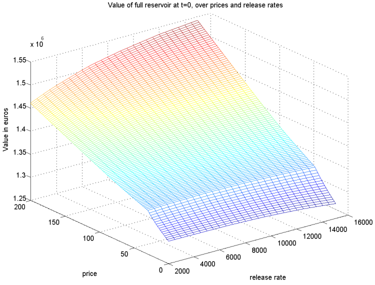
$$V^{\imath} \simeq x(w, r, \tau)P + y(w, r, \tau).$$

- No special boundary conditions are required at $w^{\min}, w^{\max}, r^{\min}, r^{\max}$.

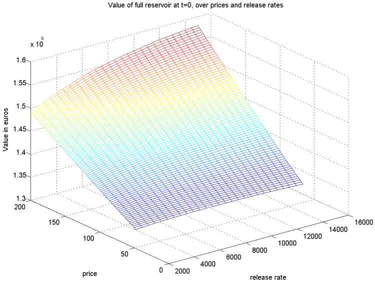
Outline

Base Case

- Results for base case parameter values, as presented in Tables 2 and 3
- Ramping restrictions set at 3000 CFS-hr
- Spike regime is only defined for prices above 46.54 EUR/MWH
- Decision variable is z, but hydro plant can only ramp up or down by changing the release rate, r.
- The next figures plots hydro plant value given today's price and release rate, with a full reservoir, based on optimal choices for ramping rates in all subsequent periods.



(a) Base regime

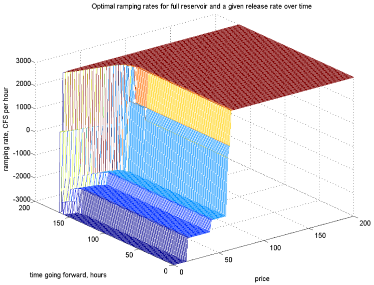


(b) Spike regime

Figure 3: Value over Price and Release Rate, Base Case, (Ramping Restrictions 3000 CFS-hr)

Base case results commentary

- Value increases with price and release rate for prices above variable cost ($c = 20$ EUR/MWh)
- For $P < c$, value declines with release rate.
- Value in spike regime exceeds value in base regime - but not by much.



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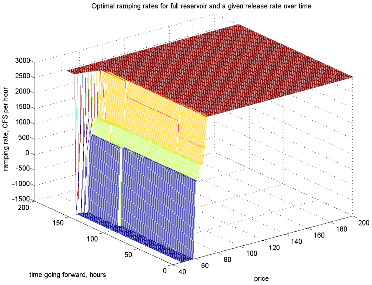
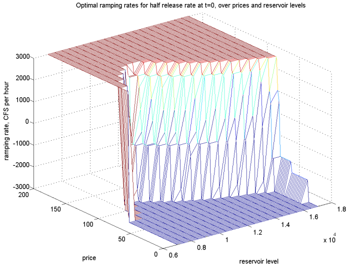


(b) Spike regime

Figure 4: Optimal Ramping Rate over Price and Time, Base Case (Ramping Restrictions 3000 CFS-hr)

Base case results commentary

- Ramp down when price is low and ramp up when price is high.
- In spike regime, mostly ramp up at the maximum.
- Constant strategy over time except at the boundary.
- At the boundary $t = T$, ramp up for all prices.



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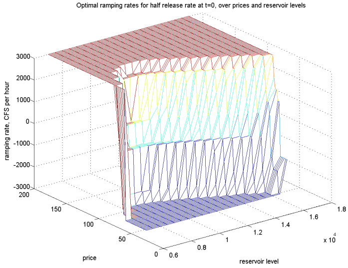

(b) Spike regime

Figure 5: Optimal Ramping Rate over Price and Reservoir Level, Base Case (Ramping Restrictions 3000 CFS-hr)

Base case results commentary

- Optimal ramping rate depends on reservoir level.
- When the reservoir is full, in general there is a wider range of prices over which it is optimal to ramp up.
- At lower water reservoir level the operator should be more inclined to let the reservoir fill up again.

Outline

Effect of ramping restrictions

- We try a range of ramping restrictions from 250 CFS-hr to no restrictions.
- The maximum impact is a decline in value of 8.3% for the most restrictive (250 CFS-hr) compared to no restrictions.
- The next two graphs show the impact on value and on optimal strategy.

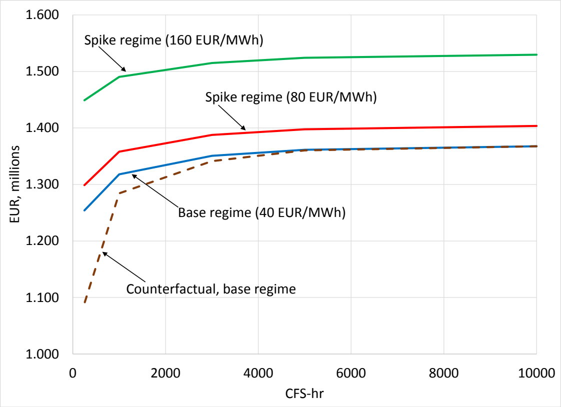

Figure 6: The Impact of Ramping Rate Restrictions on Value

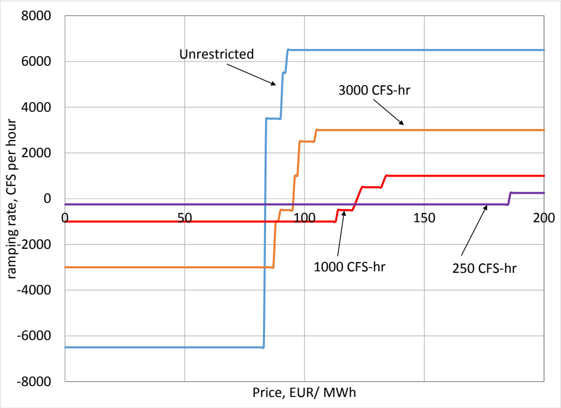

Figure 7: The Impact of Ramping Rate Restrictions on Optimal Ramping Strategy

Effect of ramping rate restrictions

- The previous figure shows that the operator switches from down to up ramping at a higher price under ramping restrictions compared to no restrictions.
- Under ramping restrictions it is more valuable to maintain the water in the reservoir.
- If an operator starts ramping up when ramping restrictions are in place, it may be costly since the decision cannot be quickly reversed.
- Ramping restrictions result in hysteresis of optimal actions.
- Counterfactual in Figure 6 shows value under the most severe restrictions if the operator did not follow the optimal control - i.e. the restrictions are met but there is not change on when to switch from down-ramping to up-ramping.

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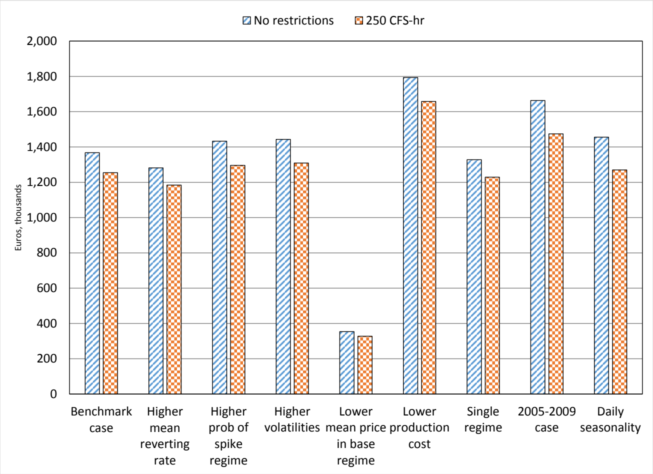

Figure 8: Sensitivity Cases, Base Regime, full reservoir, $P_0 = 40$ EUR/MWh, Comparing value for no restrictions (left bars) with 250 CFS-hr restriction (right bars)

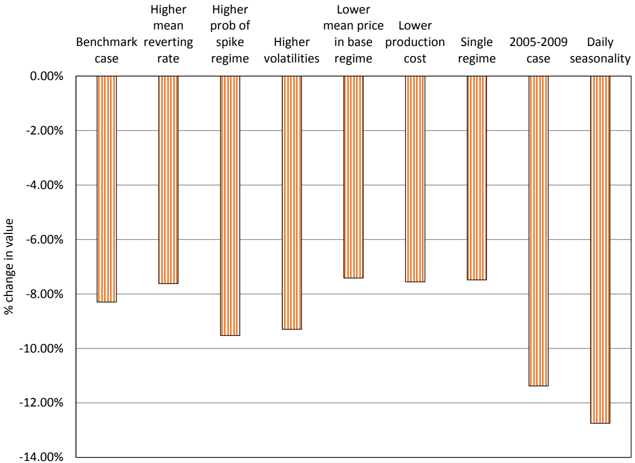

Figure 9: Sensitivity Cases, Base Regime, full reservoir, $P_0 = 40$ EUR/MWh, Percent change in value for 250 CFS-hr restriction compared to no restriction

Summary of sensitivities

- A higher speed of mean reversion reduces value and reduces the relative impact of ramping restrictions.
- Higher probability of being in the spike regime increases value and increases the relative impact of ramping restrictions.
- Higher volatilities increase value and increase the relative impact of ramping restrictions.
- Lower production cost increases value and reduces the impact of ramping restrictions.
- 2005-2009 case uses parameter estimates in a more volatile period with more price spikes. Value of the hydro plant is increased and relative impact of ramping restrictions is increased.

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Observation about ramping restrictions

- Ramping restrictions have a large impact in cases when the operator would like to change water release rates frequently - such as when prices are volatile and spikes are frequent.
- Ramping rates have a large impact when there is a greater possibility that price will drop below variable costs. In these circumstances it is important to be able to ramp down quickly.

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Outline

Daily price cycle

- Electricity prices typically follow regular daily cycles, rising during the hours of peak demand.
- This is ignored in the price process estimated by Janczura and Weron which is used in this paper.
- To determine the impact of a regular cycle we add a deterministic cyclical component to the SDE representing price.

