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RELATIVE WATSON-CRICK PRIMITIVITY OF WORDS

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ABSTRACT

We introduce the concept of relative Watson-Crick primitivity of words and its generalization, the relative θ-primitivity of words, where θ is a morphic or an antimorphic involution. Similar to relatively prime integers which do not share any common factors, we call two words u and v relatively θ-primitive if they do not share a common θ-primitive root. We study some combinatorial properties of relatively θ-primitive words, as well as establish relations between each of the two words u and v and the result of some binary word operation between u and v, from this perspective.

Keywords: Primitive words, relatively primitive words, θ-primitive words, antimorphic involution

1. Introduction

Periodicity and its opposite, primitivity of words, are fundamental properties of words in combinatorics on words and formal language theory. In addition, the detection of repetitions in strings plays an important role in, e.g., pattern matching and text compression [2, 3, 14]. On the other hand, tandem repeats in DNA, that is, sequences of two or more contiguous, approximate copies of a pattern, occur in the genomes of both eukaryotic and prokaryotic organisms, and have biological and medical significance. In this paper we bring together these concepts from two different fields with the notion of relatively prime numbers from number theory, to define the relative Watson-Crick primitivity of words.

Recall that, in the framework of formal language theory, a single DNA strand – a sequence of nucleotides that can be of four different kinds, Adenine, Cytosine, Guanine or Thymine –, can be modelled as a word w over the alphabet $\Delta = \{A, C, G, T\}$. DNA strands can be either single-stranded or double-stranded, with the latter being formed when two single strands that are Watson-Crick complementary bind to each other, to form the familiar double-helix. The fact that the Watson-Crick complement of a DNA single strand is its *reverse complement*, wherein A complements and binds to T and viceversa, while G complements and binds to C and viceversa, has been traditionally modelled by a morphic involution combined with the mirror image [13] or, most often, by an antimorphic involution θ [6]. An antimorphic involution is a function θ on an alphabet Σ^* that is an antimorphism, $\theta(xy) = \theta(y)\theta(x)$, $\forall x, y \in \Sigma^*$ (this models the "reverse" part), and an involution, $\theta^2 = id$, the identity (this models the "complement" part, in that the complement of the complement of a letter is the original). With this formalism, the Watson-Crick complement of a word u is its image $\theta_\Delta(u)$ through the involution $\theta_\Delta : \Delta \longrightarrow \Delta$ defined as $\theta_\Delta(A) = T$ and $\theta_\Delta(G) = C$, and naturally extended to an antimorphism of Σ^*. In this paper we call an (anti)morphic involution an involution that is either a morphism or an antimorphism.

Due to the fact that an (anti)morphic involution is a bijection, a word u and its image $\theta(u)$ through an (anti)morphic involution θ are retrievable from one another, and can be considered in some sense "identical". This idea, of extending the notion of identity to more general functions such as (anti)morphic involutions, led to natural generalizations of classical notions in combinatorics of words to notions such as pseudo-periodicity and pseudo-primitivity [4, 11], pseudo-palindromes [1, 9], Watson-Crick conjugate and commutative words [8], pseudoknot-bordered words [10], pseudo-repetitions [5], etc.

In this paper, we introduce the concept of *relative Watson-Crick primitivity* of words and its generalization, the *relative θ-primitivity of words*, where θ is any (anti)morphic involution. In the same way in which the concept of primitive word was inspired by the notion of prime number, this is a concept inspired by the notion of relatively prime integers in number theory. In this setting, two words are said to be relatively θ-primitive if they do not share a common θ-primitive root. Note that, if θ is the identity function on Σ, extended to a morphic involution on Σ^*, then we obtain a particular case, that of *relatively primitive words*, i.e., words that do not share a common primitive root.

The paper is organized as follows: In Section 2, we introduce some basic definitions, notations and results that are used throughout the paper. In Section 3, we introduce the concept of relatively θ-primitive words for (anti)morphic involutions θ, and study some combinatorial properties of this relation. The relation between the result of a binary word operation between two words u and v, and either u or v, is studied in Section 4, for various binary operations such as shuffle, perfect shuffle, and θ-catenation. We end with concluding remarks in Section 5.

