State complexity of union and intersection of star on k regular languages☆

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Abstract

In the paper, we continue our study on state complexity of combined operations. We study the state complexities of $L_1^* \cup L_2^*$, $\bigcup_{i=1}^{k} L_i^*$, $L_1^* \cap L_2^*$, and $\bigcap_{i=1}^{k} L_i^*$ for regular languages L_i, $1 \le i \le k$. We obtain the exact bounds for these combined operations and show that the bounds are different from the mathematical compositions of the state complexities of their component individual operations.

Keywords: state complexity, combined operations, regular languages, finite automata

1. Introduction

State complexity is a type of descriptional complexity based on finite automaton model. It is the study of the number of states of finite automata. The research on state complexity can be recalled to 1950's [20]. Up to today, motivated by new applications of regular languages that require automata of very large sizes, state complexity has received increased attention. Many results on the state complexity of individual operations, such as union, intersection, catenation, star, reversal, shuffle, power, proportional removal, and cyclic shift have been obtained [1, 4, 5, 6, 11, 13, 14, 15, 19, 24, 25, 26].

On the basis of these results on individual operations, the research on state complexity of combined operations was initiated in 2007 [22]. This is because, in practice, the operation to be performed is often a combination of several individual operations in some order. Since 2007, there have been a number of publications on the topic of state complexity of combined operations. Most of

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the papers focused on the combinations composed of two individual operations, e.g. $(L_1 \cup L_2)^*$, $(L_1 \cap L_2)^*$, $(L_1L_2)^*$, $(L_1 \cup L_2)^R$, $(L_1 \cap L_2)^R$, $(L_1L_2)^R$, etc [2, 3, 7, 8, 9, 10, 16, 17, 22]. These combinations can be viewed as basic combined operations. The research on their state complexities is helpful for the work on the combined operations whose structures are more complex.

The state complexity of a combined operation is usually not a simple mathematical composition of the state complexities of its component individual operations, but much lower [22]. For example, let L be a regular language accepted by an n-state deterministic finite automaton (DFA). The state complexity of L^* is $\frac{3}{4}2^n$ and the state complexity of L^R of the reversal is 2^n. Then the mathematical composition of these two state complexities for the combined operation $(L^R)^*$ is $\frac{3}{4}2^{2^n}$. However, the state complexity of $(L^R)^*$ is only 2^n [8]. Recently, it has also been proved that there does not exist a general algorithm to compute the state complexities of combined operations even if all the state complexities of individual operations are known [23]. Thus, the state complexity of each combined operation should be studied separately.

In [22], the state complexities of two combined operations were investigated: $(L(M) \cup L(N))^*$ and $(L(M) \cap L(N))^*$, where M and N are m-state and n-state DFAs, respectively. An interesting question is what are the state complexities of these combined operations if we change the orders of the component individual operations. Therefore, in this paper, we study the state complexities of four particular combined operations that are $L_1^* \cup L_2^*$, $\bigcup_{i=1}^{k} L_i^*$, $L_1^* \cap L_2^*$ and $\bigcap_{i=1}^{k} L_i^*$. The combined operations $L_1^* \cup L_2^*$ and $L_1^* \cap L_2^*$ can be viewed as special cases of $\bigcup_{i=1}^{k} L_i^*$ and $\bigcap_{i=1}^{k} L_i^*$, respectively. Since they are not only basic combined operations but also the basis for the study on the latter two operations on k operands, we investigate their state complexities separately.

We show that the state complexities of $L_1^* \cup L_2^*$ and $L_1^* \cap L_2^*$ are both $\frac{9}{16}2^{m+n} - \frac{3}{4}2^m - \frac{3}{4}2^n + 2$ for $m, n \geq 2$, which are less than the mathematical compositions of the state complexities of their component operations by $\frac{3}{4}2^m + \frac{3}{4}2^n - 2$. The languages L_1 and L_2 are accepted by m-state and n-state DFAs, respectively.

For $\bigcup_{i=1}^{k} L_i^*$ and $\bigcap_{i=1}^{k} L_i^*$, we prove that their state complexities are also the same:

$$(\frac{3}{4})^k 2^g - \sum_{i=1}^{k}[\prod_{j=1}^{i-1}(\frac{3}{4}2^{n_j} - 1) \prod_{t=i+1}^{k}(\frac{3}{4}2^{n_t})] + 1$$

for $n_i \geq 2$, where L_i is an n_i-state DFA language, $1 \leq i \leq k$, $k \geq 2$, and $g = \sum_{i=1}^{k} n_i$. The state complexities are less than the mathematical compositions by $\sum_{i=1}^{k}[\prod_{j=1}^{i-1}(\frac{3}{4}2^{n_j} - 1) \prod_{t=i+1}^{k}(\frac{3}{4}2^{n_t})] - 1$.

In the next section, we introduce the basic definitions and notations used in the paper. In Sections 3, 4, 5 and 6, we investigate the state complexities of $L_1^* \cup L_2^*$, $\bigcup_{i=1}^{k} L_i^*$, $L_1^* \cap L_2^*$, and $\bigcap_{i=1}^{k} L_i^*$, respectively. In Section 7, we conclude the paper.

2. Preliminaries

A DFA is denoted by a 5-tuple $A = (Q, \Sigma, \delta, s, F)$, where Q is the finite set of states, Σ is the finite input alphabet, $\delta : Q \times \Sigma \to Q$ is the state transition function, $s \in Q$ is the initial state, and $F \subseteq Q$ is the set of final states. A DFA is said to be complete if $\delta(q, a)$ is defined for all $q \in Q$ and $a \in \Sigma$. All the DFAs we use in this paper are assumed to be complete. We extend δ to $Q \times \Sigma^* \to Q$ in the usual way.

In this paper, the state transition function δ is often extended to $\hat{\delta} : 2^Q \times \Sigma \to 2^Q$. The function $\hat{\delta}$ is defined by $\hat{\delta}(R, a) = \{\delta(r, a) \mid r \in R\}$, for $R \subseteq Q$ and $a \in \Sigma$. We just write δ instead of $\hat{\delta}$ if there is no confusion.

A word $w \in \Sigma^*$ is accepted by a finite automaton if $\delta(s, w) \cap F \neq \emptyset$. Two states in a DFA A are said to be *equivalent* if and only if for every word $w \in \Sigma^*$, if A is started in either state with w as input, it either accepts in both cases or rejects in both cases. A language is said to be *regular* if and only if it is accepted by a DFA. The language accepted by a DFA A is denoted by $L(A)$. The reader may refer to [12, 21, 27] for more details about regular languages and finite automata.