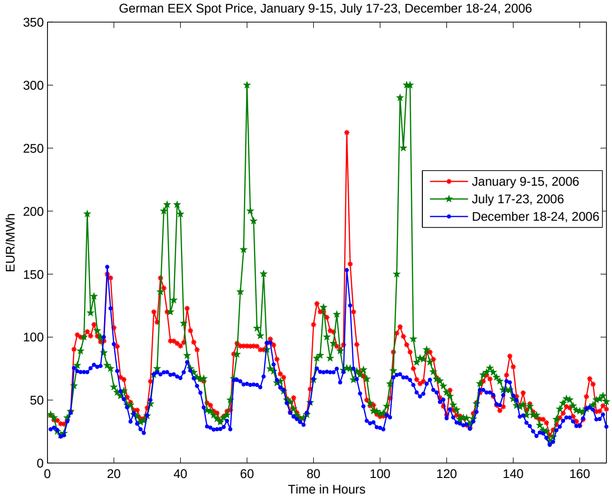


Figure 10: German EEX Spot Price, January 9-15, July 17-23, December 18-24, 2006

Stochastic price process with a daily cycle

$$dP = [\eta(\mu(t) - P) - \Lambda_1 \sigma_1 \sqrt{P}]dt + \sigma_1 \sqrt{P} d\hat{Z} + P(\xi^{12} - 1)d\hat{X}_{12}.$$

$$\mu(t) = \mu_1 + \phi \sin\left(\frac{2\pi(t - t_0)}{24}\right).$$

$$dP = \sigma_2(P - m)d\hat{z} + P(\xi^{21} - 1)d\hat{X}_{21}, \; P > m.$$

where

- $\mu(t)$ is the long-term equilibrium price with the daily price cycle;
- μ_1 is the equilibrium price without the daily price fluctuation;
- ϕ is the daily price trend;
- t_0 is the time of the daily peak of the equilibrium price;

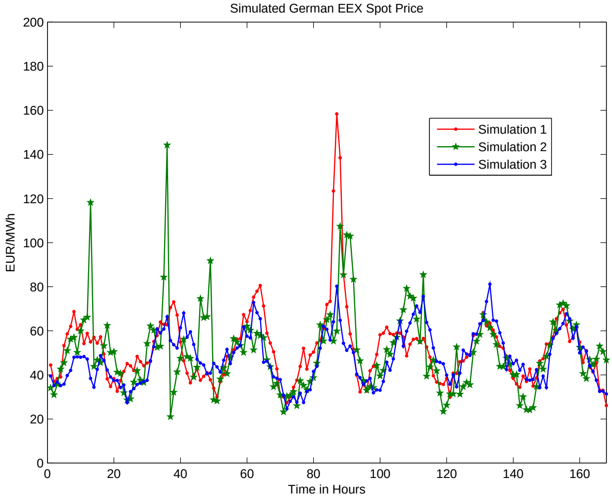
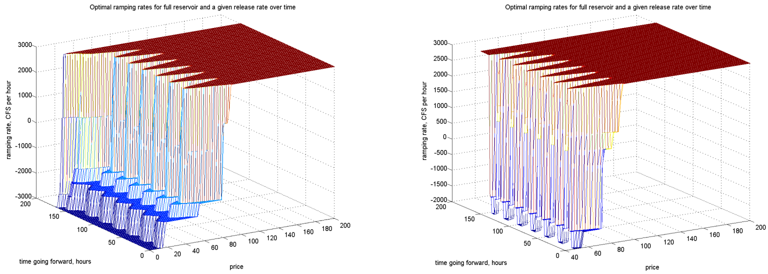


Figure 11: Simulated German EEX Spot Price



(a) Base regime

(b) Spike regime

Figure 12: Optimal Ramping Rate over Price and Time including a Daily Cycle), Ramping restrictions of 3000 CFS-hr

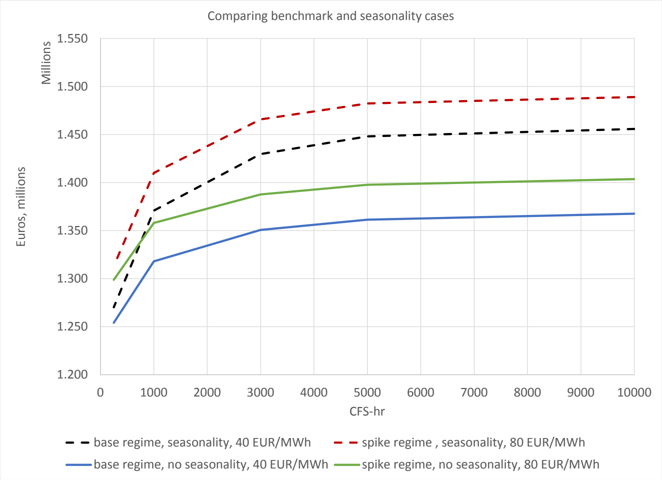


Figure 13: Comparing the Impact of Ramping Restrictions with and without Daily Seasonality, Dashed Lines Show Cases with Seasonality

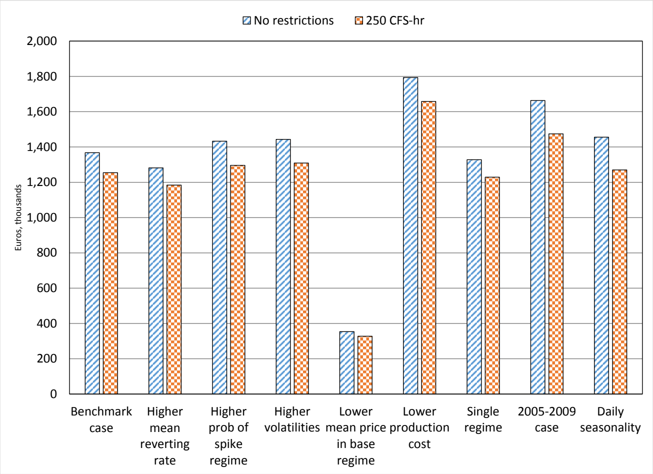


Figure 14: Repeat of earlier diagram, Sensitivity Cases, Base Regime, full reservoir, $P_0 = 40$ EUR/MWh, Comparing value for no restrictions (left bars) with 250 CFS-hr restriction (right bars)

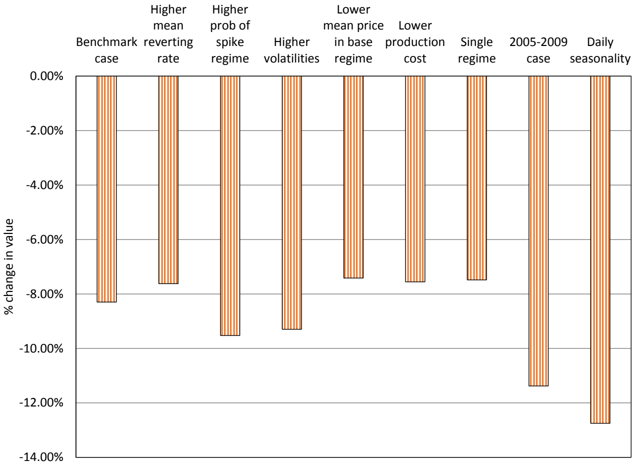


Figure 15: Repeat of earlier diagram, Sensitivity Cases, Base Regime, full reservoir, $P_0 = 40$ EUR/MWh, Percent change in value for 250 CFS-hr restriction compared to no restriction

Concluding remarks

- A benefit of hydro power is its ability to respond quickly to changing demand by ramping up and down water release rates.
- Restrictions imposed on ramping rates will reduce the value of hydro power assets to the owner, but the impact is case specific. The impact is more severe for:
  - Higher volatility
  - More time in the spike regime
  - Higher variable costs
- The optimal operating policy changes when ramping constraints are imposed. The operator waits for a higher price to ramp up.

Concluding remarks

- Policy decisions about imposing ramping rate restrictions need to be taken with an understanding of all of the costs and benefits including environmental benefits and costs. This paper looks at only one part of the cost-benefit analysis.
- Future work is needed on estimating electricity price processes using hourly data and estimating a risk-neutral process in the Q-measure.
- Consideration needs to be given to recent changes in electricity markets - increased use of wind and solar as well as the appearance of negative prices.
- Another avenue of future research include stochastic water flows as well as longer term trends for less abundant water supplies.

Monotone Schemes for Two Factor HJB Equations: Nonzero Correlation

Peter Forsyth[1]

[1]Cheriton School of Computer Science
University of Waterloo

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Outline

Need to guarantee numerical scheme converges to viscosity solution

- Sufficient conditions (Barles, Souganidis (1991))
  - Monotone, consistent (in the viscosity sense) and ℓ_∞ stable
- Examples known where seemingly reasonable (non-monotone) discretizations converge to incorrect solution

Up to now, we have looked at

- One stochastic factor, several path dependent factors
- Easy to construct a monotone scheme
- But suppose we have two (or more) stochastic factors

  → Not so easy to construct monotone schemes if we have nonzero correlation

Example I: two factor uncertain volatility

Suppose we have two stochastic factors S_1, S_2 (equities).

Risk neutral processes:

$$\begin{aligned} dS_1 &= (r - q_1)S_1 \ dt + \sigma_1 S_1 \ dW_1, \\ dS_2 &= (r - q_2)S_2 \ dt + \sigma_2 S_2 \ dW_2, \\ r &= \text{ risk free rate} \\ q_i &= \text{ dividend rate} \\ \sigma_i &= \text{ volatility} \\ dW_1 \ dW_1 &= \rho \ dt = \text{ correlation} \end{aligned} \tag{1}$$

HJB PDE

No arbitrage value of a contingent claim $\mathcal{U}(S_1, S_2, \tau = T - t)$

$$\mathcal{U}_\tau = \mathcal{L}(\sigma_1, \sigma_2, \rho)\, \mathcal{U}$$

Initial condition

$$\mathcal{U}(S_1, S_2, 0) \;\; = \;\; \mathcal{W}(S_1, S_2) = \text{ payoff}$$

where

$$\begin{aligned}\mathcal{L}(\sigma_1, \sigma_2, \rho)\, \mathcal{U} \;\; = \;\; & \frac{\sigma_1^2 S_1^2}{2}\, \mathcal{U}_{S_1 S_1} + \frac{\sigma_2^2 S_2^2}{2}\, \mathcal{U}_{S_2 S_2} \\ & +(r - q_1)\, \mathcal{U}_{S_1} + (r - q_2)\, \mathcal{U}_{S_2} - r\mathcal{U} \\ & + \underbrace{\rho \sigma_1 \sigma_1 S_1 S_2\, \mathcal{U}_{S_1 S_2}}_{cross\;\; derivative\;\; term}\end{aligned}$$

Uncertain Volatilites, Correlation

Suppose σ_1, σ_2, ρ are uncertain

Define the set of controls Q

$$Q = (\sigma_1, \sigma_2, \rho)$$

With the set of admissible controls $\mathcal{Z}$

$$\begin{aligned}\mathcal{Z} &= [\sigma_{1,\mathrm{min}}, \sigma_{1,\mathrm{max}}] \times [\sigma_{2,\mathrm{min}}, \sigma_{2,\mathrm{max}}] \times [\rho_{\mathrm{min}}, \rho_{\mathrm{max}}] \\ &\sigma_{1,\mathrm{min}} \geq 0, \quad \sigma_{2,\mathrm{min}} \geq 0 \\ &-1 \leq \rho_{\mathrm{min}} \leq 1, \quad -1 \leq \rho_{\mathrm{max}} \leq 1.\end{aligned}$$

Worst case price short, $\mathcal{L}^{\mathcal{Q}} \equiv \mathcal{L}(\sigma_1, \sigma_2, \rho)$

$$\mathcal{U}_\tau = \sup_{Q \in \mathcal{Z}} \mathcal{L}^{\mathcal{Q}} \mathcal{U}$$

Worst case long

$$\mathcal{U}_\tau = \inf_{Q \in \mathcal{Z}} \mathcal{L}^{\mathcal{Q}} \mathcal{U}$$