2. Basic definitions and notations

An alphabet Σ is a finite non-empty set of symbols, and Σ^* denotes the set of all words over Σ including the empty word λ, while Σ^+ is the set of all non-empty words over Σ. The length of a word $u \in \Sigma^*$ (i.e., the number of symbols in a word) is denoted by $|u|$. We denote by $|u|_a$ the number of occurrences of a letter a in u. By Σ^m we denote the set of all words of length $m > 0$ over Σ. A language L is a subset of Σ^*.

The complement of a language $L \subseteq \Sigma^*$ is $L^c = \Sigma^* \backslash L$. A word is called *primitive* if it cannot be expressed as a power of another word. Let Q denote the set of all primitive words. For every word $w \in \Sigma^+$ there exists a unique word $\rho(w) \in \Sigma^+$, called the primitive root of w, such that $\rho(w) \in Q$ and $w = \rho(w)^n$ for some $n \geq 1$.

A function $\theta : \Sigma^* \to \Sigma^*$ is said to be a *morphism* if for all words $u, v \in \Sigma^*$ we have that $\theta(uv) = \theta(u)\theta(v)$, an *antimorphism* if $\theta(uv) = \theta(v)\theta(u)$, and an *involution* if θ^2 is an identity on Σ^*. The (anti)morphism θ is said to be *literal* if $|\theta(a)| = 1$ for all $a \in \Sigma$, *uniform* if $|\theta(a)| = |\theta(b)|$ for all $a, b \in \Sigma$, and *non-erasing* if $\theta(a) \neq \lambda$ for any $a \in \Sigma$. A θ-*power* of a word u is a word of the form $w = u_1 u_2 \cdots u_n$ for $n \geq 1$ where $u_1 = u$ and $u_i \in \{u, \theta(u)\}$ for $2 \leq i \leq n$. A word is called θ-*primitive* if it cannot be expressed as a θ-power of another word [4, 11]. Let Q_θ denote the set of all θ-primitive words. As shown in [4], for every word $w \in \Sigma^+$ and (anti)morphic involution θ, there exists a unique θ-primitive word $t \in \Sigma^+$ such that $w \in t\{t, \theta(t)\}^*$, i.e., $\rho_\theta(w) = t$.

For an (anti)morphic involution θ, a word $u \in \Sigma^*$ is called a θ-*palindrome* [9] if $u = \theta(u)$, and P_θ denotes the set of all θ-palindromes. Given two words $u, v \in \Sigma^+$, we say that u θ-*commutes* with v if $uv = \theta(v)u$, see [8]. The word u is said to be a θ-*conjugate* of v if there exists $w \in \Sigma^+$ such that $uw = \theta(w)v$, see [8]. Note that, unlike their classical counterparts which are symmetric, the relation "is a θ-conjugate of" is symmetric for morphic involutions θ [8], but not symmetric for antimorphic involutions θ, and the relation "θ-commutes with" is in general not symmetric.

For a binary relation $\mathcal{R}$, a language L is said to be $\mathcal{R}$-*independent* if for any $u, v \in L$, $u\mathcal{R}v$ implies $u = v$. Let us recall the following results regarding conjugacy, commutativity, θ-conjugacy, and θ-commutativity of words, which are used in this paper.

Proposition 1. *[12] If $uv = vw$ where $u, v, w \in \Sigma^*$ and $u \neq \lambda$, then $u = xy$, $v = (xy)^k x$, $w = yx$ for some $x, y \in \Sigma^*$ and $k \geq 0$.*

Proposition 2. *[12] If $uv = vu$ where $u, v \in \Sigma^+$, then u and v are powers of a common word.*

The following analogous results from [8] provide a characterization of words that θ-commute or are θ-conjugate.

