Hedging Costs for Variable Annuities under Regime-Switching

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Some History

In Canada, variable annuities have a long history

- Historically known as *segregated funds*
- In 2000, a group of us at UofWaterloo organized a workshop (in Toronto) on segregated funds

Me: *"bla, bla, bla, and now we determine the no-arbitrage price by solving the following PDE"*

Actuary from Insurance company X: *"But the market is not complete, and you can't hedge."*

Me: *"But you have to hedge your exposure to these guarantees."*

Actuary from X: *"The risk to us is nothing. Everybody knows, the market is never down over any ten year period."*

What happened?

- Insurance company X takes multi-billion dollar hit to balance sheet in 2008. Did not hedge variable annuities.

Cost of hedging

To be clear: I am going to discuss the cost of hedging of a particular class of variable annuities

- Guaranteed Lifelong Withdrawal and Death Benefits (GLWDB)
- Separate the *cost of hedging* from retail consumer behaviour
- Worst case for the hedger
  - $\rightarrow$ Holder carries out *loss maximizing withdrawal strategy*
- Unfortunately referred to as the *optimal* withdrawal strategy
- But it may not be optimal for anyone.

Overview of GLWDBs

Response to declining availability of defined benefit pension plans (DB).

Attempt to replicate a DB plan (i.e. lifelong guaranteed cash flows, with possible increase if market does well).

Contract bootstrapped by initial payment to insurance company, S_0

- Virtual withdrawal account $W(t)$ and death benefit account $D(t)$ set to S_0
- S_0 invested in risky assets, value $S(t)$.
- Fund management fee and guarantee fee withdrawn from risky asset account $S(t)$
- At a series of event times, t_i (usually yearly) various actions can be triggered.

Event actions at t_i

Withdrawal Event Holder can withdraw

$$\begin{aligned} \text{withdrawal amount} \quad &\in \quad [0, G * W(t_i^-)] \\ & \quad\quad G = \text{ spec'd contract rate} \\ & \quad\quad W = \text{ Withdrawal account} \end{aligned}$$

Death benefit account D and risky asset account S reduced by withdrawal amount.
Note: Contract amount can be withdrawn even if $S = 0$.

Surrender Event Holder withdraws an amount $> G * W(t_i^-)$

- Penalty charged as fraction of withdrawal
- $W(t_i^+), D(t_i^+)$ reduced proportionately
- Total amount withdrawn cannot exceed

$$G * W(t_i^-) + S(t_i^-)$$

Events c't'd

Ratchet Event Withdrawal account can ratchet up, i.e.

$$W(t_i^+) = \max(S(t_i^-), W(t_i^-)) \tag{1}$$

Note: W can never decrease[1], even if market crashes.

Bonus Event If holder does not withdraw, withdrawal account increased

$$\begin{aligned} W(t_i^+) &= (1+B)W(t_i^-) \\ B &= \text{ bonus rate} \end{aligned}$$

[1] except if the holder surrenders

Death Benefits, Assumptions

If you die, then your estate gets

$$\max(D(t), S(t)) \tag{2}$$

Estate guaranteed to get back initial payment (less withdrawals)

We assume

- Mortality risk is diversifiable, i.e. determine cost of hedging for a large number of contracts of similarly aged clients.
- Risky asset follows a *regime switching* process
  - Contracts are long-term (30 years)
  - Can impose views on possible future states of the economy
- Separate *the cost of hedging* from *retail consumer behaviour*

Computational Procedure

Let $V(S, W, D, t)$[2] be the cost of hedging of this guarantee.

Assume that no contract holders will be alive at $t = T$

$$V(S, W, D, T) = 0$$

Work backwards to today ($t = 0$).

- $t_{i+1}^- \rightarrow t_i^+$: solve regime switching PDE
  - Include fee withdrawals and death benefits
  - Cost of hedging $\rightarrow \mathbb{Q}$ measure.

Advance solution (backwards in time) across the event time

$$V(S^-, W^-, D^-, t_i^-) \;=\; V(S^+, W^+, D^+, t_i^+) + \text{ cash flows}$$

Then, solve PDE $t_i^- \rightarrow t_{i-1}^+$, etc.

[2] Assume single regime for ease of exposition

Across Event Times

Let γ be the impulse control applied to the system at t_i.