Aside: a useful result

Consider the objective function

$$\max_{(\sigma_1,\sigma_2,\rho)\in\mathcal{Z}}\left(\frac{\sigma_1^2 S_1^2}{2}\,\mathcal{U}_{S_1S_1}+\rho\sigma_1\sigma_1 S_1 S_2\,\mathcal{U}_{S_1S_2}+\frac{\sigma_2^2 S_2^2}{2}\,\mathcal{U}_{S_2S_2}\right). \quad (2)$$

Proposition 1

Suppose that $\frac{\partial^2\mathcal{U}}{\partial S_i \partial S_j}$ exist $\forall i,j$. The optimal value of the objective function in (2) can be determined by examining values only on the boundary of $\mathcal{Z}$, denoted by $\partial\mathcal{Z}$.

$$\max_{(\sigma_1,\sigma_2,\rho)\in\mathcal{Z}}\Big(\textit{Equation (2)}\Big)=\max_{(\sigma_1,\sigma_2,\rho)\in\partial\mathcal{Z}}\Big(\textit{Equation (2)}\Big)$$

Discretization

Localize computational domain

$$(S_1, S_2) = [0, (S_1)_{\max}] \times [0, (S_2)_{\max}]$$

Define a set of nodes, timesteps

$$\{(S_1)_1, (S_1)_2, \ldots, (S_1)_{N_1}\} \quad ; \quad \{(S_2)_1, (S_2)_2, \ldots, (S_2)_{N_2}\}$$
$$\tau^n = n\Delta\tau, \; n = 0, \ldots, N_\tau$$

And

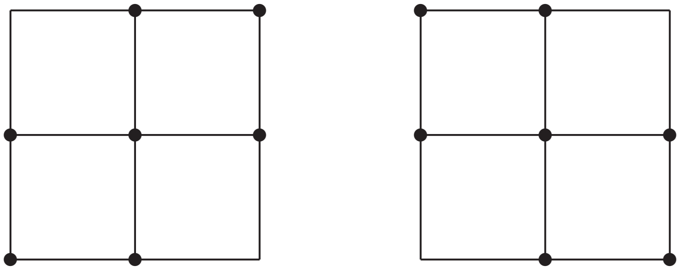
$$\Delta(S_k)_{\max} = \max_i \Delta(S_k)_i, \quad \Delta(S_k)_{\min} = \min_i \Delta(S_k)_i,$$
$$\Delta(S_k)_i = (S_k)_{i+1} - (S_k)_i$$
$$k = 1, 2$$

With a discretization parameter h

$$\Delta(S_1)_{\max} = C_1 h, \; \Delta(S_2)_{\max} = C_2 h,$$
$$\Delta(S_1)_{\min} = C_1^{'} h, \; \Delta(S_2)_{\min} = C_2^{'} h, \; \Delta\tau = C_3 h$$
$$C_1, C_2, C_1^{'}, C_2^{'}, C_3 > 0$$

First Attempt:First Attempt: Fixed Stencil

Finite difference of cross-derivative term



(a) $\rho \geq 0$ (b) $\rho < 0$

Other terms:

- Three point second derivative finite difference
- Central/forward/backward for first derivative terms
- Try to produce a positive coefficent scheme

Positive Coefficient Scheme[1]

$\mathcal{U}_{i,j}^n \equiv$ approximate solution at $((S_1)_i, (S_2)_j, \tau^n)$

Discretization operator L_f^Q (fixed stencil)

$$
\begin{aligned}
L_f^Q \mathcal{U}_{i,j}^n = {} & (\alpha_{i,j}^{S_1} - \gamma_{i,j})\mathcal{U}_{i-1,j}^n + (\beta_{i,j}^{S_1} - \gamma_{i,j})\mathcal{U}_{i+1,j}^n + (\alpha_{i,j}^{S_2} - \gamma_{i,j})\mathcal{U}_{i,j-1}^n \\
& + (\beta_{i,j}^{S_2} - \gamma_{i,j})\mathcal{U}_{i,j+1}^n + \mathbf{1}_{\rho \geq 0}(\gamma_{i,j}\mathcal{U}_{i+1,j+1}^n + \gamma_{i,j}\mathcal{U}_{i-1,j-1}^n) \\
& + \mathbf{1}_{\rho < 0}(\gamma_{i,j}\mathcal{U}_{i+1,j-1}^n + \gamma_{i,j}\mathcal{U}_{i-1,j+1}^n) \\
& - (\alpha_{i,j}^{S_1} + \beta_{i,j}^{S_1} + \alpha_{i,j}^{S_2} + \beta_{i,j}^{S_2} - 2\gamma_{i,j} + r)\mathcal{U}_{i,j},
\end{aligned}
$$

Definition 1 (Positive Coefficient Discretization)

L_f^Q is a positive coefficient discretization if $\forall Q \in \mathcal{Z}$

$$
\begin{aligned}
& \alpha_{i,j}^{S_k} - \gamma_{i,j} \geq 0, \quad \beta_{i,j}^{S_k} - \gamma_{i,j} \geq 0 \;\; ; \quad k = 1, 2 \\
& \gamma_{i,j} \geq 0
\end{aligned}
$$

[1] Note that α, β, γ are functions of the control Q.

Monotone Schemes

Consider fully implicit timestepping:

$$\mathcal{U}_{i,j}^{n+1} = \mathcal{U}_{i,j}^{n} + \Delta\tau \max_{Q \in Z} \mathcal{L}_f^Q \mathcal{U}_{i,j}^{n+1} \tag{3}$$

which we can write as

$$\mathcal{G}_{i,j}(\mathcal{U}_{i,j}^{n+1}, \mathcal{U}_{i,j}^{n}, \mathcal{U}_{i+1,j}^{n+1}, \ldots) = \mathcal{U}_{i,j}^{n+1} - \mathcal{U}_{i,j}^{n} - \Delta\tau \max_{Q \in Z} \mathcal{L}_f^Q \mathcal{U}_{i,j}^{n+1} = 0 \tag{4}$$

Definition 2 (Monotone Scheme)

Scheme (3) is monotone if $\mathcal{G}_{i,j}(\mathcal{U}_{i,j}^{n+1}, \mathcal{U}_{i,j}^{n}, \mathcal{U}_{i+1,j}^{n+1}, \ldots)$ is a nonincreasing function of $(\mathcal{U}_{i,j}^{n}, \mathcal{U}_{i+1,j}^{n+1}, \ldots)$.

Theorem 3 (Positive Coefficient Scheme, see (Forsyth and Labahn (2007))

A positive coefficient scheme is monotone.

Conditions for a Positive Coefficient Scheme: Fixed Stencil

Recall that the positive coefficient property has to hold $\forall Q \in \mathcal{Z}$ (i.e. $\alpha_{i,j}, \beta_{i,j}, \gamma_{i,j}$ are functions of Q)

- The problem is the cross-derivative term
- For general $\mathcal{Z}$, this requires severe restrictions on the grid spacing
- May be impossible to satisfy

Alternative: wide stencil method

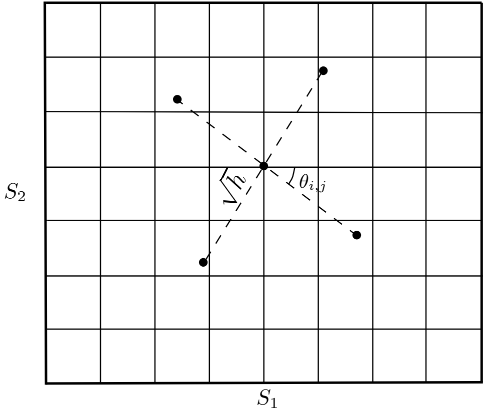
Wide Stencil Method

Wide stencil[2]

- Grid spacing $O(h)$
- At each node, do virtual rotation, eliminate x-derivative term, finite difference on rotated grid
- Values are interpolated from *real grid*
- Size of virtual stencil $O(\sqrt{h})$
    - We interpolate data for stencil from actual grid
    - Stencil is $O(\sqrt{h}) \rightarrow$ guarantees consistency

[2] Debrebant and Jakobsen (2013) factor the diffusion tensor

Local Rotation



Note: local rotation angle $\theta_{i,j}$ depends on

- Node location, i.e. (S_i, S_j)
- Control Q at this node

Wide Stencil II

Why is this called a wide stencil method?

- Size of (virtual) stencil $O(\sqrt{h})$
- Grid spacing $O(h)$
- Relative stencil length

$$\frac{\sqrt{h}}{h} \to \infty \text{ as } h \to 0$$

What happens near the boundaries?

Simple application of wide stencil

$\rightarrow$ Stencil may require data outside computational domain

Wide Stencil: near boundaries

If we need data $S_1 > (S_1)_{\max}$ or $S_2 > (S_2)_{\max}$

- Localization
  - $\rightarrow$ Use artificial boundary conditions at $(S_1)_{\max}, (S_2)_{\max}$ based on asymptotic form of solution
- Use same asymptotic form for data needed from wide stencil
- Errors small if $(S_1)_{\max}, (S_2)_{\max}$ sufficiently large

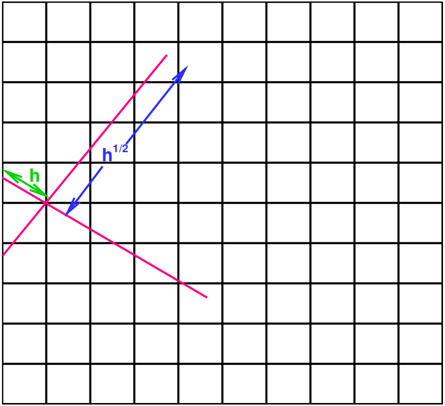
But, what about near $S_1 = 0, S_2 = 0$?

- Wide stencil may need data for $S_1 < 0$ or $S_2 < 0$

Solution:

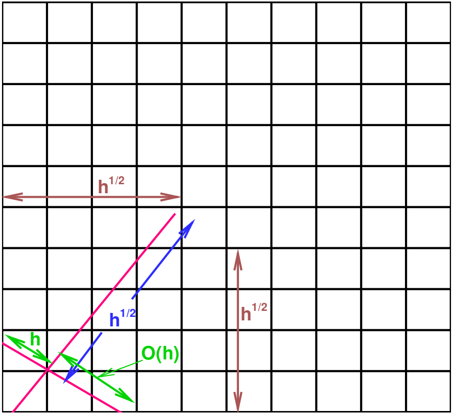
- Shrink stencil arm so that we do not go outside domain

Shrink Stencil Arm



if $(S_1)_i > \sqrt{h}$ or $(S_2)_j > \sqrt{h}) \Rightarrow$ discretization is consistent $O(\sqrt{h})$

What about lower left corner?



Discretization of 2nd order derivative inconsistent here $O(1)$

- Region $(S_1, S_2) \in [0, \sqrt{h}] \times [0, \sqrt{h}]$
- Equation coefficient $O(h) \rightarrow$ consistent discretization of PDE!