The *state complexity* of a regular language L, denoted by $sc(L)$, is the number of states of the minimal complete DFA that accepts L. The state complexity of a class S of regular languages, denoted by $sc(S)$, is the supremum among all $sc(L)$, $L \in S$. The state complexity of an operation on regular languages is the state complexity of the resulting languages from the operation as a function of the state complexity of the operand languages. Thus, in a certain sense, the state complexity of an operation is a worst-case complexity.

3. State complexity of $L_1^* \cup L_2^*$

We first consider the state complexity of $L_1^* \cup L_2^*$, where L_1 and L_2 are regular languages accepted by m-state and n-state DFAs, respectively. It has been proved that the state complexity of L_1^* is $\frac{3}{4}2^m$ and the state complexity of $L_1 \cup L_2$ is mn [18, 26]. The mathematical composition of them is $\frac{9}{16}2^{m+n}$. In the following, we show this upper bound of the state complexity of $L_1^* \cup L_2^*$ can be lowered.

Theorem 3.1. *For any m-state DFA $M = (Q_M, \Sigma, \delta_M, s_M, F_M)$ and n-state DFA $N = (Q_N, \Sigma, \delta_N, s_N, F_N)$ such that $|F_M - \{s_M\}| = k \geq 1$, $|F_N - \{s_N\}| = l \geq 1$, $m \geq 2$, $n \geq 2$, there exists a DFA of at most*

$$(2^{m-1} + 2^{m-k-1})(2^{n-1} + 2^{n-l-1}) - (2^{m-1} + 2^{m-k-1}) - (2^{n-1} + 2^{n-l-1}) + 2$$

states that accepts $L(M)^* \cup L(N)^*$.

Proof. Let $M = (Q_M, \Sigma, \delta_M, s_M, F_M)$ be a DFA of m states, $m \geq 2$. Denote $F_M - \{s_M\}$ by F_0. Then $|F_0| = k \geq 1$. Let $N = (Q_N, \Sigma, \delta_N, s_N, F_N)$ be another DFA of n states, $n \geq 2$. Denote $F_N - \{s_N\}$ by F_1 and $|F_1| = l \geq 1$. Let $M' = (Q_{M'}, \Sigma, \delta_{M'}, s_{M'}, F_{M'})$ be a DFA where

$$\begin{aligned}
&s_{M'} \notin Q_M \text{ is a new initial state,}\\
&Q_{M'} = \{s_{M'}\} \cup \{P \mid P \subseteq (Q_M - F_0) \ \& \ P \neq \emptyset\}\\
&\qquad \cup \{R \mid R \subseteq Q_M \ \& \ s_M \in R \ \& \ R \cap F_0 \neq \emptyset\},\\
&F_{M'} = \{s_{M'}\} \cup \{R \mid R \subseteq Q_M \ \& \ s_M \in R \ \& \ R \cap F_M \neq \emptyset\},
\end{aligned}$$

and for $R \subseteq Q_M$ and $a \in \Sigma$,

$$\delta_{M'}(s_{M'}, a) = \begin{cases} \{\delta_M(s_M, a)\}, & \text{if } \delta_M(s_M, a) \cap F_0 = \emptyset; \\ \{\delta_M(s_M, a)\} \cup \{s_M\}, & \text{otherwise,} \end{cases}$$

$$\delta_{M'}(R, a) = \begin{cases} \{\delta_M(R, a)\}, & \text{if } \delta_M(R, a) \cap F_0 = \emptyset; \\ \{\delta_M(R, a)\} \cup \{s_M\}, & \text{otherwise.} \end{cases}$$

It is clear that M' accepts $L(M)^*$. In the second term of the union for $Q_{M'}$ there are $2^{m-k} - 1$ states. And in the third term, there are $(2^k - 1)2^{m-k-1}$ states. So M' has $2^{m-1} + 2^{m-k-1}$ states in total.

Symmetrically, we can construct a DFA $N' = (Q_{N'}, \Sigma, \delta_{N'}, s_{N'}, F_{N'})$ of $2^{n-1} + 2^{n-l-1}$ states that accepts $L(N)^*$. Now we construct another DFA $A = (Q, \Sigma, \delta, s, F)$ where

$$\begin{aligned}
&s = \langle s_{M'}, s_{N'} \rangle,\\
&Q = \{\langle i, j \rangle \mid i \in Q_{M'} - \{s_{M'}\}, j \in Q_{N'} - \{s_{N'}\}\} \cup \{s\},\\
&\delta(\langle i, j \rangle, a) = \langle \delta_{M'}(i, a), \delta_{N'}(j, a) \rangle, \ \langle i, j \rangle \in Q, \ a \in \Sigma,\\
&F = \{\langle i, j \rangle \in Q \mid i \in F_{M'} \text{ or } j \in F_{N'}\}.
\end{aligned}$$

We can see that

$$L(A) = L(M') \cup L(N') = L(M)^* \cup L(N)^*.$$

Note $\langle s_{M'}, j \rangle \notin Q$, for $j \in Q_{N'} - \{s_{N'}\}$, and $\langle i, s_{N'} \rangle \notin Q$, for $i \in Q_{M'} - \{s_{M'}\}$, because there is no transition going into $s_{M'}$ and $s_{N'}$ in the DFA M' and N', respectively. There are $(2^{m-1} + 2^{m-k-1}) + (2^{n-1} + 2^{n-l-1}) - 2$ such states. Thus, the number of states of minimal DFA that accepts $L(M)^* \cup L(N)^*$ is no more than

$$(2^{m-1} + 2^{m-k-1})(2^{n-1} + 2^{n-l-1}) - (2^{m-1} + 2^{m-k-1}) - (2^{n-1} + 2^{n-l-1}) + 2.$$

□

If s_M and s_N are the only final states of M and N, respectively, $(k = l = 0)$, then $L(M)^* = L(M)$ and $L(N)^* = L(N)$.

Corollary 3.1. *For any m-state DFA $M = (Q_M, \Sigma, \delta_M, s_M, F_M)$ and n-state DFA $N = (Q_N, \Sigma, \delta_N, s_N, F_N)$, $m \geq 2$, $n \geq 2$, there exists a DFA A of at most*

$$\frac{9}{16}2^{m+n} - \frac{3}{4}2^m - \frac{3}{4}2^n + 2$$

states such that $L(A) = L(M)^* \cup L(N)^*$.

Proof. Let k and l be defined as in the previous proof. There are four cases in the following.

(I) $k = l = 0$. In this case, $L(M)^* = L(M)$ and $L(N)^* = L(N)$. Then A simply needs at most $m \cdot n$ states, which is less than $\frac{9}{16}2^{m+n} - \frac{3}{4}2^m - \frac{3}{4}2^n + 2$ when $m, n \geq 2$.