- Action due to the holder (e.g. surrender) or contract (e.g. ratchet)

Let

$$
\begin{aligned}
\mathbf{x} &= (S, W, D) = \text{ state} \\
\mathbf{x}^+(\mathbf{x}(t_i^-), \gamma(\mathbf{x}(t_i^-)) &= \text{ state after control is applied} \\
& \quad \text{conditional on } \mathbf{x} = \mathbf{x}(t_i^-) \\
C(\mathbf{x}(t_i^-), \gamma(\mathbf{x}(t_i^-)) &= \text{ cash flow after control is applied} \\
& \quad \text{conditional on } \mathbf{x} = \mathbf{x}(t_i^-)
\end{aligned}
$$

Move solution across event times

$$
V(\mathbf{x}, t_i^-) = V(\mathbf{x}^+(\mathbf{x}, \gamma), t_i^+) + C(\mathbf{x}, \gamma(\mathbf{x}))
$$

Fair fee

Let α be the fee for this guarantee

We can parameterize the solution as a function of this fee, i.e.

$$V = V(\mathbf{x}, t; \alpha)$$

The fee α^* which covers the cost of hedging can be determined by solving

$$V(S_0, S_0, S_0, 0; \alpha^*) = S_0$$

since no up-front fee is charged.[3]

[3] α^* found by a Newton iteration, each iteration requires a PDE solve.

Cost of hedging

Once the control γ is given

- Cost of hedging completely determined
- E.g. delta hedging can be carried out, delta determined from PDE solve under $\mathbb{Q}$ measure

Note: we have made no assumptions (up to now) about how the control γ is determined.

We have decoupled the specification of the control from the cost of hedging.

Worst Case Cost of Hedging

Under a worst case scenario, the cost of hedging is given by

$$V(\mathbf{x}, t_i^-) = \max_{\gamma} \left\{ V(\mathbf{x}^+(\mathbf{x}, \gamma), t_i^+) + C(\mathbf{x}, \gamma(\mathbf{x})) \right\}$$

No-arbitrage price if retail customers could buy/sell annuities.

But, the market is not complete

- Upper bound to the cost of hedging these annuities
- Unlikely that a retail customer would choose to follow this strategy[4]

[4] Empirical studies in Japanese market show moneyness of guarantee explains much policy holder behaviour (Knoller et al (2013))

More General Approach

Assume control is determined by a completely separate process.
Example:

- Assume policy holder acts so as to maximize
  - After tax cash flows (e.g. Moenig and Bauer)
  - A utility function of the cash flows
  - etc.

In a PDE context

- We solve a completely separate PDE system (under the $\mathbb{P}$ measure)
- This PDE system represents the value function being maximized by the policy holder, $\bar{V}(\mathbf{x}, t)$)
- Solve backwards in time $\rightarrow$ optimal control

Optimal control: consumption utility

Let $\mathbb{U}(\cdot)$ be a consumption utility function.

The control $\bar{\gamma}$ is determined by maximizing the policy holder value function $\bar{V}(\cdot)$

$$
\begin{aligned}
\bar{V}(\mathbf{x}, t_i^-) &= \bar{V}(\mathbf{x}^+(\mathbf{x}, \bar{\gamma}), t_i^+) + \mathbb{U}(C(\mathbf{x}, \bar{\gamma}(\mathbf{x}))) \\
\bar{\gamma} &= \arg\max_{\gamma} \left\{ \bar{V}(\mathbf{x}^+(\mathbf{x}, \gamma), t_i^+) + \mathbb{U}(C(\mathbf{x}, \gamma(\mathbf{x}))) \right\}
\end{aligned}
$$

This control is then fed into the cost of hedging $V(\cdot)$

$$
V(\mathbf{x}, t_i^-) = V(\mathbf{x}^+(\mathbf{x}, \bar{\gamma}), t_i^+) + C(\mathbf{x}, \bar{\gamma}(\mathbf{x}))
$$

Numerical Example: $\mathbb{Q}$ measure regime switching[5]

Parameter				Value
Volatility	σ_1	σ_2	0.0832	0.2141
Risk-free rate	r_1	r_2	0.0521	0.0521
Rate of transition	$q^{\mathbb{Q}}_{1\to 2}$	$q^{\mathbb{Q}}_{2\to 1}$	0.0525	0.1364
Initial regime	I			1
Initial investment	$S(0)$			100
Contract rate	G			0.05
Bonus rate	B			0.05
Initial age	x_0			65
Expiry time	T			57
Mortality data				Padiska et al (2005)
Ratchets				Triennial
Withdrawals				Annual