Convergence of wide stencil method $\mathcal{L}_w^Q$

Lemma 4 (Ma and Forsyth (2015))

The fully implicit wide stencil scheme

$$\mathcal{U}_{i,j}^{n+1} = \mathcal{U}_{i,j}^{n} + \Delta\tau \sup_{Q \in \mathcal{Z}} \mathcal{L}_w^Q \mathcal{U}_{i,j}^{n+1}$$

is consistent (in the viscosity sense), ℓ_∞ stable and monotone.

Theorem 5 (Convergence)

The wide stencil method converges to the viscosity solution of the uncertain volatility HJB PDE.

Proof.

The HJB PDE satisfies the strong comparison property (Guyon and Henry-Labordere (2011)). Result follows from Lemma 4 and (Barles and Souganidis (1993)). □

Hybrid Method

Algorithm 1 Hybrid Discretization Method $(\mathcal{L}_H^Q)_{i,j}$

for $i = 1, \ldots, N-1$; $j = 1, \ldots, N_2$ **do**
  if $(\mathcal{L}_f^Q)_{i,j}$ monotone $\forall Q \in \mathcal{Z}$ **then**
    Use fixed stencil at this node $(\mathcal{L}_H^Q)_{i,j} = (\mathcal{L}_f^Q)_{i,j}$
  else
    Use wide stencil at this node $(\mathcal{L}_H^Q)_{i,j} = (\mathcal{L}_w^Q)_{i,j}$
  end if
end for

Fixed stencil used as much as possible (more accurate).

- We do not enforce any grid conditions
- We simply check to see if the monotonicity conditions are satisfied at a given node
- Algorithm 1 only done once at start

Fully Implicit Timestepping

$$\mathcal{U}_{i,j}^{n+1} = \mathcal{U}_{i,j}^{n} + \Delta\tau \sup_{Q \in \mathcal{Z}} \mathcal{L}_H^Q \, \mathcal{U}_{i,j}^{n+1}$$

$$\sup_{Q \in \mathcal{Z}} \left[-(1 - \Delta\tau \mathcal{L}_H^Q) \, \mathcal{U}_{i,j}^{n+1} + \mathcal{U}_{i,j}^{n} \right] = 0$$

Define:

$$\mathbf{U}^n = \left(\mathcal{U}_{1,1}^n, \mathcal{U}_{2,1}^n, \ldots, \mathcal{U}_{N_1,1}^n, \ldots, \mathcal{U}_{1,N_2}^n, \ldots, \mathcal{U}_{N_1,N_2}^n \right)$$
$$\mathbf{U}_\ell^n = \mathcal{U}_{i,j}^n, \quad \ell = i + (j-1) N_1 .$$

Similarly the vector of optimal controls is

$$\mathcal{Q} = (Q_{1,1}, \ldots, Q_{N_1 N_2})$$

The nonlinear algebraic equations are then[3]

$$\sup_{Q \in \mathcal{Z}} \left\{ -\mathbf{A}(\mathcal{Q}) \mathbf{U}^{n+1} + \mathbf{C}(\mathcal{Q}) \right\} = 0, \tag{5}$$

$\mathbf{A} =$ matrix of discretized equations ; $\mathbf{C}(\mathcal{Q}) =$ rhs vector

[3] Row ℓ of $\mathbf{A}, \mathbf{C}$ depends only on Q_ℓ

Policy Iteration[4]

Algorithm 2 Policy Iteration

Let $(\hat{\mathbf{U}})^0$ = Initial estimate for $\mathbf{U}^{n+1}$
for $k = 0, 1, 2, \ldots$ until converge **do**
    $\mathcal{Q}_\ell^k = \underset{\mathcal{Q}_\ell \in \mathcal{Z}}{\arg\max} \left\{ -[\mathbf{A}(\mathcal{Q})]\hat{\mathbf{U}}^k + \mathbf{C}(\mathcal{Q}) \right\}_\ell$
    Solve $[\mathbf{A}(\mathcal{Q}^k)]\hat{\mathbf{U}}^{k+1} = \mathbf{C}(\mathcal{Q}^k)$
    if converged **then**
        break from the iteration
    end if
end for

[4]Use ILU-PCG method to solve matrix, complexity = $O((N_1 N_2)^{5/4})$.

Policy Iteration II

Let $(\hat{\mathbf{U}})^0$ = Initial estimate for $\mathbf{U}^{n+1}$
for $k = 0, 1, 2, \ldots$ until converge **do**
    $\mathcal{Q}_\ell^k = \arg\max\limits_{\mathcal{Q}_\ell \in \mathcal{Z}} \left\{ -[\mathbf{A}(\mathcal{Q})]\hat{\mathbf{U}}^k + \mathbf{C}(\mathcal{Q}) \right\}_\ell$
    Solve $[\mathbf{A}(\mathcal{Q}^k)]\hat{\mathbf{U}}^{k+1} = \mathbf{C}(\mathcal{Q}^k)$
    if converged **then**
        break from the iteration
    end if
end for

Theorem 6 (Convergence of Policy Iteration)

If $\forall \mathcal{Q} \in \mathcal{Z}$, $[\mathbf{A}(\mathcal{Q})]$ *is an* $\mathcal{M}$ *matrix, then Policy iteration converges to the unique solution of equation (5).*

For wide stencil nodes

- The rotation angle is a function of $\mathcal{Q}$
  - $\rightarrow$ The stencil changes at each policy iteration

But, we can still prove policy iteration converges!

- Positive coefficient $\rightarrow$ $\mathbf{A}(Q)$ is an $\mathcal{M}$ matrix

Numerical Example (nonconvex payoff)

Butterfly on maximum (worst case short)

$$\begin{aligned} S_{\max} &= \max(S_1, S_2), \\ \text{Payoff} &= \max(S_{\max} - K_1, 0) + \max(S_{\max} - K_2, 0) \\ &\quad - 2\max(S_{\max} - (K_1 + K_2)/2, 0). \end{aligned}$$

Parameter	Value
Time to expiry (T)	0.25
r	0.05
σ_1	[.3, .5]
σ_2	[.3, .5]
ρ	[.3, .5]
K_1	34
K_1	46

Grid/timesteps

Refine Level	Timesteps	S_1 nodes	S_2 nodes	$\partial\mathcal{Z}$ nodes
1	25	91	91	24
2	50	181	181	46
3	100	361	361	90
4	200	721	721	178

For fixed stencil, analytic expression for global maximum of objective function on $\partial\mathcal{Z}$.

For wide stencil, need to discretize control and do linear search on $\partial\mathcal{Z}$.

Convergence study

	Hybrid Scheme		Pure Wide Stencil	
Refine	Value	Diff	Value	Diff
1	2.7160		2.6371	
2	2.6946	0.0214	2.6397	0.0026
3	2.6880	0.0066	2.6650	0.0252
4	2.6862	0.0018	2.6744	0.0094

Table: Butterfly call on max of two, worst case short, value at $(S_1 = S_2 = 40, t = 0)$

Refine	Hybrid Scheme	Pure Wide	Fraction Fixed (Hybrid)
1	4.0	3.7	0.38
2	3.8	3.7	0.42
3	3.6	3.6	0.44
4	3.3	3.3	0.45

Table: Average number of policy iterations per timestep

Summary: Uncertain Volatility

- Cross derivative term $\rightarrow$ difficult to construct monotone scheme
- Wide stencil method
  $\rightarrow$ Unconditionally monotone, but only first order
- Hybrid scheme: use fixed stencil as much as possible
  $\rightarrow$ Converges faster than pure wide stencil

Example II: mean variance portfolio allocation: Heston stochastic volatility

Long term investor can allocate wealth into two assets:

Amount in Risk-free bond B

$$dB = r\ B\ dt$$
$$r = \text{ risk-free rate}$$

Amount in risky-asset S

$$\frac{dS}{S} = (r + \xi V)\ dt + \sqrt{V}\ dZ_1$$
$$\xi V = \text{ market price of volatility risk}$$

Variance process V

$$dV(t) = \kappa(\theta - V(t))\ dt + \sigma\sqrt{V(t)}\ dZ_2$$
$$\sigma = \text{ vol of vol ; } \kappa = \text{ Mean-reversion speed}$$
$$\theta = \text{ mean variance ; } \rho\ dt = dZ_1\ dZ_2$$

SDE for Total Wealth

The investor's total wealth $W = S + B$ follows the process

$$dW(t) = (r + p\xi V(t)) W(t)\, dt + p\sqrt{V} W(t)\, dZ_1.$$

$$p = \left(\frac{S}{W}\right) = \text{ fraction invested in risky asset}$$

Constraints on control p

- Trading must stop if $W = 0$
- Leverage is constrained: $p \le p_{\max}$

Objective: determine optimal control $p(W, V, t)$ which generates points on the efficient frontier

$$\sup_p \Big\{ \underbrace{E^{p(\cdot)}[W(T)]}_{ExpectedValue} - \lambda \underbrace{Var^{p(\cdot)}[W(T)]}_{Variance} \Big\}$$

• Varying $\lambda \in [0, \infty)$ traces out the efficient frontier

Reformulate MV Problem $\Rightarrow$ Dynamic Programming

Embedding technique[5] for fixed λ, if $p^*(\cdot)$ maximizes

$$\sup_{p(\cdot)\in\mathbb{Z}} \left\{ \underbrace{E^p[W(T)]}_{ExpectedValue} - \lambda \underbrace{Var^p[W(T)]}_{Variance} \right\} ,$$

$\mathbb{Z}$ is the set of admissible controls

$\rightarrow \exists\ \gamma$ such that $p^*(\cdot)$ minimizes

$$\inf_{p(\cdot)\in\mathbb{Z}} E^{p(\cdot)}\left[\left(W(T) - \frac{\gamma}{2} \right)^2 \right] .$$

[5] Zhou and Li (2000), Li and Ng (2000)

Value Function $\mathcal{U}(w, v, \tau)$[6]

$$\mathcal{U}(w, v, \tau) = \inf_{p(\cdot)\in\mathbb{Z}} E^{p(\cdot)}_{\tau,w,v}\left[\left(W(T) - \frac{\gamma}{2}\right)^2\right]$$

$$w = \text{ wealth } ; \; v = \text{ local variance } ; \; \tau = T - t$$

HJB PDE for optimal allocation strategy $p(\cdot)$:

$$\mathcal{U}_\tau = \inf_{p\in\mathcal{Z}}\Bigg\{(r + p\xi v)w\,\mathcal{U}_w + \kappa(\theta - v)\,\mathcal{U}_v + \left(\frac{(p\sqrt{v}w)^2}{2}\right)\mathcal{U}_{ww} + \overbrace{(p\rho\sigma\sqrt{v}w)\,\mathcal{U}_{wv}}^{x-derivative\ \ term} + \left(\frac{\sigma^2 v}{2}\right)\mathcal{U}_{vv}\Bigg\},$$

$$\mathcal{U}(w, v, 0) = \left(w - \frac{\gamma}{2}\right)^2.$$

Given $p(\cdot)$, compute $E^{p(\cdot)}[W_T]$, $Var^{p(\cdot)}[W_T]$

[6] For a fixed γ, this gives one point on the efficient frontier.