(II) $k \geq 1$, $l = 0$. We can see that $L(M)^* \cup L(N)^* = L(M)^* \cup L(N)$. The state complexity of $L(M)^* \cup L(N)$ has been proved to be $\frac{3}{4}2^m \cdot n - n + 1$ in [10] which is less than the upper bound in Corollary 3.1 when $m, n \geq 2$.

(III) $k = 0$, $l \geq 1$. The case is symmetric to Case (II).

(IV) $k \geq 1$, $l \geq 1$. The claim is clearly true by Theorem 3.1.

□

Next, we show that the upper bound $\frac{9}{16}2^{m+n} - \frac{3}{4}2^m - \frac{3}{4}2^n + 2$ can be reached when $m, n \geq 2$.

Theorem 3.2. *Given two integers $m \geq 2$, $n \geq 2$, there exist a DFA M of m states and a DFA N of n states such that any DFA accepting $L(M)^* \cup L(N)^*$ needs at least*

$$\frac{9}{16}2^{m+n} - \frac{3}{4}2^m - \frac{3}{4}2^n + 2$$

states.

Proof. Let $M = (Q_M, \Sigma, \delta_M, 0, \{m-1\})$ be a DFA, where $Q_M = \{0, 1, \ldots, m-1\}$, $\Sigma = \{a, b, c, d\}$ and the transitions of M are

$$\begin{aligned}
&\delta_M(i, a) = i + 1 \bmod m,\ i = 0, 1, \ldots, m-1,\\
&\delta_M(0, b) = 0,\ \delta_M(i, b) = i + 1 \bmod m,\ i = 1, \ldots, m-1,\\
&\delta_M(i, c) = i,\ i = 0, 1, \ldots, m-1,\\
&\delta_M(i, d) = i,\ i = 0, 1, \ldots, m-1.
\end{aligned}$$

The transition diagram of M is shown in Figure 1.
Let $N = (Q_N, \Sigma, \delta_N, 0, \{n-1\})$ be another DFA, where $Q_N = \{0, 1, \ldots, n-1\}$ and

$$\begin{aligned}
&\delta_N(i, a) = i,\ i = 0, 1, \ldots, n-1,\\
&\delta_N(i, b) = i,\ i = 0, 1, \ldots, n-1,\\
&\delta_N(i, c) = i + 1 \bmod n,\ i = 0, 1, \ldots, n-1,\\
&\delta_N(0, d) = 0,\ \delta_M(i, d) = i + 1 \bmod n,\ i = 1, \ldots, n-1.
\end{aligned}$$

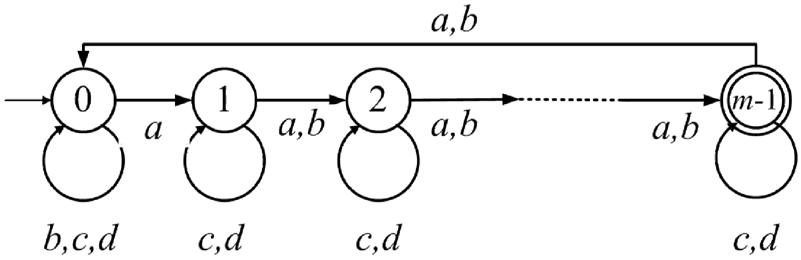

Figure 1: Witness DFA M for Theorems 3.2 and 5.2

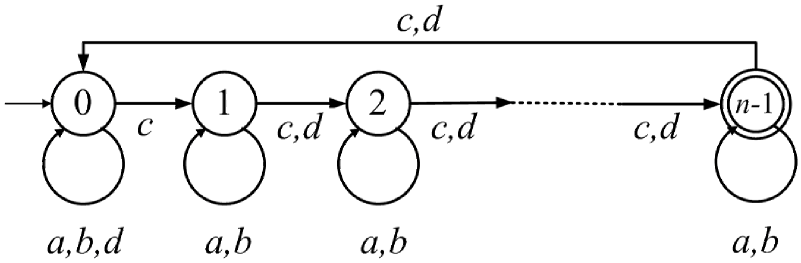

Figure 2: Witness DFA N for Theorems 3.2 and 5.2

The transition diagram of N is shown in Figure 2.

It has been proved in [26] that the minimal DFA that accepts the star of an m-state DFA language has $\frac{3}{4}2^m$ states in the worst case. M (N) is a modification of the worst-case example given in [26] by adding c- and d-loops (a- and b-loops) to every state. So we can design a $\frac{3}{4}2^m$-state, minimal DFA $M' = (Q_{M'}, \Sigma, \delta_{M'}, s_{M'}, F_{M'})$ that accepts $L(M)^*$, where

$$\begin{aligned}
&s_{M'} \notin Q_M \text{ is a new initial state,}\\
&Q_{M'} = \{s_{M'}\} \cup \{P \mid P \subseteq \{0, 1, \ldots, m-2\} \ \&\ P \neq \emptyset\}\\
&\qquad \cup \{R \mid R \subseteq \{0, 1, \ldots, m-1\} \ \&\ 0 \in R \ \&\ m-1 \in R\},\\
&F_{M'} = \{s_{M'}\} \cup \{R \in Q_{M'} \mid R \subseteq \{0, 1, \ldots, m-1\} \ \&\ m-1 \in R\},
\end{aligned}$$

and for $R \subseteq Q_M$, $R \in Q_{M'}$ and $a \in \Sigma$,

$$\delta_{M'}(s_{M'}, a) = \{\delta_M(0, a)\},$$

$$\delta_{M'}(R, a) = \begin{cases} \delta_M(R, a), & \text{if } m-1 \notin \delta_M(R, a);\\ \delta_M(R, a) \cup \{0\}, & \text{otherwise.} \end{cases}$$

In a similar way, a $\frac{3}{4}2^n$-state, minimal DFA $N' = (Q_{N'}, \Sigma, \delta_{N'}, s_{N'}, F_{N'})$ can be constructed to accept $L(N)^*$.

Then we construct the DFA $A = (Q, \Sigma, \delta, s, F)$ that accepts $L(M)^* \cup L(N)^*$

exactly as described in the proof of Theorem 3.1, where

$$s = \langle s_{M'}, s_{N'} \rangle,$$
$$Q = \{\langle i, j \rangle \mid i \in Q_{M'} - \{s_{M'}\}, j \in Q_{N'} - \{s_{N'}\}\} \cup \{s\},$$
$$\delta(\langle i, j \rangle, a) = \langle \delta_{M'}(i, a), \delta_{N'}(j, a) \rangle, \ \langle i, j \rangle \in Q, \ a \in \Sigma,$$
$$F = \{\langle i, j \rangle \in Q \mid i \in F_{M'} \text{ or } j \in F_{N'}\}.$$

Now we need to show that A is a minimal DFA.