Proposition 3. *[8] Let $u, w \in \Sigma^+$ be two words such that u is a θ-conjugate of w, that is, $uv = \theta(v)w$ for some $v \in \Sigma^+$.*

(I) *If θ is a morphic involution, then there exists $x, y \in \Sigma^*$ such that $u = xy$ and one of the following holds:*
  (a) $w = y\theta(x)$ and $v = (\theta(xy)xy)^i \theta(x)$ for some $i \geq 0$.
  (b) $w = \theta(y)x$ and $v = (\theta(xy)xy)^i \theta(xy)x$ for some $i \geq 0$.

(II) *If θ is an antimorphic involution then either $u = \theta(w)$, or there exist $x, y \in \Sigma^*$ such that $u = xy$ and $w = y\theta(x)$.*

Proposition 4. *[8] Let $u, v \in \Sigma^+$ be two words such that u θ-commutes with v, that is, $uv = \theta(v)u$.*

(i) *If θ is a morphic involution, then one of the following holds:*
(a) $u = \alpha^n$, $v = \alpha^m$ *for* $\alpha \in P_\theta$, $m, n \geq 1$.
(b) $u = \theta(\alpha)[\alpha\theta(\alpha)]^n$, $v = [\alpha\theta(\alpha)]^m$ *for some* $m \geq 1$ *and* $n \geq 0$.

(ii) *If θ is an antimorphic involution, then* $u = \alpha(\beta\alpha)^n$, $v = (\beta\alpha)^m$ *for some* $\alpha, \beta \in P_\theta$, $m \geq 1$ *and* $n \geq 0$.

3. Relatively θ-primitive words

In this section we introduce and investigate the concept of relatively θ-primitive words for a given (anti)morphic involution θ.

Definition 5. Let $u, v \in \Sigma^*$ and let θ be an (anti)morphic involution on Σ^*. Then $(u, v)_\theta$ is defined as:

$$(u, v)_\theta = \begin{cases} x & \text{if } \rho_\theta(u) = \rho_\theta(v) = x, \quad x \in \Sigma^+ \\ \lambda & \text{otherwise.} \end{cases}$$

Note that if $(u, v)_\theta \neq \lambda$, then $(u, v)_\theta$ is the common θ-primitive root of u and v.

Definition 6. Let $u, v \in \Sigma^*$ and let θ be an (anti)morphic involution on Σ^*. The words u, v are said to be relatively θ-primitive if $(u, v)_\theta = \lambda$, and this is denoted by $u \perp_\theta v$.

For $x \in \Sigma^+$, if $\rho_\theta(u) = \rho_\theta(v) = x$, then u and v are not relatively θ-primitive, and this is denoted by $u \not\perp_\theta v$.

If θ_Δ is the Watson-Crick antimorphic involution on Δ^*, where Δ is the DNA alphabet, then two words that are relatively θ_Δ-primitive are called *relatively Watson-Crick primitive*. If θ is the identity function on Σ extended to a morphism of Σ^*, then two words that are relatively θ-primitive are called *relatively primitive*, and this is denoted by $u \perp v$.

Example 7. Let $\Sigma = \{a, b, c\}$ and θ be an antimorphic involution such that $\theta(a) = b$ and vice versa, and $\theta(c) = c$. Let $u = abccababc$. Then for $v_1 = abcc$, $(u, v_1)_\theta = \lambda$, i.e., u and v_1 are relatively θ-primitive. However, for $v = abcabc$, $(u, v)_\theta = abc$ since $\rho_\theta(u) = \rho_\theta(v) = abc$, so u and v are not relatively θ-primitive.

Observe that, for all $u \in \Sigma^*$, $(u, \lambda)_\theta = (\lambda, u)_\theta = \lambda$. The following lemma follows directly from Definition 6.

Lemma 8. *Let $x, y, u, v \in \Sigma^+$ and let θ be an (anti)morphic involution of Σ^*. Then the following statements hold.*

(i) $(x, x)_\theta = y$ *where* $\rho_\theta(x) = y$.
(ii) $(u, v)_\theta = v$ *iff* $\rho_\theta(u) = v$.
(iii) *If* $(u, v)_\theta = x$, *then* $1 \leq |x| \leq min\{|u|, |v|\}$.
(iv) $(u, v)_\theta = (v, u)_\theta$.