Discretization

Write HJB PDE as

$$\sup_{p\in\mathcal{Z}} \{\mathcal{U}_\tau - (r + p\xi v)w\mathcal{U}_w - \mathcal{L}^p\mathcal{U}\} = 0, \qquad (6)$$

where

$$\mathcal{L}^p\mathcal{U} = \kappa(\theta - v)\mathcal{U}_v + \frac{1}{2}(p\sqrt{v}w)^2\mathcal{U}_{ww} + p\rho\sigma\sqrt{v}w\mathcal{U}_{wv} + \frac{1}{2}\sigma^2 v\mathcal{U}_{vv}.$$

Define the Lagrangian derivative $\frac{D\mathcal{U}}{D\tau}(p)$ by

$$\frac{D\mathcal{U}}{D\tau}(p) = \mathcal{U}_\tau - (r + p\xi v)w\mathcal{U}_w,$$

$\rightarrow$ rate of change of $\mathcal{U}$ along the characteristic $w = w(\tau)$, depends on risky asset fraction p

$$\frac{dw}{d\tau} = -(r + p\xi v)w.$$

Semi-Lagrangian form

Rewrite equation (6) (use Lagrangian derivative)

$$\sup_{p \in \mathcal{Z}} \left\{ \frac{D\mathcal{U}}{D\tau} - \mathcal{L}^p \mathcal{U} \right\} = 0. \tag{7}$$

Construct finite difference grid

$$\{w_i, v_j\}_{i=1,\ldots,N_1 \; ; \; j=1,\ldots,N_2}$$

Solve characteristic equation (backwards) from τ^{n+1} to τ^n, for fixed w_i^{n+1}.

Point at foot of characteristic (backwards in time from (w_i, v_j))

$$(w_{i^*}, v_j) = (w_i e^{(r + p\xi v_j)\Delta\tau^n}, v_j), \tag{8}$$

Discretization[7]

Lagrangian derivative:

$$\frac{D\mathcal{U}}{D\tau}(p) \approx \frac{\mathcal{U}_{i,j}^{n+1} - \mathcal{U}_{i^*,j}^{n}(p)}{\Delta\tau^n}, \tag{9}$$

$\mathcal{U}_{i^*,j}^{n}(p) = \mathcal{U}(w_{i^*}(p), b_j, \tau^n)$: w_{i^*} depends on control p through equation (8).

Final discretization (h is the mesh size parameter)

$$\sup_{p \in \mathcal{Z}_h} \left\{ \frac{\mathcal{U}_{i,j}^{n+1}}{\Delta\tau^n} - \frac{\mathcal{U}_{i^*,j}^{n}(p)}{\Delta\tau^n} - L_h^p \mathcal{U}_{i,j}^{n} \right\} = 0$$

$$L_h^p = \text{ wide stencil discretization}$$

$$\mathcal{Z}_h = \text{ discretized control set}$$

[7] No fixed stencil will be monotone for all $p \in \mathcal{Z}$

Numerical details

- Policy iteration used to solve nonlinear discretized equations at each timestep
- Linear interpolation used at foot of characteristic

Scheme is

- Monotone, consistent, stable

Construct efficient frontier

- Pick γ, solve HJB PDE for controls, store at each grid/timestep
- Use stored controls, solve linear PDEs for expected value, variance
- This is one point on frontier
- Repeat for different value of γ

Numerical Example

κ	θ	σ	ρ	ξ
5.07	0.0457	0.48	-0.767	1.605

Table: $\mathbb{P}$ measure Heston parameters[8]

Investment Horizon T	10
The risk free rate r	0.03
Leverage constraint $p_{\max}$	2
Initial wealth w_0	100
Initial variance v_0	0.0457

[8] Aıt-Sahalia, Kimmel, Journal of Financial Economics (2007)

Discretization Details

Refinement	Timesteps	W Nodes	V Nodes	$\mathcal{Z}_h$ Nodes
0	160	112	57	8
1	320	223	113	16
2	640	445	225	32
3	1280	889	449	64

First Attempt

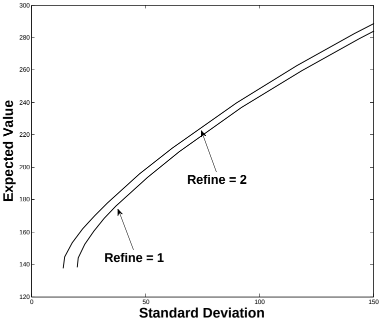


Figure: Close-up of efficient frontier: small standard deviations

But, when *std dev* $\rightarrow 0$, we know the optimal strategy

- Invest all wealth in bond
- Point on frontier $(Var^p[W(T)], E^p[W(T)]) = (0, 134.99)$
- Convergence is very slow near $Var^p[W(T)] = 0$

Why is convergence so slow?

Recall that

$$\mathcal{U}(w, v, \tau) = \inf_{p(\cdot)\in\mathbb{Z}} E^{p(\cdot)}_{\tau,w,v}\left[\left(W(T) - \frac{\gamma}{2}\right)^2\right]$$

Consider:

$$W_{opt} = \frac{\gamma}{2} e^{-r(T-t)}$$

Proposition 2

If $W(t^*) = W_{opt}$, then an optimal strategy is to set $p = 0$ [9] *$\forall t > t^*$. This implies that*

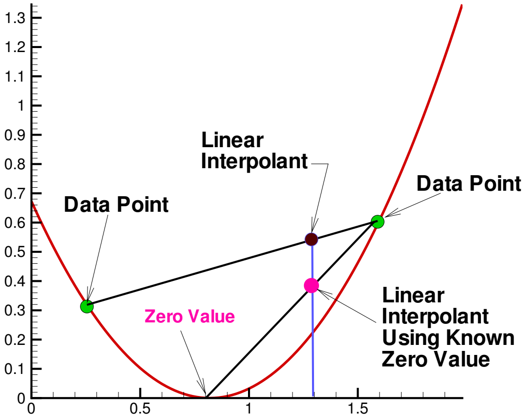
$$\mathcal{U}(\frac{\gamma e^{-r\tau}}{2}, v, \tau) = 0$$

[9] Proof: $E[(W(T) - \gamma/2)^2] = 0$ with certainty.

Culprit: linear interpolation

Recall we have to interpolate at the foot of the characteristic for the Semi-Lagrangian method

- Linear interpolation will diffuse solution near this optimal point



A better interpolant

Solution to inaccurate linear interpolation

→ When interpolating, take into account we know exact solution at point $(\frac{\gamma e^{-r\tau}}{2}, v)$.

Note:

- Optimal strategy at $(w, v) = (\frac{\gamma e^{-r\tau}}{2}, v)$ is $p = 0$
- PDE degenerates to first order PDE when $p = 0$
- Simple backward differencing for $\mathcal{U}_w$ would be very diffusive
- Semi-Lagrangian timestepping allows us to interpolate with knowledge of exact solution when $w = W_{opt}(t)$

Improved linear interpolation

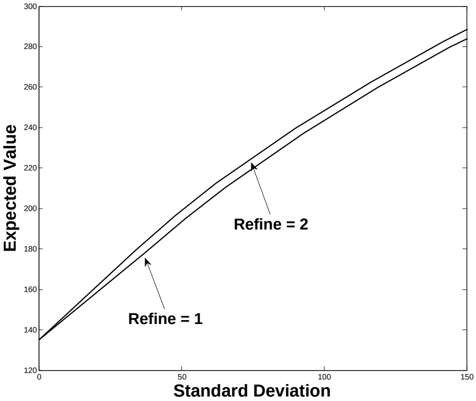


Figure: Close-up of efficient frontier: small standard deviations. Linear interpolation using $\mathcal{U}(\frac{\gamma e^{-r\tau}}{2}, v, \tau) = 0$

One more trick

Recall that for each point on the frontier (fixed γ) we solve HJB PDE and store controls at each point

Compute $(Var^p[\cdot], E^p[\cdot])$

1. PDE Method
   - Use stored controls, and linear PDEs to determine $(Var^p[\cdot], E^p[\cdot])$
2. Alternative: Hybrid PDE and Monte Carlo method
   - Use stored controls, solve SDES, compute $(Var^p[\cdot], E^p[\cdot])$ by Monte Carlo

Note: controls computed using HJB PDE in both cases

Convergence: Pure PDE vs. Hybrid(PDE + MC)

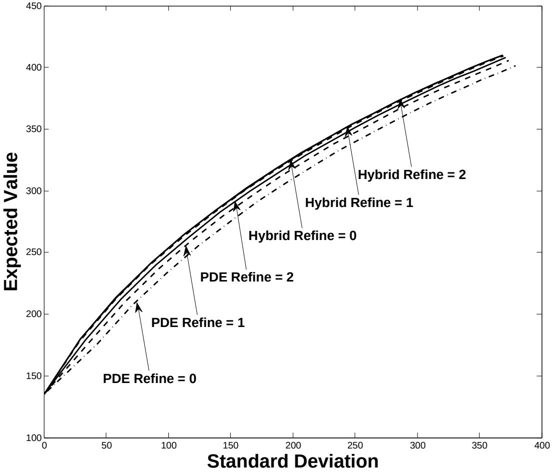


Figure: Comparison of Hybrid (PDE + Monte Carlo) and pure PDE approach

Efficient Frontier (vary speed of mean reversion κ)

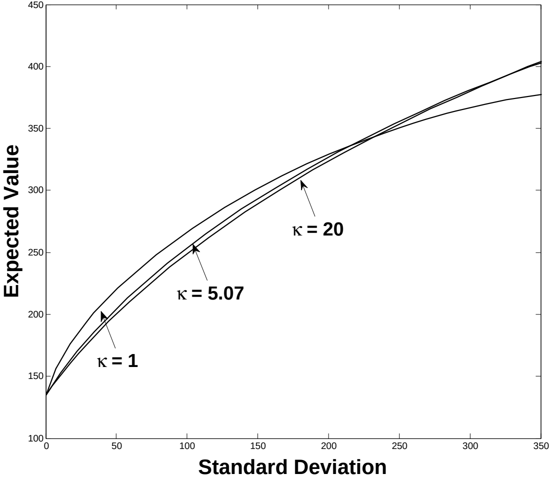


Figure: $T = 10$ years. $w_0 = 100$. $\kappa = 5 \simeq 2.5$ months mean reversion time. Curves for $\kappa = 5, 20$ very close.

Long Term Investment: Stochastic Volatility Unimportant?

Fix γ (parameter that traces out efficient frontier)

A Assume constant volatility GBM[10], compute and store optimal strategy

B Assume stochastic volatility[11], compute and store optimal strategy

Assume real world follows stochastic volatility, compute result using MC simulations, for both **A** and **B**

	$\gamma = 540$		$\gamma = 1350$	
	Mean	Stndrd Dev	Mean	Stndrd Dev
GBM Control **A**	212.68	58.42	329.13	207.23
Stoch Vol Control **B**	213.99	58.53	331.28	207.37

Table: $\kappa T > 20$, stochastic vol well approximated by GBM.