(I) All the states in Q are reachable.
For an arbitrary state $\langle i, j \rangle$ in Q, there always exists a string $w_1 w_2$ such that $\delta(\langle s_{M'}, s_{N'} \rangle, w_1 w_2) = \langle i, j \rangle$, where

$$\delta_{M'}(s_{M'}, w_1) = i, \ w_1 \in \{a, b\}^*,$$
$$\delta_{N'}(s_{N'}, w_2) = j, \ w_2 \in \{c, d\}^*.$$

(II) Any two different states $\langle i_1, j_1 \rangle$ and $\langle i_2, j_2 \rangle$ in Q are distinguishable. Without loss of generality, assume that $i_1 \neq i_2$. Since $i_1, i_2 \in Q_{M'}$, there exists a word w such that $\delta_{M'}(i_1, w) \in F_{M'}$ and $\delta_{M'}(i_2, w) \notin F_{M'}$. Then the two states $\langle i_1, j_1 \rangle$ and $\langle i_2, j_2 \rangle$ can be distinguished by the string wd^n because

$$\delta(\langle i_1, j_1 \rangle, wd^n) \in F,$$
$$\delta(\langle i_2, j_2 \rangle, wd^n) \notin F,$$

Since all the states in A are reachable and distinguishable, the DFA A is minimal. Thus, any DFA that accepts $L(M)^* \cup L(N)^*$ has at least $\frac{9}{16}2^{m+n} - \frac{3}{4}2^m - \frac{3}{4}2^n + 2$ states. □

This result gives a lower bound for the state complexity of $L(M)^* \cup L(N)^*$. It coincides with the upper bound in Corollary 3.1. So we have the following Theorem 3.3.

Theorem 3.3. *For any integer $m \geq 2$, $n \geq 2$, $\frac{9}{16}2^{m+n} - \frac{3}{4}2^m - \frac{3}{4}2^n + 2$ states are both sufficient and necessary in the worst case for a DFA to accept $L(M)^* \cup L(N)^*$, where M is an m-state DFA and N is an n-state DFA.*

When $m = 1$, $n \geq 2$, $L(M)$ is either $\emptyset$ or Σ^*. Then the state complexity of $L(M)^* \cup L(N)^*$ is the same as that of $L(N)^*$ which is $\frac{3}{4}2^n$. When $m = n = 1$,

$$L(M)^* \cup L(N)^* = \begin{cases} \{\varepsilon\}, & \text{if } L(M) = L(N) = \emptyset; \\ \Sigma^*, & \text{otherwise.} \end{cases}$$

The state complexity of $L(M)^* \cup L(N)^*$ is 2 in this case.

4. State complexity of $\bigcup_{i=1}^{k} L_i^*$

In this section, we investigate the state complexity of $\bigcup_{i=1}^{k} L_i^*$, where L_i is a regular language accepted by an n_i-state DFA, $1 \le i \le k$, $k \ge 2$. Since the state complexity of L_i^* is $\frac{3}{4}2^{n_i}$ and the state complexity of $L_i \cup L_{i+1}$ is $n_i n_{i+1}$ [18, 26], the mathematical composition of them gives an upper bound $\prod_{i=1}^{k} \frac{3}{4}2^{n_i}$ to the state complexity of $\bigcup_{i=1}^{k} L_i^*$. In the following, we first show that the upper bound can also be lowered.

Theorem 4.1. *For any n_i-state DFA $N_i = (Q_{N_i}, \Sigma, \delta_{N_i}, s_{N_i}, F_{N_i})$ such that $|F_{N_i} - \{s_{N_i}\}| = l_i \ge 1$, $n_i \ge 2$, $1 \le i \le k$, $k \ge 2$, there exists a DFA of at most*

$$\prod_{i=1}^{k}(2^{n_i-1}+2^{n_i-l_i-1})-\sum_{i=1}^{k}[\prod_{j=1}^{i-1}(2^{n_j-1}+2^{n_j-l_j-1}-1)\prod_{t=i+1}^{k}(2^{n_t-1}+2^{n_t-l_t-1})]+1$$

states that accepts $\bigcup_{i=1}^{k} L(N_i)^*$.

Proof. Let $N_i = (Q_{N_i}, \Sigma, \delta_{N_i}, s_{N_i}, F_{N_i})$ be a DFA of n_i states, $n_i \ge 2$, $1 \le i \le k$, $k \ge 2$. Denote $F_{N_i} - \{s_{N_i}\}$ by T_i. Then $|T_i| = l_i \ge 1$. We construct the DFA $N_i' = (Q_{N_i'}, \Sigma, \delta_{N_i'}, s_{N_i'}, F_{N_i'})$ for $L(N_i)^*$ in a similar manner to the proof of Theorem 3.1, where

$$\begin{aligned}
&s_{N_i'} \notin Q_{N_i} \text{ is a new initial state,}\\
&Q_{N_i'} = \{s_{N_i'}\} \cup \{P \mid P \subseteq (Q_{N_i} - T_i) \ \&\ P \neq \emptyset\}\\
&\qquad \cup \{R \mid R \subseteq Q_{N_i} \ \&\ s_{N_i} \in R \ \&\ R \cap T_i \neq \emptyset\},\\
&F_{N_i'} = \{s_{N_i'}\} \cup \{R \mid R \subseteq Q_{N_i} \ \&\ s_{N_i} \in R \ \&\ R \cap F_{N_i} \neq \emptyset\},
\end{aligned}$$

and for $R \subseteq Q_{N_i}$, $R \in Q_{N_i'}$ and $a \in \Sigma$,

$$\delta_{N_i'}(s_{N_i'}, a) = \begin{cases} \{\delta_{N_i}(s_{N_i}, a)\}, & \text{if } \delta_{N_i}(s_{N_i}, a) \cap T_i = \emptyset;\\ \{\delta_{N_i}(s_{N_i}, a)\} \cup \{s_{N_i}\}, & \text{otherwise,} \end{cases}$$

$$\delta_{N_i'}(R, a) = \begin{cases} \{\delta_{N_i}(R, a)\}, & \text{if } \delta_{N_i}(R, a) \cap T_i = \emptyset;\\ \{\delta_{N_i}(R, a)\} \cup \{s_{N_i}\}, & \text{otherwise.} \end{cases}$$

Clearly, N_i' accepts $L(N_i)^*$. There are $2^{n_i-l_i} - 1$ states in the second term of the union for $Q_{N_i'}$ and $(2^{l_i} - 1)2^{n_i-l_i-1}$ states in the third term. So N_i' has $2^{n_i-1} + 2^{n_i-l_i-1}$ states in total.