(V) *For all $u, v \in Q_\theta$ such that $u \neq v$, $u \perp_\theta v$.*
(VI) *If $(u, v)_\theta = x$ then $(u^i, v^j)_\theta = x$ for all $i, j \geq 1$.*

From Lemma 8 (I) , it is clear that the relation $\perp_\theta$ is not reflexive on Σ^*. By Lemma 8 (IV), the relation $\perp_\theta$ is symmetric on Σ^*. We also note that the relation $\perp_\theta$ is not transitive on Σ^*. Indeed, for $u, v, w \in \Sigma^+$, $u \perp_\theta v$ and $v \perp_\theta w$ do not necessarily imply that $u \perp_\theta w$, as demonstrated by the following example.

Example 9. Let $\Sigma = \{a, b, c\}$ and θ be an antimorphic involution such that $\theta(a) = b$ and vice-versa, and $\theta(c) = c$. Then for $u = a$, $v = c$ and $w = ab$, $u \perp_\theta v$, $v \perp_\theta w$ but $u \not\perp_\theta w$ as $\rho_\theta(u) = \rho_\theta(w) = a$.

However, the relation $\perp_\theta$ is transitive on Q_θ. Note that the relation $\not\perp_\theta$ is reflexive on Σ^+ and symmetric on Σ^*. The following lemma shows that the relation $\not\perp_\theta$ is transitive.

Lemma 10. *Let θ be an (anti)morphic involution over Σ^*. If $(u, v)_\theta = x$, and $(v, w)_\theta = y$, with $x, y \in \Sigma^+$, then $x = y$ and $(u, w)_\theta = x$.*

Proof. Since $(u, v)_\theta = x$, $\rho_\theta(u) = \rho_\theta(v) = x$. Also $(v, w)_\theta = y$ implies $\rho_\theta(v) = \rho_\theta(w) = y$, where $x, y \in \Sigma^+$. Since the θ-primitive root of a word is unique, this further implies that $x = y$ and $(u, w)_\theta = x$. □

It is clear from (I) and (IV) of Lemma 8, and Lemma 10 that the relation $\not\perp_\theta$ is an equivalence relation on Σ^+.

The concept of a common θ-primitive root can be extended to $n \geq 2$ words, by defining the common θ-primitive root of words $u_1, u_2, \ldots, u_n \in \Sigma^+$ to be $(u_1, u_2, \ldots, u_n)_\theta = x$ iff $\rho_\theta(u_i) = x$ for all $1 \leq i \leq n$, and to be λ if no such x exists. The following result is an immediate consequence of Lemma 10.

Corollary 11. *Let θ be an (anti)morphic involution over Σ^*. If $(u_1, u_2)_\theta = v_1$, $(u_2, u_3)_\theta = v_2$, …, $(u_{n-1}, u_n)_\theta = v_{n-1}$, where $n \geq 2$ and $v_i \in \Sigma^+$ for $1 \leq i \leq n-1$ then $v_1 = v_2 = \cdots = v_{n-1} = (u_1, u_2, \ldots, u_n)_\theta$.*

In the following proposition, we prove that if $y \in \Sigma^+$ and $x \in Q_\theta$, then either x and y are relatively θ-primitive, or x is the θ-primitive root of y.

Proposition 12. *Let θ be an (anti)morphic involution over Σ^*, $x \in Q_\theta$ and let y be a word over Σ^+. Then either $x \perp_\theta y$ or $\rho_\theta(y) = x$.*

Proof. If $y \in Q_\theta$ then either $x = y$ or $x \perp_\theta y$. If $y \notin Q_\theta$ then either $x \perp_\theta y$ or $\rho_\theta(y) = x$. □

It was shown in [4] that every θ-primitive word is primitive but the converse is not always true. Similarly, we now prove that if x and y are relatively θ-primitive, then x and y are relatively primitive.