[10] Constant volatility = mean value from stochastic vol model

[11] $\kappa \simeq 5,\ T = 10$ years

Conclusions

- If the control appears only in the first derivative term
  - → Semi-Lagrangian timestepping simple and effective
- Similar timestepping method can be used for impulse control.
- Control appearing in 2nd derivative terms
  - → For non-zero correlation, need monotone discretization (wide stencil)
  - → Non-linear algebraic equations easily solved using Policy iteration
- Low accuracy control (e.g. GBM for stoch vol, coarse control set discretization)
  - → Accurate value function. Why?
- Challenges:
  - Higher dimensions
  - Wide stencil only 1*st* order
  - Solution of local optimization problem at each node (need global optimum)

American Options under Regime Switching

Peter Forsyth

University of Waterloo
Cheriton School of Computer Science

Coruna, September, 2015

Outline

Regime Switching Examples

HJB Equation

Direct Control vs. Penalty

Iterative Methods

Numerical Examples

Floating Point Considerations

Conclusions

Regime Switching: Motivation

Basic idea: stochastic process consists of different regimes

- Stocks: low, high volatility regimes
- Commodities: low, high mean reversion values

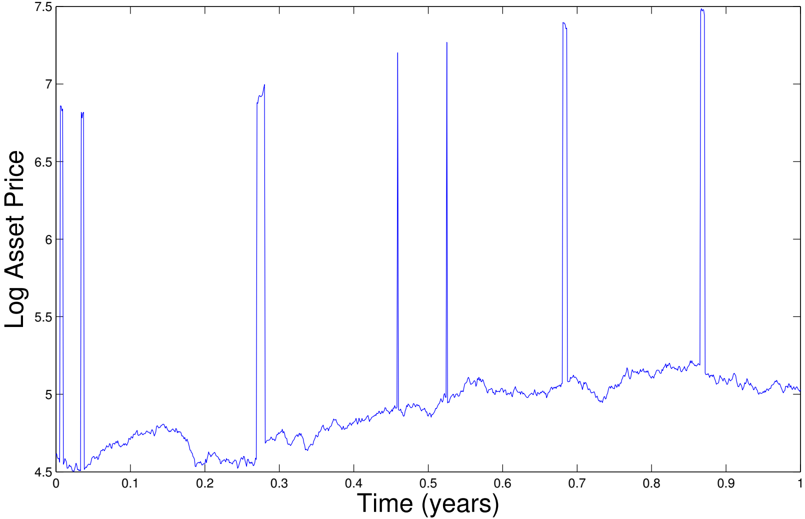
Intuitive economic interpretation, produces smiles (equities), spike phenomena (electricity), jumps.

Computationally inexpensive compared to a stochastic volatility, jump diffusion model.

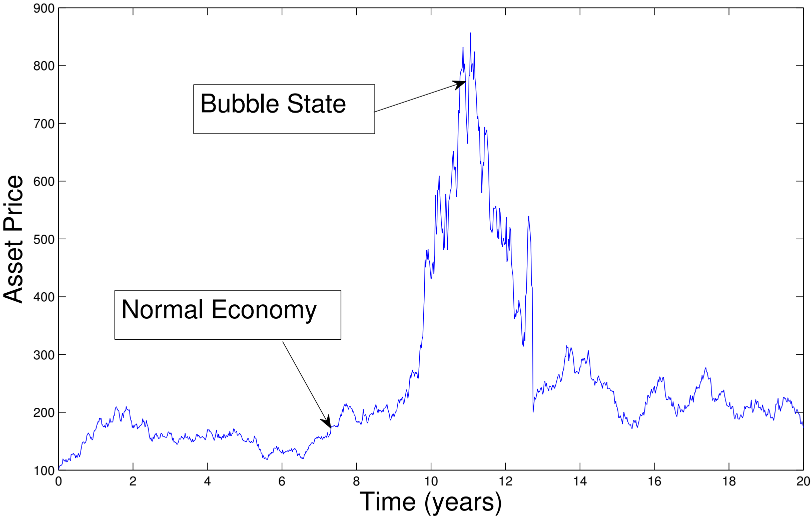
Example applications:

Long term insurance guarantees	Convertible bonds
Electricity markets	Interest rate derivatives
Natural gas	Foreign exchange
Trading strategies	Optimal Forest Harvesting
Stock loans	

Electricity Prices: GBM plus spikes



Stock Price Bubbles



Objective

Many methods have been proposed for pricing American options under regime switching.
$\rightarrow$ Which ones actually work?

We will consider this problem as a special case of an abstract stochastic control problem (an HJB equation).

- This will allow us to analyze most existing methods, as well as consider some new techniques.
- Our abstract methods can be immediately applied to more general control problems, assuming regime switching and/or jumps.
- Easy extension to Markov modulated jump diffusion.
- Singular control problems also fall under the abstract formulation.

Requirements for our methods

We are looking for general purpose methods with the properties

(a) Unconditionally stable
(b) Do not require special forms for drift, diffusion, transition rates

Note that most semi-analytic methods violate (b)

- Commodities (e.g. natural gas): price, time dependent mean reversion levels, jump terms, common
- Convertible bonds: transition rates (default intensities) are usually asset price dependent.

For example, FFT methods can be very fast, but have problems if the PDE cannot be transformed to having constant coefficients.

Regime Switching: Equity Model

$\mathbb{P}$ measure process

$$dS = \mu_j^{\mathbb{P}} \; S \; dt + \sigma^j \; S \; dZ + \sum_{k=1}^{K} (\xi_{jk} - 1) \; S \; dX_{jk} \; ; \; j = 1, \ldots, K \; ,$$

σ^j volatility in regime j

dZ increment of a Wiener process

$\mu_j^{\mathbb{P}}$ drift in regime j

where dX controls the transition of the Markov chain

$$dX_{jk} = \begin{cases} 1 \text{ with probability} & \lambda_{jk}^{\mathbb{P}} \; dt + \delta_{jk} \\ 0 \text{ with probability} & 1 - \lambda_{jk}^{\mathbb{P}} \; dt - \delta_{jk} \end{cases}$$

When a transition $j \to k$ occurs, $S \to \xi_{jk} S$. ξ_{jk} are deterministic functions of (S, t).

No arbitrage value: contingent claim

Let $\mathcal{V}_j$ be the value of the option in regime j.
Construct hedging portfolio

$$P = -\mathcal{V}_j + e\ S + \sum_{k=1}^{K-1} w_k F_k$$

$$e = \text{ units of the underlying asset}$$

$$F_k = \text{ price of additional hedging instrument}$$

$$w_k = \text{ units of asset priced } F_k$$

If assets with prices $\{S, F_1, \ldots, F_{K-1}\}$ form a non-redundant set (Kennedy, 2008; Forsyth, Vetzal, JEDC, 2014), then perfect hedge possible.

Define risk neutral transition rates λ_{jk}, and

$$\lambda_{jj} = -\sum_{\substack{k=1 \\ k \neq j}}^{K} \lambda_{jk} \ \ ; \ \ \rho_j = \sum_{\substack{k=1 \\ k \neq j}}^{K} \lambda_{jk}(\xi_{jk} - 1) \ \ ; \ \ \lambda_j = \sum_{\substack{k=1 \\ k \neq j}}^{K} \lambda_{jk} \ .$$

HJB Equation

Define operators

$$
\begin{aligned}
\mathcal{L}_j \mathcal{V}_j &= \frac{\sigma_j^2 S^2}{2} D_{SS} \mathcal{V}_j + (r - \rho_j) S D_S \mathcal{V}_j - (r + \lambda_j) \mathcal{V}_j \\
\mathcal{J}_j \mathcal{V} &= \sum_{\substack{k=1 \\ k \neq j}}^{K} \frac{\lambda_{jk}}{\lambda_j} \mathcal{V}_k(\xi_{jk} S, \tau) \ ,
\end{aligned}
$$

where r is the risk free rate, D_S, D_{SS} differential operators.

American option price given by coupled system of PDEs:

$$
\min \left[\ \mathcal{V}_{j,\tau} - \mathcal{L}_j \mathcal{V}_j - \lambda_j \mathcal{J}_j \mathcal{V}, \ \ \mathcal{V}_j - \mathcal{V}^* \ \right] = 0 \ ; \ j = 1, \ldots, K \ ,
$$

$$
\mathcal{V}^* = \ \text{option payoff}
$$

Discretization

Define a set of nodes $\{S_1, S_2, ..., S_{i_{\max}}\}$, $\tau^n = n\Delta\tau$.

Let $V_{i,j}^n$ be the approximate solution at (S_i, τ^n), regime j.

Define vector of size $N = K \times i_{\max}$

$$V^n \quad = \quad [V_{1,1}^n, ..., V_{i_{\max},1}^n, \ldots, V_{1,K}^n, ..., V_{i_{\max},K}^n]'$$

Define $\mathcal{L}_j^h$ as the discrete form of $\mathcal{L}_j$. Use forward, backward, central differencing

$$\begin{aligned} (\mathcal{L}_j^h V^n)_{ij} &= \alpha_i V_{i-1,j}^n + \beta_i V_{i+1,j}^n - (\alpha_i + \beta_i + r + \lambda_i) V_{i,j}^n \\ & \alpha_i \geq 0 \;\; ; \;\; \beta_i \geq 0 \end{aligned}$$

- Central differencing as much as possible (Wang and Forsyth, 2008)
- Positive coefficient discretization

Jump Terms

The interesting parts of this problem are the coupling terms between regimes (regime $j \to k$):

$$[\mathcal{J}_j^h V^n]_{i,j} = \sum_{\substack{k=1 \\ k \neq j}}^{K} \frac{\lambda_{jk}}{\lambda_j} I_{i,j,k}^h V^n ,$$

where $I_{i,j,k}^h V^n$ is a linear interpolation operator in regime k

$$\begin{aligned} I_{i,j,k}^h V^n &= w V_{m,k}^n + (1-w) V_{m+1,k}^n , \qquad w \in [0,1] \\ &\simeq \mathcal{V}_k(\min(S_{\max}, \xi_{jk} S_i), \tau^n) . \end{aligned}$$

- Jumps localized to $[0, S_{\max}]$.

Explicit American constraint and regime coupling

Simplest approach (first order in time)

$$\left(\frac{1}{\Delta\tau} - \mathcal{L}_j^h\right)\hat{V}_{i,j}^{n+1} = \frac{V_{i,j}^n}{\Delta\tau} + \lambda_j[\mathcal{J}_j^h V^n]_{i,j}$$

$$V_{i,j}^{n+1} = \max(\hat{V}_{i,j}^{n+1}, \mathcal{V}_i^*) \qquad (1)$$

Proposition (Unconditional stability)

If a positive coefficient method is used to form $\mathcal{L}_j^h$, and linear interpolation is used in $\mathcal{J}_j^h$, then scheme (1) is unconditionally stable.

Proof.

This is easily proven using maximum analysis. □

Remark (Computational Cost)

This method requires solution of K decoupled tridiagonal systems in each timestep, and is consequently very inexpensive.