Now let $A = (Q, \Sigma, \delta, s, F)$ be another DFA, where

$$\begin{aligned}
&s = \langle s_{N_1'}, s_{N_2'}, \dots, s_{N_k'}\rangle,\\
&Q = \{\langle p_1, p_2, \dots, p_k\rangle \mid p_i \in Q_{N_i'} - \{s_{N_i'}\}, 1 \le i \le k\} \cup \{s\},\\
&\delta(\langle p_1, p_2, \dots, p_k\rangle, a) = \langle \delta_{N_1'}(p_1, a), \delta_{N_2'}(p_2, a), \dots, \delta_{N_k'}(p_k, a)\rangle,\ a \in \Sigma,\\
&F = \{\langle p_1, p_2, \dots, p_k\rangle \in Q \mid \exists i (p_i \in F_{N_i'}, 1 \le i \le k)\}.
\end{aligned}$$

It is easy to see that

$$L(A) = \bigcup_{i=1}^{k} L(N_i') = \bigcup_{i=1}^{k} L(N_i)^*.$$

Note that the state $\langle p_1, \ldots, p_{i-1}, s_{N_i'}, p_{i+1}, \ldots, p_k \rangle \notin Q$ if $p_j \in Q_{N_j'} - \{s_{N_j'}\}$, $1 \le i, j \le k$, $j \ne i$, because there is no ingoing transition to the new initial state $s_{N_i'}$ in the DFA N_i'. There are

$$\sum_{i=1}^{k}[\prod_{j=1}^{i-1}(2^{n_j-1} + 2^{n_j-l_j-1} - 1) \prod_{t=i+1}^{k} (2^{n_t-1} + 2^{n_t-l_t-1})] - 1$$

such states in total. Thus, we obtain the upper bound shown in Theorem 4.1. □

Next, we consider the case when $l_i = 0$, $1 \le i \le k$, combine it with Theorem 4.1, and get a general upper bound.

Corollary 4.1. *Let $N_i = (Q_{N_i}, \Sigma, \delta_{N_i}, s_{N_i}, F_{N_i})$ be an arbitrary n_i-state DFA, where $n_i \ge 2$, $1 \le i \le k$, $k \ge 2$. Denote $\sum_{i=1}^{k} n_i$ by g. Then there exists a DFA of at most*

$$(\frac{3}{4})^k 2^g - \sum_{i=1}^{k}[\prod_{j=1}^{i-1}(\frac{3}{4}2^{n_j} - 1) \prod_{t=i+1}^{k} (\frac{3}{4}2^{n_t})] + 1$$

states that accepts $\bigcup_{i=1}^{k} L(N_i)^*$.

Proof. Let l_i be defined as in the proof of Theorem 4.1. When $l_i = 0$, s_{N_i} is the only final state in N_i and we know that $L(N_i)^* = L(N_i)$. Thus, in the construction of the resulting DFA A for $\bigcup_{i=1}^{k} L(N_i)^*$, the DFA N_i can be used to replace N_i', which reduces the size of the state set of A. When every $l_i \ge 1$, the corollary is true by Theorem 4.1. □

Next, we show that the upper bound in Theorem 4.1 is reachable when every $n_i \ge 2$.

Theorem 4.2. *Given an integer $n_i \ge 2$, there exists a DFA N_i of n_i states such that any DFA accepting $\bigcup_{i=1}^{k} L(N_i)^*$ needs at least*

$$(\frac{3}{4})^k 2^g - \sum_{i=1}^{k}[\prod_{j=1}^{i-1}(\frac{3}{4}2^{n_j} - 1) \prod_{t=i+1}^{k} (\frac{3}{4}2^{n_t})] + 1$$

states, where $1 \le i \le k$, $k \ge 2$, and $g = \sum_{i=1}^{k} n_i$.

Proof. Let $N_i = (Q_{N_i}, \Sigma, \delta_{N_i}, 0, \{n_i - 1\})$ be a DFA, where $Q_{N_i} = \{0, 1, \ldots, n_i - 1\}$, $\Sigma = \{a_{i,j} \mid 1 \le i \le k, j \in \{1, 2\}\}$ and the transitions of N_i are

$$\begin{aligned}
&\delta_{N_i}(p, a_{i,1}) = p + 1 \bmod n_i,\ p = 0, 1, \ldots, n_i - 1, \\
&\delta_{N_i}(0, a_{i,2}) = 0,\ \delta_{N_i}(p, a_{i,2}) = p + 1 \bmod n_i,\ p = 1, \ldots, n_i - 1, \\
&\delta_{N_i}(p, c) = p,\ c \in \Sigma - \{a_{i,1}, a_{i,2}\},\ p = 0, 1, \ldots, n_i - 1.
\end{aligned}$$

The transition diagram of N_i is similar to Figure 1.

As we mentioned before, it has been shown in [26] that the minimal DFA that accepts the star of an n_i-state DFA language has $\frac{3}{4}2^{n_i}$ states in the worst case. N_i is also a modification of the witness DFA shown in [26] by adding c-loops to every state, where $c \in \Sigma - \{a_{i,1}, a_{i,2}\}$. So we can design a $\frac{3}{4}2^{n_i}$-state, minimal DFA $N'_i = (Q_{N'_i}, \Sigma, \delta_{N'_i}, s_{N'_i}, F_{N'_i})$ that accepts $L(N_i)^*$, where

$$\begin{aligned}
&s_{N'_i} \notin Q_{N_i} \text{ is a new initial state,} \\
&Q_{N'_i} = \{s_{N'_i}\} \cup \{P \mid P \subseteq \{0, 1, \ldots, n_i - 2\} \ \&\ P \neq \emptyset\} \\
&\qquad \cup \{R \mid R \subseteq \{0, 1, \ldots, n_i - 1\} \ \&\ 0 \in R \ \&\ n_i - 1 \in R\}, \\
&F_{N'_i} = \{s_{N'_i}\} \cup \{R \in Q_{N'_i} \mid R \subseteq \{0, 1, \ldots, n_i - 1\} \ \&\ n_i - 1 \in R\},
\end{aligned}$$

and for $R \subseteq Q_{N_i}$, $R \in Q_{N'_i}$ and $a \in \Sigma$,

$$\delta_{N'_i}(s_{N'_i}, a) = \{\delta_{N_i}(0, a)\},$$

$$\delta_{N'_i}(R, a) = \begin{cases} \delta_{N_i}(R, a), & \text{if } n_i - 1 \notin \delta_{N_i}(R, a); \\ \delta_{N_i}(R, a) \cup \{0\}, & \text{otherwise.} \end{cases}$$