Lemma 13. *Let θ be an (anti)morphic involution over Σ^*. If $x \perp_\theta y$ then $x \perp y$.*

Proof. If $x \not\perp y$ then $x = p^m$ and $y = p^n$ for some $p \in \Sigma^+$ and $m, n \geq 1$. This implies that $\rho_\theta(x) = \rho_\theta(y) = \rho_\theta(p)$, which further implies that $(x, y)_\theta = \rho_\theta(p)$, with $p \in \Sigma^+$, that is, $x \not\perp_\theta y$ - a contradiction. Hence $x \perp y$. □

The converse of Lemma 13 need not hold, as demonstrated by the following example.

Example 14. Let $\Sigma = \{a, b, c\}$ and θ be an antimorphic involution such that $\theta(a) = c$ and vice versa, and $\theta(b) = b$. Let $x = acbbac$ and $y = acb$. Note that, $x, y \in Q$ and hence $x \perp y$. But $x = y\theta(y)$ for $y \in Q_\theta$ and hence $\rho_\theta(x) = y$, i.e., $x \not\perp_\theta y$.

In the following, we consider two words v and w such that v θ-commutes with w, and give conditions under which a word u is relatively θ-primitive with words v and w. We say that two nonempty sets of nonempty words $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_m\}$ are relatively θ-primitive, and we denote this by $\{u_1, u_2, \ldots u_n\} \perp_\theta \{v_1, v_2, \ldots v_m\}$, where $n, m \geq 1$ and $u_i, v_j \in \Sigma^+$ for $1 \leq i \leq n$, $1 \leq j \leq m$, iff u_i and v_j are relatively θ-primitive for all $1 \leq i \leq n$ and $1 \leq j \leq m$.

Proposition 15. *Let θ be a morphic involution over Σ^* and $u, v, w \in \Sigma^+$ be such that v θ-commutes with w, i.e., $vw = \theta(w)v$. Then,*

(I) $u \perp_\theta \{v, \theta(v)\}$ *implies* $u \perp_\theta w$

(II) $u \perp_\theta \{w, \theta(w)\}$ *implies* $u \perp_\theta v$.

Proof. To prove (I), as v θ-commutes with w and θ is morphic, by Proposition 4 (I), we have that either $\rho_\theta(v) = \rho_\theta(w)$ or $\theta(\rho_\theta(v)) = \rho_\theta(w)$. Assume, for the sake of contradiction, $u \not\perp_\theta w$, i.e., $\rho_\theta(u) = \rho_\theta(w) = x$, with $x \in \Sigma^+$.

Then either $\rho_\theta(u) = \rho_\theta(w) = \rho_\theta(v) = x$, or $\rho_\theta(u) = \rho_\theta(w) = \theta(\rho_\theta(v)) = x$. The former contradicts $u \perp_\theta v$. The latter implies $\rho_\theta(u) = \rho_\theta(\theta(v))$ which contradicts $u \perp_\theta \theta(v)$. Both cases lead to contradictions, hence $u \perp_\theta w$. Statement (II) can be proved similarly. □

The above result does not necessarily hold if θ is an antimorphic involution, as demonstrated by the following example.

Example 16. Let $\Sigma = \{a, b, c, d\}$ and θ be an antimorphic involution such that $\theta(a) = b$, $\theta(c) = d$ and vice versa. Let $x = acdbab$, $u = x\theta(x)$, $v = abx = \theta(x)ab$ and $w = x$. It is easy to verify that v θ-commutes with w, and $u \perp_\theta \{v, \theta(v)\}$ but $u \not\perp_\theta w$.

Proposition 17. *For an (anti)morphic involution θ over Σ^* and $u, v, w \in \Sigma^+$, $((u, v)_\theta, w)_\theta = (u, (v, w)_\theta)_\theta$.*

Proof. If $u \perp_\theta v$ then $(u, v)_\theta \perp_\theta w$ and $((u, v)_\theta, w)_\theta = \lambda$. We have the following two cases:

Case (1): If $v \perp_\theta w$ then $u \perp_\theta (v, w)_\theta$ and $(u, (v, w)_\theta)_\theta = \lambda$.