Direct Control Formulation

Rewrite HJB equation in control form, with control φ

$$\max_{\varphi \in \{0,1\}} \left[\Omega \, \varphi(\mathcal{V}^* - \mathcal{V}_j) - (1-\varphi)(\mathcal{V}_{j,\tau} - \mathcal{L}_j \mathcal{V}_j - \lambda_j \mathcal{J}_j \mathcal{V}) \right] = 0 \ , \quad (2)$$

- Scaling factor $\Omega > 0$ introduced

$\hookrightarrow$ No effect for exact solution

- Discretize (2), replace $\mathcal{L}_j$ by $\mathcal{L}_j^h$, etc.

$\hookrightarrow$Crank-Nicolson timestepping

- Scaling factor important for discretized equations

$\hookrightarrow$ Iterative methods compare two non-zero terms in $\max(\cdot)$ expression

$\hookrightarrow$ Note terms in $\max(\cdot)$ have different units

Penalty Formulation

Control form of penalized equations:

$$\mathcal{V}^{\varepsilon}_{j,\tau} \;=\; \mathcal{L}_j \mathcal{V}^{\varepsilon}_j + \lambda_j \mathcal{J}_j \mathcal{V}^{\varepsilon} + \max_{\varphi \in \{0,1\}} \left[\varphi \frac{(\mathcal{V}^* - \mathcal{V}^{\varepsilon}_j)}{\varepsilon} \right] .$$

This is mathematically equivalent to

$$\max_{\varphi \in \{0,1\}} \Bigg[\varphi \left\{ (\mathcal{V}^* - \mathcal{V}^{\varepsilon}_j) - \varepsilon (\mathcal{V}^{\varepsilon}_{j,\tau} - \mathcal{L}_j \mathcal{V}^{\varepsilon}_j - \lambda_j \mathcal{J}_j \mathcal{V}^{\varepsilon}) \right\} - (1-\varphi)(\mathcal{V}^{\varepsilon}_{j,\tau} - \mathcal{L}_j \mathcal{V}^{\varepsilon}_j - \lambda_j \mathcal{J}_j \mathcal{V}^{\varepsilon}) \Bigg] = 0 \; , \qquad (3)$$

- Discretize (3), replace $\mathcal{L}_j$ by $\mathcal{L}^h_j$, etc.

If $\varepsilon = \Delta\tau / C$, then this is a consistent discretization of the American option problem control problem

Viscosity Solution

Remark (Convergence to the Viscosity Solution)

Regime switching is a special case of the more general systems of Variational Inequalities (VIs) considered in (Crepey, 2010), where it is shown that such VIs have unique, continuous viscosity solutions. Note that the definition of a viscosity solution must be generalized for systems of PDEs.

Form of the Discretized Equations

If the American constraint, and regime coupling terms implicit
↪ A nonlinear set of equations are solved each timestep
↪ For both penalty and direct control methods, equations have same form.

$$[\mathcal{A}(Q) - \mathcal{B}(Q)]\ U \;=\; \mathcal{C}(Q)$$

$$\text{with} \quad Q_\ell \;=\; \arg\max_{Q \in Z} \Big[-\mathcal{A}(Q)U + \mathcal{B}(Q)U + \mathcal{C}(Q) \Big]_\ell \, .$$

where Q is the vector of controls

$$Q \;=\; [\varphi_{1,1}^{n+1}., \varphi_{i_{\max},1}^{n+1}, \ldots, \varphi_{1,K}^{n+1}, ..., \varphi_{i_{\max},K}^{n+1}]'$$

and U is the vector of solution values at τ^{n+1}

$$U \;=\; [V_{1,1}^{n+1}, ..., V_{i_{\max},1}^{n+1}, \ldots, V_{1,K}^{n+1}, ..., V_{i_{\max},K}^{n+1}]'$$

Splitting

Remark (Dependence on Control)

It is important to note that $[\mathcal{A}]_{\ell,m}, [\mathcal{B}]_{\ell,m}, [\mathcal{C}]_{\ell}$ depend only on Q_ℓ. Consequently, we can write the equations in more compact form

$$\max_{Q \in Z} \left\{ -\mathcal{A}(Q)U + \mathcal{B}(Q)U + \mathcal{C}(Q) \right\} = 0$$

$$Z \quad \textit{is the set of admissible controls} \qquad (4)$$

$\mathcal{A}$ contains only terms which couple nodes within the same regime.
$\hookrightarrow$ $\mathcal{A}$ is block tridiagonal, easy to factor.

$\mathcal{B}$ contains terms which couple different regimes.

- $[\mathcal{A} - \mathcal{B}]U = \mathcal{C}$ may be non-trivial to solve

Properties of $\mathcal{A}, \mathcal{B}$

Since a positive coefficient discretization is used, we have the following results

- $\mathcal{B} \geq 0$
- $(\mathcal{A} - \mathcal{B})$ and $\mathcal{A}$ are strictly diagonally dominant $\mathcal{M}$ matrices

Remark ($\mathcal{M}$ matrices)

We remind the audience that a matrix $\mathcal{A}$ is an $\mathcal{M}$ matrix if the offdiagonals are nonpositive, and $\mathcal{A}^{-1} \geq 0$.

Policy Iteration

Algorithm 1 Policy Iteration

$U^0 =$ Initial solution vector of size N
for $k = 0, 1, 2, \ldots$ until converge **do**
    $Q_\ell^k = \underset{Q_\ell \in Z}{\arg\max} \left\{ [-\mathcal{A}(Q) + \mathcal{B}(Q)]U^k + \mathcal{C}(Q) \right\}_\ell$
    Solve $[\mathcal{A}(Q^k) - \mathcal{B}(Q^k)]U^{k+1} = \mathcal{C}(Q^k)$
    if converged **then**
        break from the iteration
    end if
end for

Theorem (Convergence of Policy Iteration)

If $[\mathcal{A}(Q) - \mathcal{B}(Q)]$ is an $\mathcal{M}$ matrix, Policy iteration converges to the unique solution of equation (4).

Fixed Point Policy Iteration

We would like to avoid solving the full Policy matrix on each iteration.

Algorithm 2 Fixed Point-Policy Iteration

$U^0 =$ Initial solution vector of size N

for $k = 0, 1, 2, \ldots$ until converge **do**

$$Q_\ell^k = \arg\max_{Q_\ell \in Z} \left\{ -\left[\mathcal{A}(Q) - \mathcal{B}(Q)\right] U^k + \mathcal{C}(Q) \right\}_\ell$$

Solve $\mathcal{A}(Q^k)U^{k+1} = \mathcal{B}(Q^k)U^k + \mathcal{C}(Q^k)$

if converged **then**

break from the iteration

end if

end for

The following results are proven in (Huang, Forsyth, Labahn, 2012, SINUM)

Theorem (Convergence of Fixed Point-Policy Iteration)

If $\mathcal{A}(Q)$ is an $\mathcal{M}$ matrix, $\mathcal{B}(Q) \geq 0$, and $\exists C_4 < 1$ such that

$$\|\mathcal{A}(Q^k)^{-1}\mathcal{B}(Q^{k-1})\|_\infty \leq C_4 \quad \textit{and} \quad \|\mathcal{A}(Q^k)^{-1}\mathcal{B}(Q^k)\|_\infty \leq C_4 \ ,$$

then the fixed point-policy iteration in Algorithm 2 converges.

Corollary

The fixed point-policy iteration converges unconditionally for the penalty discretization and converges for the direct control discretization if the scaling factor satisfies

$$\Omega \quad > \quad \theta \cdot \hat{\lambda} \qquad \textit{where} \quad \hat{\lambda} = \max_j \lambda_j \ .$$

$$\theta = \begin{cases} 1 & \textit{fully implicit} \\ 1/2 & \textit{Crank Nicolson} \end{cases}$$

Local Policy Iteration

Salmi and Toivanen (2010) suggest the following idea for American options under jump diffusion (single regime)

- Lag jump terms one iteration
- Solve American LCP (Linear Complementarity) problem with frozen jump terms
- Convergence is proven using properties of LCP

We can write this algorithm within our general framework

- We can also obtain a general convergence result, applicable to our abstract control problem
- This idea can be applied to many HJB problems
- Regime switching is a special case of this general result

Local Policy Iteration

Algorithm 3 Local Policy Iteration

1: $U^0 =$ Initial solution vector of size N
2: **for** $k = 0, 1, 2, \ldots$ until converge **do**
3: Solve : $\max_{Q \in Z} \left\{ -\mathcal{A}(Q)U^{k+1} + \mathcal{B}(Q)U^k + \mathcal{C}(Q) \right\} = 0$
4: **if** converged **then**
5: break from the iteration
6: **end if**
7: **end for**

Remark (Local Nonlinear Solution)

Note that line 3 of this algorithm requires the solution of the nonlinear local control problem with the regime coupling terms (that is $\mathcal{B}U^k$) lagged one iteration.

Convergence: Local Policy Iteration

The following results are given in (Huang, Forsyth, Labahn, 2011, SISC)

Theorem (Convergence of Local Policy Iteration)

If $\mathcal{A}(Q)$ is an M matrix, $\mathcal{B}(Q) \geq 0$ and

$$\max_{Q \in Z} \|\mathcal{A}(Q)^{-1}\mathcal{B}(Q)\|_\infty \leq C_5 < 1 \ , \tag{5}$$

then the local policy iteration (3) converges. Furthermore, if U^* is the solution to equation (4), and $E^k = U^k - U^*$, then

$$\|E^{k+1}\|_\infty \leq C_5 \|E^k\|_\infty \ . \tag{6}$$

Convergence Rate for Regime Switching

Using the convergence theorem for local policy iteration and the properties of $\mathcal{A}(Q)$ and $\mathcal{B}(Q)$, we easily obtain the following

Corollary

Local Policy Iteration for either the penalty or direct control formulation of the regime switching, American option problem converges at the rate

$$\frac{\|E^{k+1}\|_\infty}{\|E^k\|_\infty} \leq \frac{\theta\hat{\lambda}\Delta\tau}{1+\theta(r+\hat{\lambda})\Delta\tau} \quad \textit{where} \quad \hat{\lambda} = \max_j \lambda_j$$

$$\theta = 1, \ \textit{fully implicit}; \ \theta = 1/2, \ \textit{CN}$$

• Note that since usually $\hat{\lambda}\Delta\tau$ is small (e.g. $< 10^{-2}$), convergence is rapid.

• However we are left with the problem of having to solve the local nonlinear problem at each iteration.

A Bad Idea: iterated optimal stopping

Recently, several authors have suggested an iterated optimal stopping approach in several contexts (including American options under regime switching)

- Freeze regime coupling terms
- Obtain entire solution (over all timesteps) with these frozen terms
- Update coupling terms and repeat

Note: each iteration requires storage of the entire solution (all mesh points) for all timesteps.

We can show that iterated optimal stopping

- Converges slower than Local Policy iteration
- Takes more storage

$\Rightarrow$ Forget it!