Then we construct the DFA $A = (Q, \Sigma, \delta, s, F)$ that accepts $\bigcup_{i=1}^{k} L(N_i)^*$ exactly as described in the proof of Theorem 4.1, where

$$\begin{aligned}
&s = \langle s_{N'_1}, s_{N'_2}, \ldots, s_{N'_k} \rangle, \\
&Q = \{\langle p_1, p_2, \ldots, p_k \rangle \mid p_i \in Q_{N'_i} - \{s_{N'_i}\}, 1 \le i \le k\} \cup \{s\}, \\
&\delta(\langle p_1, p_2, \ldots, p_k \rangle, a) = \langle \delta_{N'_1}(p_1, a), \delta_{N'_2}(p_2, a), \ldots, \delta_{N'_k}(p_k, a) \rangle,\ a \in \Sigma, \\
&F = \{\langle p_1, p_2, \ldots, p_k \rangle \in Q \mid \exists i (p_i \in F_{N'_i}, 1 \le i \le k)\}.
\end{aligned}$$

In the following, we show that the DFA A is minimal.

(I) All the states in Q are reachable.
For an arbitrary state $\langle p_1, p_2, \ldots, p_k \rangle$ in Q, there always exists a string $w_1 w_2 \cdots w_k$ such that $\delta(s, w_1 w_2 \cdots w_k) = \langle p_1, p_2, \ldots, p_k \rangle$, where

$$\delta_{N'_i}(s_{N'_i}, w_i) = p_i,\ w_i \in \{a_{i,1}, a_{i,2}\}^*,\ 1 \le i \le k.$$

(II) Any two different states $\langle p_1, p_2, \ldots, p_k \rangle$ and $\langle q_1, q_2, \ldots, q_k \rangle$ in Q are distinguishable.

Without loss of generality, we assume that $p_i \neq q_i$, $1 \le i \le k$. Then there exists a word w_i such that

$$\delta(\langle p_1, p_2, \ldots, p_k\rangle, a_{1,2}^{n_1} a_{2,2}^{n_2} \cdots a_{i-1,2}^{n_{i-1}} w_i a_{i+1,2}^{n_{i+1}} \cdots a_{k,2}^{n_k}) \in F,$$
$$\delta(\langle q_1, q_2, \ldots, q_k\rangle, a_{1,2}^{n_1} a_{2,2}^{n_2} \cdots a_{i-1,2}^{n_{i-1}} w_i a_{i+1,2}^{n_{i+1}} \cdots a_{k,2}^{n_k}) \notin F.$$

where $w_i \in \{a_{i,1}, a_{i,2}\}^*$, $\delta_{N_i'}(p_i, w_i) \in F_{N_i'}$ and $\delta_{N_i'}(q_i, w_i) \notin F_{N_i'}$.

Since all the states in A are reachable and pairwise distinguishable, A is a minimal DFA. Thus, any DFA that accepts $\bigcup_{i=1}^{k} L(N_i)^*$ has at least $(\frac{3}{4})^k 2^g - \sum_{i=1}^{k}[\prod_{j=1}^{i-1}(\frac{3}{4}2^{n_j} - 1)\prod_{t=i+1}^{k}(\frac{3}{4}2^{n_t})] + 1$ states, where $g = \sum_{i=1}^{k} n_i$. □

This lower bound coincides with the upper bound in Corollary 4.1. Thus, we obtain Theorem 4.3.

Theorem 4.3. *For any integer $n_i \ge 2$,*

$$(\frac{3}{4})^k 2^g - \sum_{i=1}^{k}[\prod_{j=1}^{i-1}(\frac{3}{4}2^{n_j} - 1)\prod_{t=i+1}^{k}(\frac{3}{4}2^{n_t})] + 1$$

states are both sufficient and necessary in the worst case for a DFA to accept $\bigcup_{i=1}^{k} L(N_i)^*$, where N_i is an n_i-state DFA, $1 \le i \le k$, $k \ge 2$, and $g = \sum_{i=1}^{k} n_i$.

5. State complexity of $L_1^* \cap L_2^*$

The state complexity of intersection on regular languages has been proved to be the same as that of union [18, 26]. Thus, the mathematical composition of the state complexities of star and intersection for $L(M)^* \cap L(N)^*$ is also $\frac{9}{16}2^{m+n}$. In this section, we show that the state complexity of $L(M)^* \cap L(N)^*$ is $\frac{9}{16}2^{m+n} - \frac{3}{4}2^m - \frac{3}{4}2^n + 2$ which is the same as the state complexity of the combined operation $L(M)^* \cup L(N)^*$.

Theorem 5.1. *For any m-state DFA $M = (Q_M, \Sigma, \delta_M, s_M, F_M)$ and n-state DFA $N = (Q_N, \Sigma, \delta_N, s_N, F_N)$ such that $|F_M - \{s_M\}| = k \ge 1$, $|F_N - \{s_N\}| = l \ge 1$, $m \ge 2$, $n \ge 2$, there exists a DFA of at most*

$$(2^{m-1} + 2^{m-k-1})(2^{n-1} + 2^{n-l-1}) - (2^{m-1} + 2^{m-k-1}) - (2^{n-1} + 2^{n-l-1}) + 2$$

states that accepts $L(M)^* \cap L(N)^*$.

Proof. We can construct the DFA A for $L(M)^* \cap L(N)^*$ which is the same as in the proof of Theorem 3.1, except that the set of final states of A is

$$F = \{\langle i, j\rangle \in Q \mid i \in F_{M'} \ \&\ j \in F_{N'}\}.$$

Thus, after removing the $(2^{m-1}+2^{m-k-1})+(2^{n-1}+2^{n-l-1})-2$ unreachable states $\langle s_{M'}, j\rangle \notin Q$, for $j \in Q_{N'}-\{s_{N'}\}$, and $\langle i, s_{N'}\rangle \notin Q$, for $i \in Q_{M'}-\{s_{M'}\}$, the number of states of A is still no more than

$$(2^{m-1}+2^{m-k-1})(2^{n-1}+2^{n-l-1})-(2^{m-1}+2^{m-k-1})-(2^{n-1}+2^{n-l-1})+2.$$

□

Now we consider the cases when M or N has no other final state except s_M or s_N. The following corollary shows a general upper bound of the state complexity of $L(M)^* \cap L(N)^*$.