Case (2): If $v \not\perp_\theta w$, i.e., $\rho_\theta(v) = \rho_\theta(w) = x$, with $x \in \Sigma^+$, then $(u, (v, w)_\theta)_\theta = (u, x)_\theta = \lambda$, since $u \perp_\theta v$. Thus, $u \perp_\theta (v, w)_\theta$ and $(u, (v, w)_\theta)_\theta = \lambda$.

If, on the other hand, $u \not\perp_\theta v$, then let $\rho_\theta(u) = \rho_\theta(v) = x'$ with $x' \in \Sigma^+$. Then we have the following two cases:

Case (1): If $v \perp_\theta w$ then this case is similar to Case (2) above.

Case (2): If $v \not\perp_\theta w$, i.e., $\rho_\theta(v) = \rho_\theta(w) = x'$ then $(u, (v, w)_\theta)_\theta = ((u, v)_\theta, w)_\theta = x'$. □

Note that, if θ is a morphic involution and two words u, v have a common θ-primitive root, $(u, v)_\theta = x$, $x \in \Sigma^+$, then the words $\theta(u)$ and $\theta(v)$ also have a common θ-primitive root, namely $\theta(x)$, that is $(\theta(u), \theta(v))_\theta = \theta(x)$. Similarly, if u and v are relatively θ-primitive, $u \perp_\theta v$, then this implies that also $\theta(u)$ and $\theta(v)$ are relatively θ-primitive, $\theta(u) \perp_\theta \theta(v)$.

In contrast, if θ is an antimorphic involution, $u \perp_\theta v$ does not necessarily imply that $\theta(u) \perp_\theta \theta(v)$. Indeed, let $u = x\theta(x)x$ and $v = \theta(x)x$ for some θ-primitive word $x \in Q_\theta$. Then u and v are relatively θ-primitive, $u \perp_\theta v$, since u has the θ-primitive root x and v has the θ-primitive root $\theta(x)$. However, $\theta(u) = \theta(x)x\theta(x)$ and $\theta(v) = \theta(x)x$ which imply that $\theta(u)$ and $\theta(v)$ have the common θ-primitive root $\theta(x)$ and are thus not relatively θ-primitive, $\theta(u) \not\perp_\theta \theta(v)$.

Similarly, if θ is an antimorphic involution, $(u, v)_\theta = x$ does not necessarily imply that $(\theta(u), \theta(v))_\theta = \theta(x)$. Indeed, let $u = x\theta(x)$ and $v = x$, with $x \in Q_\theta$. Then $\rho_\theta(u) = \rho_\theta(v) = x$ and thus $(u, v)_\theta = x$. However, $\theta(u) = x\theta(x)$ and $\theta(v) = \theta(x)$ which means that $\theta(u)$ has the θ-primitive root x, while $\theta(v)$ has the θ-primitive root $\theta(x)$. That is, not only $\theta(u)$ and $\theta(v)$ do not have the common θ-primitive root $\theta(x)$, they do not have any common θ-primitive root at all, and are in fact relatively θ-primitive, $\theta(u) \perp_\theta \theta(v)$.

We know from Proposition 2, that for words u and v, $uv = vu$ iff u and v are powers of a common word, i.e., u and v share the common primitive root. The following lemma state some conditions under which two words u and v share a common θ-primitive root for a morphic involution θ.

Proposition 18. *Let θ be a morphic involution over Σ^* and let $u, v \in \Sigma^+$. If for some $x \in \Sigma^+$ we have that $(uv, vu)_\theta = x$ then $(u, v)_\theta = x$, and conversely.*

Proof. Let us assume that $(uv, vu)_\theta = x$, i.e., $\rho_\theta(uv) = \rho_\theta(vu) = x$.