Numerical Example: Three Regimes

Payoff: Put, Butterfly

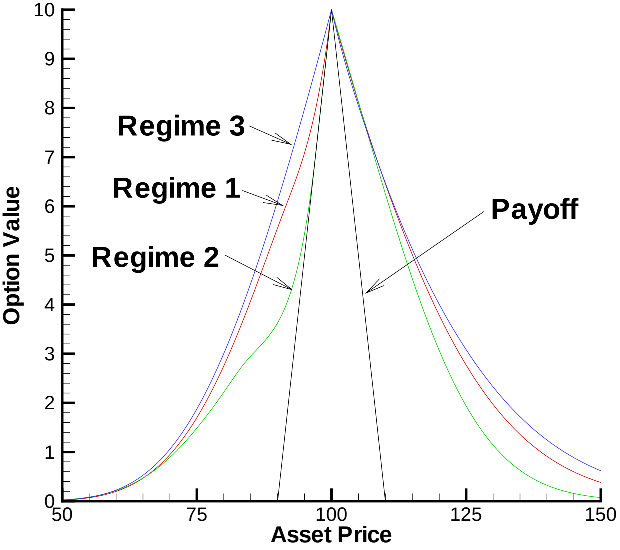
Expiry Time	.50
Exercise	American
Strike: Put	100
Butterfly Parameters K_1, K_2	90, 110
Risk free rate r	.02
Penalty Parameter ε	$10^{-6}\Delta\tau$
Scale factor Ω	$1/\varepsilon$
$S_{\max}$	5000
Relative Convergence Tolerance	10^{-8}

$$\boldsymbol{\lambda} = \begin{bmatrix} -3.2 & 0.2 & 3.0 \\ 1.0 & -1.08 & .08 \\ 3.0 & 0.2 & -3.2 \end{bmatrix} \; ; \; \boldsymbol{\xi} = \begin{bmatrix} 1.0 & 0.90 & 1.1 \\ 1.2 & 1.0 & 1.3 \\ 0.95 & 0.8 & 1.0 \end{bmatrix} \; ; \; \boldsymbol{\sigma} = \begin{bmatrix} .2 \\ .15 \\ .30 \end{bmatrix} \; .$$

$\lambda_{i,j}$ transition probability regime $i \rightarrow j$. $S \rightarrow S\xi_{i,j}$ when transition $i \rightarrow j$ occurs.

American Butterfly

- Contract can only be early exercised as a unit.
- Severe numerical test case used by several authors.



Sequence of Grids/timesteps

Refine Level	S Nodes	Timesteps (Put)	Timesteps (Butterfly)	Unknowns
0	51	34	34	153
1	101	66	67	303
2	201	130	132	603
3	401	256	261	1203
4	801	507	519	2403
5	1601	1010	1033	4803
6	3201	2015	2062	9603
7	6401	4023	4118	19203

Table: Grid/timestep data for convergence study, regime switching example. On each grid refinement, new fine grids are inserted between each two coarse grid nodes, and the timestep control parameter is halved.

Implicit coupling vs. explicit coupling (butterfly)

	Explicit Coupling		Fully Implicit		Crank Nicolson[1]	
Refine	Value	Ratio	Value	Ratio	Value	Ratio
0	3.9168371	N/A	4.4089970	N/A	4.4442032	N/A
1	4.1594341	N/A	4.4352395	N/A	4.4526625	N/A
2	4.2819549	1.98	4.4352395	1.8	4.4582809	1.5
3	4.3512466	1.8	4.4554933	2.4	4.4598662	3.5
4	4.3916690	1.7	4.4580458	2.3	4.4602286	4.4
5	4.4158036	1.7	4.4592283	2.2	4.4603215	3.9
6	4.4308340	1.6	4.4597991	2.1	4.4603452	3.9
7	4.4404872	1.55	4.4600781	2.04	4.4603512	3.9

Table: Value at $t = 0$, $S = 93$, Regime 2. Ratio is the ratio of successive changes as the grid is refined. Butterfly payoff.

Implicit coupling $3 - 5$ iterations per step

$\hookrightarrow$ More efficient than explicit coupling

[1]With Rannacher timestepping

Full Policy Iteration

Recall basic step of full policy iteration

$$[\mathcal{A}(Q^k) - \mathcal{B}(Q^k)]U^{k+1} = \mathcal{C}(Q^k)$$

compared with fixed point policy iteration

$$\mathcal{A}(Q^k)U^{k+1} = \mathcal{B}(Q^k)U^k + \mathcal{C}(Q^k)$$

Full policy iteration requires solving $[\mathcal{A} - \mathcal{B}]U^{k+1} = \mathcal{C}$
$\hookrightarrow$ This may be costly.

Fixed point policy requires solving $\mathcal{A}U^{k+1} = \mathcal{B}U^k + \mathcal{C}$
$\hookrightarrow$ This is easy (tridiagonal)

Full Policy Iteration vs. Fixed Point Policy

Linear Solution Method	Outer Itns per step	Inner Itns per Outer Itn	CPU time (Normalized)
	Full Policy Iteration, Algorithm 1		
Direct (Min degree)	2.4	N/A	48.50
GMRES (ILU(0))	2.4	1.91	4.85
Simple Iteration[2]	2.4	2.06	1.53
	Fixed Point Policy Iteration, Algorithm 2		
Direct (tridiagonal)	3.22	N/A	1.0

Table: Grid refinement level 5. Regime switching, American option, penalty formulation, put payoff. All methods used the same number of timesteps. Crank Nicolson timestepping used.

- Fixed point policy iteration is the winner

[2] $\mathcal{A}(Q^k)U^{m+1} = \mathcal{B}(Q^k)U^m + \mathcal{C}(Q^k)$; $m = 1, \ldots$

Comparison: Direct Control, Penalty Method

Refinement	Number of iterations/step			
	Direct Control			Penalty
	$\Omega = 100$	$\Omega = 10^4$	$\Omega = 10^6/(\Delta\tau)$	$1/\varepsilon = 10^6/(\Delta\tau)$
0	5.40	5.40	5.40	5.40
1	4.75	4.75	4.75	4.75
2	4.25	4.25	4.25	4.25
3	3.99	3.75	3.75	3.75
4	3.97	3.70	3.50	3.55
5	4.12	3.75	3.17	3.22
6	4.65	4.26	3.00	3.04
7	6.48	5.19	3.00	3.03

Table: Number of fixed point-policy iterations per timestep. Crank Nicolson timestepping used. American put. Fixed point policy iteration.

- Scaling is important for the Direct Control method

$\hookrightarrow$ A good choice is $\Omega = 1/\varepsilon$, where ε is the penalty parameter.

Floating Point Considerations

Both penalty and direct control methods require computing a term

$$\text{Penalty} \quad \frac{V_{i,j} - \mathcal{V}_i^*}{\varepsilon}$$

$$\text{Direct Control} \quad \Omega(V_{i,j} - \mathcal{V}_i^*)$$

$$\Omega = \frac{1}{\varepsilon} = \frac{C}{\Delta\tau}$$

Near the exercise region

- We subtract two nearly equal numbers, and divide by a small number
- Round off error will cause a nonconvergence of the iteration if C is too large

Floating Point Analysis: $\varepsilon = \Delta\tau / C$

We can estimate the maximum value for $C = C_{\max}$ which can be used.
$\hookrightarrow$ If $C > C_{\max}$, then nonlinear iteration will not converge, due to roundoff contamination.

$$C_{\max} \simeq \frac{tolerance}{2\delta}$$

$tolerance =$ is the nonlinear convergence tolerance $\simeq 10^{-8}$

$\delta =$ is the unit roundoff $\simeq 10^{-16}$; $\Rightarrow C_{\max} \simeq 10^{8}$

On the other hand if C is too small, then the fixed point policy iteration may not converge for the direct control method.

$$C_{\min} \simeq \hat{\lambda}\Delta\tau$$

Note that accuracy (for a fixed grid size) will degrade for the penalty method for C small
$\hookrightarrow$ We should choose C as large as possible for the penalty method

Effect of Ω or $1/\varepsilon$

Ω or $1/\varepsilon$	Direct Control	Penalty
$C/(\Delta\tau) = 10^9/(\Delta\tau)$	***	***
$10^8/(\Delta\tau)$	7.618333108	7.618333108
$10^7/(\Delta\tau)$	7.618333108	7.618333108
$10^6/(\Delta\tau)$	7.618333108	7.618333107
$10^5/(\Delta\tau)$	7.618333108	7.618333106
$10^4/(\Delta\tau)$	7.618333108	7.618333088
$10^3/(\Delta\tau)$	7.618333108	7.618332912
$10^2/(\Delta\tau)$	7.618333108	7.618331174
$10^1/(\Delta\tau)$	7.618333108	7.618314664
$1/(\Delta\tau)$	7.618333108	7.618144290
...	...	
$10^{-6}/(\Delta\tau)$	7.618333108	
$10^{-7}/(\Delta\tau)$	***	

Table: Value of the American put, $t = 0$, $S = 100$. $***$ indicates algorithm failed to converge. Level 5 grid refinement. Fixed point policy iteration. $C_{\max} = 10^8$, $C_{\min} = 10^{-3}$. Bounds are conservative.

Direct Control vs. Penalty

A good conservative rule of thumb: choose

$$\Omega = \frac{1}{\varepsilon} = \frac{C}{\Delta\tau}$$

$$C = 10^{-2} C_{\text{max}} = 10^{-2} \frac{(\textit{tolerance})}{2\delta}$$

- With this choice of C, both penalty and direct control have similar accuracy, number of iterations
- However, we can choose a very wide range for C for the direct control formulation, and still have virtually no effect on the solution
- The direct control method appears to be superior to the penalty method in this regard.

Local Policy vs. Fixed Point Policy

Method	Outer Itns per timestep	Inner Itns per Outer Itn	Normalized CPU time
American Butterfly			
Fixed point policy	3.23	N/A	1.0
Local policy	3.20	1.75	1.44
American Put			
Fixed point policy	3.17	N/A	1.0
Local policy	3.16	1.73	1.41

Table: Penalty formulation. Refinement level five.

• Number of outer iterations is almost the same for both Fixed Point Policy and Local Policy
↪ Does not seem worthwhile to solve the inner nonlinear problem to convergence
↪ Fixed Point Policy has the edge here

Conclusions

- Do not use iterated optimal stopping
  - $\rightarrow$ Local Policy Iteration converges faster and uses less storage
- Fixed Point Policy Iteration is the best method, Local Policy Iteration the runnerup
- Both Penalty and Direct Control methods require a dimensionless scaling parameter
  - We can estimate the sizes of this parameter (to ensure convergence in inexact arithmetic, see (Huang et al, APNUM 2013))
  - Direct Control method is less sensitive to this parameter compared with the Penalty method
- Our formulation is based on an abstract control problem
  - We can immediately apply these same ideas to more complex problems, e.g. singular control (Huang et. al (2012), IMAJ Num Anal), optimal switching (power plant operation).
- Local Policy Iteration may be useful for these more complex control problems