Corollary 5.1. *For any m-state DFA $M=(Q_M,\Sigma,\delta_M,s_M,F_M)$ and n-state DFA $N=(Q_N,\Sigma,\delta_N,s_N,F_N)$, $m \geq 2$, $n \geq 2$, there exists a DFA A of at most*

$$\frac{9}{16}2^{m+n}-\frac{3}{4}2^m-\frac{3}{4}2^n+2$$

states such that $L(A)=L(M)^* \cap L(N)^*$.

Proof. Let k and l be $|F_M-\{s_M\}|$ and $|F_N-\{s_N\}|$, respectively. In a similar manner to the proof of Corollary 3.1, we have

$$L(M)^* \cap L(N)^* = \begin{cases} L(M) \cap L(N), & \text{if } k=l=0; \\ L(M)^* \cap L(N), & \text{if } k \geq 1 \text{ and } l=0; \\ L(M) \cap L(N)^*, & \text{if } k=0 \text{ and } l \geq 1; \end{cases}$$

Clearly, the third case above is symmetric to the second case. The state complexities of $L(M) \cap L(N)$ and $L(M)^* \cap L(N)$ are mn and $\frac{3}{4}2^m \cdot n - n + 1$, respectively [10, 18, 26]. They are both less than the upper bound shown in Corollary 5.1. When $k, l \geq 1$, the corollary also holds by Theorem 5.1. □

Next, we show that this general upper bound of state complexity of $L(M)^* \cap L(N)^*$ can be reached by some witness DFAs.

Theorem 5.2. *Given two integers $m \geq 2$, $n \geq 2$, there exist a DFA M of m states and a DFA N of n states such that any DFA accepting $L(M)^* \cap L(N)^*$ needs at least $\frac{9}{16}2^{m+n}-\frac{3}{4}2^m-\frac{3}{4}2^n+2$ states.*

Proof. We use the same DFAs M and N as in the proof of Theorem 3.2. Their transition diagrams are shown in Figure 1 and Figure 2, respectively. Construct the DFA $M'=(Q_{M'},\Sigma,\delta_{M'},s_{M'},F_{M'})$ for $L(M)^*$ and the DFA $N'=(Q_{N'},\Sigma,\delta_{N'},s_{N'},F_{N'})$ for $L(N)^*$ in the same way as in the proof of Theorem 3.2.

Then we construct the DFA $A=(Q,\Sigma,\delta,s,F)$ that accepts $L(M)^* \cap L(N)^*$ exactly as described in the proof of Theorem 3.2 except that

$$F=\{\langle i,j\rangle \in Q \mid i \in F_{M'} \ \&\ j \in F_{N'}\}.$$

In the following, we will prove that A is a minimal DFA. We omit the proof for the reachability of an arbitrary state $\langle i,j\rangle$ in A, because it is the same as that in the proof of Theorem 3.2. Next, let us prove that any two different states $\langle i_1,j_1\rangle$ and $\langle i_2,j_2\rangle$ of A are distinguishable.

1. $i_1 \neq i_2$.

   We can find a string $w_1 w_2$ such that

$$\delta(\langle i_1, j_1\rangle, w_1 w_2) \in F,$$
$$\delta(\langle i_2, j_2\rangle, w_1 w_2) \notin F,$$

   where

$$\delta_{M'}(i_1, w_1) \in F_{M'},\ \delta_{M'}(i_2, w_1) \notin F_{M'},\ w_1 \in \{a, b\}^*,$$
$$\delta_{N'}(j_1, w_2) \in F_{N'},\ w_2 \in \{c, d\}^*.$$

2. $i_1 = i_2$, $j_1 \neq j_2$.

   There exists a string $w_1 w_2$ such that

$$\delta(\langle i_1, j_1\rangle, w_1 w_2) \in F,$$
$$\delta(\langle i_2, j_2\rangle, w_1 w_2) \notin F,$$

   where

$$\delta_{M'}(i_1, w_1) \in F_{M'},\ w_1 \in \{a, b\}^*,$$
$$\delta_{N'}(j_1, w_2) \in F_{N'},\ \delta_{N'}(j_2, w_2) \notin F_{N'},\ w_2 \in \{c, d\}^*.$$

Since every state of A is reachable from its initial state and all the states are pairwise distinguishable, A is a minimal DFA with $\frac{9}{16}2^{m+n} - \frac{3}{4}2^m - \frac{3}{4}2^n + 2$ states which accepts $L(M)^* \cap L(N)^*$. □

This lower bound coincides with the upper bound in Corollary 5.1. Thus, the bounds are tight.

Theorem 5.3. *For any integer $m \geq 2$, $n \geq 2$, $\frac{9}{16}2^{m+n} - \frac{3}{4}2^m - \frac{3}{4}2^n + 2$ states are both sufficient and necessary in the worst case for a DFA to accept $L(M)^* \cap L(N)^*$, where M is an m-state DFA and N is an n-state DFA.*

When $m = 1$, $n \geq 2$, the state complexity of $L(M)^* \cap L(N)^*$ is the same as that of $L(N)^*$ which is $\frac{3}{4}2^n$, because $L(M)$ is either $\emptyset$ or Σ^* in this case. When $m = n = 1$,

$$L(M)^* \cap L(N)^* = \begin{cases} \{\varepsilon\}, & \text{if } L(M) = \emptyset \text{ or } L(N) = \emptyset; \\ \Sigma^*, & \text{otherwise.} \end{cases}$$

Then the state complexity of $L(M)^* \cap L(N)^*$ is clearly 2 when $m = n = 1$.

6. State complexity of $\bigcap_{i=1}^{k} L_i^*$

Next, we will investigate the state complexity of $\bigcap_{i=1}^{k} L_i^*$, where L_i is an n_i-state DFA language, $1 \leq i \leq k$, $k \geq 2$. The mathematical composition of the component operations of this combined operation is $\prod_{i=1}^{k} \frac{3}{4}2^{n_i}$ which is the same as that of $\bigcup_{i=1}^{k} L_i^*$. This upper bound can also be lowered.

Theorem 6.1. *For any n_i-state DFA $N_i = (Q_{N_i}, \Sigma, \delta_{N_i}, s_{N_i}, F_{N_i})$ such that $|F_{N_i} - \{s_{N_i}\}| = l_i \geq 1$, $n_i \geq 2$, $1 \leq i \leq k$, $k \geq 2$, there exists a DFA of at most*

$$\prod_{i=1}^{k}(2^{n_i-1}+2^{n_i-l_i-1})-\sum_{i=1}^{k}[\prod_{j=1}^{i-1}(2^{n_j-1}+2^{n_j-l_j-1}-1)\prod_{t=i+1}^{k}(2^{n_t-1}+2^{n_t-l_t-1})]+1$$

states that accepts $\bigcap_{i=1}^{k} L(N_i)^*$.