If $u \in \{x, \theta(x)\}^+$ then $v \in \{x, \theta(x)\}^+$. Moreover since $\rho_\theta(uv) = x$, u must start with x, and since $\rho_\theta(vu) = x$, v must also start with x. Thus $\rho_\theta(u) = \rho_\theta(v) = x$.

If $u \notin \{x, \theta(x)\}^+$ then, in the decomposition of uv as a θ-power of x, the word u ends either inside x, or inside $\theta(x)$. In other words, $x = x_1x_2$ with $x_1, x_2 \in \Sigma^+$ and either $uv = \alpha x_1x_2\beta$ (where $u = \alpha x_1$ and $v = x_2\beta$), or $uv = \alpha\theta(x_1)\theta(x_2)\beta$ (where $u = \alpha\theta(x_1)$ and $v = \theta(x_2)\beta$). Moreover, if u ends inside x then $\alpha \in x\{x, \theta(x)\}^* \cup \{\lambda\}$ since uv begins with x, and $\beta \in \{x, \theta(x)\}^*$, while if u ends inside $\theta(x)$ then $\alpha \in x\{x, \theta(x)\}^*$, and $\beta \in \{x, \theta(x)\}^*$. Also, since vu begins with x, $vu = x_1x_2\sigma$ where $\sigma \in \{x, \theta(x)\}^*$.

Consider the first possibility, where u ends inside of x, that is, $u = \alpha x_1$, $v = x_2\beta$ for some $x_1, x_2 \in \Sigma^+$ with $x = x_1x_2$, and let $\beta \neq \lambda$. Then we have following two cases:

Case (1): $\beta = x\gamma$ where $\gamma \in \{x, \theta(x)\}^*$, i.e., $vu = x_2x_1x_2\gamma\alpha x_1 = x_1x_2\sigma$. Then $x_1x_2 = x_2x_1$. This implies, by Proposition 2, that $x_1 = s^i$ and $x_2 = s^j$ for some $s \in \Sigma^+$, $i, j \geq 1$ which further implies that $x = s^{i+j}$, a contradiction since $x \in Q_\theta$.

Case (2): $\beta = \theta(x)\gamma$ where $\gamma \in \{x, \theta(x)\}^*$, i.e., $vu = x_2\theta(x_1)\theta(x_2)\gamma\alpha x_1 = x_1x_2\sigma$. Then $x_2\theta(x_1) = x_1x_2$, i.e., the word x_2 θ-commutes with $\theta(x_1)$. Thus by Proposition 4 either $x_2, \theta(x_1) \in p^+$ for $p \in P_\theta$, or $\theta(x_1) = [q\theta(q)]^m$ and $x_2 = \theta(q)[q\theta(q)]^n$ for some $q \in \Sigma^+$, $m \geq 1$ and $n \geq 0$. This further implies that for some $k \geq 2$, $x = x_1x_2 = p^k$, that is $x \notin Q_\theta$, or that $x \in \theta(q)\{q, \theta(q)\}^+ \notin Q_\theta$, a contradiction.

Consider now the second possibility, where u ends inside $\theta(x)$, that is, $u = \alpha\theta(x_1)$, $v = \theta(x_2)\beta$, for some $x_1, x_2 \in \Sigma^+$ with $x = x_1x_2$, and let $\beta \neq \lambda$. Then, as stated before, because u starts with x, we must have $\alpha \neq \lambda$, and we have following two cases:

Case (1'): $\beta = x\gamma$ where $\gamma \in \{x, \theta(x)\}^*$, i.e., $vu = \theta(x_2)x_1x_2\gamma\alpha\theta(x_1) = x_1x_2\sigma$. Then $x_1x_2 = \theta(x_2)x_1$, i.e., x_1 θ-commutes with x_2. Thus we get a similar contradiction as that of Case (2).