[5] Parameters from O'Sullivan and Moloney (2010), calibrated to FTSE options, January, 2007

Hedging Costs: Worst Case and Contract Rate

Case	Hedging fee (bps)			
	Worst	Contract	Worst	Contract
	Death Benefit		No Death Benefit	
Initial Regime Low Vol	54	48	27	19
Initial Regime High Vol	158	113	86	52

Table: Fair hedging fee: regime switching

- **Worst:** assume holder's strategy produces highest possible hedging cost
- **Contract:** assume holder always withdraws at rate $G * W$, i.e. no surrender, no bonus

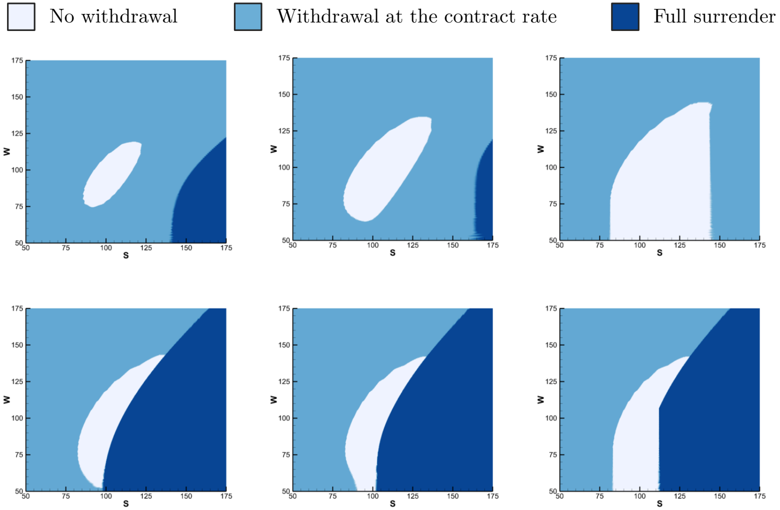

Figure: Observed loss-maximizing strategies at $D = 100$ under regime 2 (high vol). No ratchet. The subfigures, from top-left to bottom-right, correspond to $t = 1, 2, \ldots, 6$.

Control determined by utility consumption model

Assume HARA utility of consumption

$$\mathbb{U}(X) = \begin{cases} \log(aX + b) & p = 0 \\ \frac{1-p}{p}\left(\frac{aX}{1-p} + b\right)^{p} & 0 < p < 1 \\ aX & p = 1 \end{cases}$$

p, a, b are parameters.

Now, determine hedging fee, solve two systems of PDEs

- A PDE for $\bar{V}$ determines the withdrawal strategy (holder utility under $\mathbb{P}$ measure)
- B PDE for V determines the hedging cost, uses strategy from (A) ($\mathbb{Q}$ measure cash flows)

Utility based control: cost of hedging

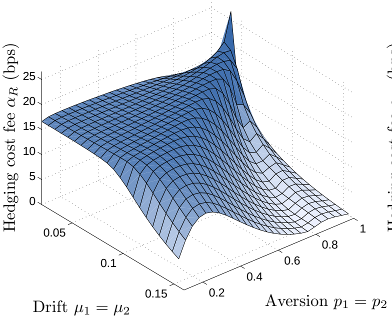

Figure: Left: initial regime low vol. Right: initial regime high vol. Effects of varying drift and risk-aversion on the hedging cost fee. No death benefit.

- Upper right maximum: parameters reduce to worst case hedging cost.
- Lower right corner: unrealistically large $\mathbb{P}$ measure drift.
- Flat region: always withdraw at contract rate G

Conclusions: Pricing GLWBs

- Cost of hedging is known once we know the control strategy of policy holder
  - Worst case cost of hedging can be determined by maximizing contract value
  - But this may not be optimal for the policy holder
- Separate control strategy from cost of hedging
- Use completely separate model to determine holder's optimal control strategy (e.g. maximize consumption utility)
- For a wide range of utility function parameters
  - Policyholder always withdraws at contract rate
  - Cost of hedging in this case significantly less than worst-case cost