Proof. The DFA A for $\bigcap_{i=1}^{k} L(N_i)^*$ can be constructed in a same way as in the proof of Theorem 4.1, except that the set of final states of A is

$$F = \{\langle p_1, p_2, \ldots, p_k\rangle \in Q \mid \forall i(p_i \in F_{N_i'}, 1 \leq i \leq k)\}.$$

Thus, the number of states of A is no more than the upper bound shown in Theorem 6.1 which is the same as that for the state complexity of $\bigcup_{i=1}^{k} L(N_i)^*$ in Theorem 4.1. □

In a similar manner to the proof of Corollary 4.1, we obtain the following corollary on the basis of Theorem 6.1, by considering the cases when N_i has no other final state except s_{N_i} $(L(N_i)^* = L(N_i))$.

Corollary 6.1. *Let $N_i = (Q_{N_i}, \Sigma, \delta_{N_i}, s_{N_i}, F_{N_i})$ be an arbitrary n_i-state DFA, where $n_i \geq 2$, $1 \leq i \leq k$, $k \geq 2$. Denote $\sum_{i=1}^{k} n_i$ by g. Then there exists a DFA of at most*

$$(\frac{3}{4})^k 2^g - \sum_{i=1}^{k}[\prod_{j=1}^{i-1}(\frac{3}{4}2^{n_j} - 1)\prod_{t=i+1}^{k}(\frac{3}{4}2^{n_t})] + 1$$

states that accepts $\bigcap_{i=1}^{k} L(N_i)^*$.

Next, we show that the upper bound in Theorem 6.1 can be reached when every $n_i \geq 2$.

Theorem 6.2. *Given an integer $n_i \geq 2$, there exists a DFA N_i of n_i states such that any DFA accepting $\bigcap_{i=1}^{k} L(N_i)^*$ needs at least*

$$(\frac{3}{4})^k 2^g - \sum_{i=1}^{k}[\prod_{j=1}^{i-1}(\frac{3}{4}2^{n_j} - 1)\prod_{t=i+1}^{k}(\frac{3}{4}2^{n_t})] + 1$$

states, where $1 \le i \le k$*,* $k \ge 2$*, and* $g = \sum_{i=1}^{k} n_i$.

Proof. We use the same DFA N_i as in the proof of Theorem 4.2. Construct the DFA $N_i' = (Q_{N_i'}, \Sigma, \delta_{N_i'}, s_{N_i'}, F_{N_i'})$ for $L(N_i)^*$ in the same way as in the proof of Theorem 4.2.

Then we construct the DFA $A = (Q, \Sigma, \delta, s, F)$ that accepts $L(M)^* \cap L(N)^*$ exactly as described in the proof of Theorem 4.2 except that

$$F = \{\langle p_1, p_2, \ldots, p_k \rangle \in Q \mid \forall i (p_i \in F_{N_i'}, 1 \le i \le k)\}.$$

Now we will show that A is minimal. The proof for the reachability of an arbitrary state in A is omitted, because it is the same as that in the proof of Theorem 4.2. Thus, we prove that any two different states $\langle p_1, p_2, \ldots, p_k \rangle$ and $\langle q_1, q_2, \ldots, q_k \rangle$ of A are distinguishable in the following.

Without loss of generality, we assume that $p_i \ne q_i$, $1 \le i \le k$. Then there exists a word $w_1 w_2 \cdots w_k$ such that

$$\delta(\langle p_1, p_2, \ldots, p_k \rangle, w_1 w_2 \cdots w_k) \in F,$$
$$\delta(\langle q_1, q_2, \ldots, q_k \rangle, w_1 w_2 \cdots w_k) \notin F.$$

where

$$w_j \in \{a_{j,1}, a_{j,2}\}^*,\ \delta_{N_j'}(p_j, w_j) \in F_{N_j'},\ 1 \le j \le k,\ j \ne i,$$
$$w_i \in \{a_{i,1}, a_{i,2}\}^*,\ \delta_{N_i'}(p_i, w_i) \in F_{N_i'},\ \delta_{N_i'}(q_i, w_i) \notin F_{N_i'}.$$

Since all the states in A can be reached and are pairwise distinguishable, the DFA A is minimal. Thus, any DFA that accepts $\bigcap_{i=1}^{k} L(N_i)^*$ has at least $(\frac{3}{4})^k 2^g - \sum_{i=1}^{k} [\prod_{j=1}^{i-1} (\frac{3}{4} 2^{n_j} - 1) \prod_{t=i+1}^{k} (\frac{3}{4} 2^{n_t})] + 1$ states, where $g = \sum_{i=1}^{k} n_i$. □

This lower bound coincides with the upper bound in Corollary 6.1. Thus, we obtain the state complexity of $\bigcap_{i=1}^{k} L(N_i)^*$.

Theorem 6.3. *For any integer* $n_i \ge 2$*,*

$$(\frac{3}{4})^k 2^g - \sum_{i=1}^{k} [\prod_{j=1}^{i-1} (\frac{3}{4} 2^{n_j} - 1) \prod_{t=i+1}^{k} (\frac{3}{4} 2^{n_t})] + 1$$

states are both sufficient and necessary in the worst case for a DFA to accept $\bigcap_{i=1}^{k} L(N_i)^*$*, where* N_i *is an* n_i*-state DFA,* $1 \le i \le k$*,* $k \ge 2$*, and* $g = \sum_{i=1}^{k} n_i$.

7. Conclusion

In this paper, we studied the state complexities of union of star and intersection of star. We obtained the state complexities of four particular combined operations that are $L_1^* \cup L_2^*$, $\bigcup_{i=1}^{k} L_i^*$, $L_1^* \cap L_2^*$ and $\bigcap_{i=1}^{k} L_i^*$ where L_i is an n_i-state DFA language, $n_i \geq 2$, $1 \leq i \leq k$, and $k \geq 2$. The state complexities of these combined operations are all less than the mathematical compositions of the state complexities of their component individual operations.

Comparing with other known state complexities of combined operations, it is interesting to see that the state complexities of $L_1^* \cup L_2$ and $L_1^* \cap L_2$ are the same, and $L_1^* \cup L_2^*$ and $L_1^* \cap L_2^*$ share the same state complexity, whereas the state complexities of $(L_1 \cup L_2)^*$ and $(L_1 \cap L_2)^*$ are different.

One possible, future topic could be the state complexities of $\bigcup_{i=1}^{k} L_i^*$ and $\bigcap_{i=1}^{k} L_i^*$ on a smaller, fixed alphabet when k is also fixed. We also expect more results on the state complexities of combined operations on k regular languages, which are more general and closer to the nature of combined operations.

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