Case (2'): $\beta = \theta(x)\gamma$ where $\gamma \in \{x, \theta(x)\}^*$, i.e., $vu = \theta(x_2)\theta(x_1)\theta(x_2)\gamma\alpha\theta(x_1) = x_1x_2\sigma$. Then $x_1x_2 = \theta(x_2)\theta(x_1)$. Thus x_1 is a θ-conjugate of $\theta(x_1)$ and, by Proposition 3, there exists $p, q \in \Sigma^*$ such that $x_1 = pq$ and either $\theta(x_1) = q\theta(p)$ and $x_2 = (\theta(p)\theta(q)pq)^i\theta(p)$, or $\theta(x_1) = \theta(q)p$ and $x_2 = (\theta(p)\theta(q)pq)^i\theta(p)\theta(q)p$ for some $i \geq 0$.

Case (2'(a)): Let $\theta(x_1) = q\theta(p)$ and $x_2 = (\theta(p)\theta(q)pq)^i\theta(p)$. Then $\theta(p)\theta(q) = q\theta(p)$, which implies that either $p = \lambda$, $q \neq \lambda$, $\theta(q) = q$, or that $q = \lambda$, $p \neq \lambda$, or that $p, q \in \Sigma^+$ and $\theta(p)$ θ-commutes with $\theta(q)$.

In the first case, $p = \lambda$, $q \neq \lambda$, we have $x_1 = q$, $\theta(x_1) = q$, $x_2 = (\theta(q)q)^i = q^{2i}$ for some $i \geq 0$. As $x_2 \neq \lambda$ we have $i \neq 0$, and $x_1x_2 = q^{2i+1}$, $q \in P_\theta \cap \Sigma^+$, which contradict the θ-primitivity of $x = x_1x_2$.

In the second case, $q = \lambda$, $p \neq \lambda$, we have $x_1 = p$, $\theta(x_1) = \theta(p)$, and $x_2 = (\theta(p)p)^i\theta(p)$ for some $i \geq 0$. This further implies $x_1x_2 = p(\theta(p)p)^i\theta(p)$, $p \in \Sigma^+$ which contradicts the θ-primitivity of $x = x_1x_2$.

In the third case, where $p, q \in \Sigma^+$ and $\theta(p)$ θ-commutes with $\theta(q)$, by Proposition 4, either $\theta(p), \theta(q) \in s^+$ for $s \in P_\theta$, or $\theta(q) = [t\theta(t)]^m$ and $\theta(p) = \theta(t)[t\theta(t)]^n$ for some $t \in \Sigma^+$, $m \geq 1$ and $n \geq 0$. This further implies that either for $k \geq 3$, $x = x_1x_2 = pq(\theta(p)\theta(q)pq)^i\theta(p) = s^k \notin Q_\theta$, or $x \in t\{t, \theta(t)\}^+ \notin Q_\theta$, both leading to contradictions.

Case(2'(b)): Let $\theta(x_1) = \theta(q)p$ and $x_2 = (\theta(p)\theta(q)pq)^i\theta(p)\theta(q)p$. Then $\theta(q)p = \theta(p)\theta(q)$, which implies that either $p = \lambda$, $q \neq \lambda$, or that $q = \lambda$, $p \neq \lambda$, $p = \theta(p)$, or that $p, q \in \Sigma^+$ and $\theta(q)$ θ-commutes with p. In all three cases we reach contradictions similar to those of Case (2'(a)).

Since the two possibilities where u ends inside of x, $u = \alpha x_1$, $v = x_2\beta$ and $\beta \neq \lambda$, or u ends inside of $\theta(x)$, $u = \alpha\theta(x_1)$, $v = \theta(x_2)\beta$ and $\beta \neq \lambda$ both led to contradictions, if these kinds of decompositions occur, we can only have $\beta = \lambda$.

Thus either u ends inside of x, with $u = \alpha x_1$ and $v = x_2$, i.e., $uv = \alpha x_1x_2 = \alpha x$ or, alternatively, u ends inside of $\theta(x)$, with $u = \alpha\theta(x_1)$ and $v = \theta(x_2)$, and $uv = \alpha\theta(x_1)\theta(x_2) = \alpha\theta(x)$.

In the first situation, if $\alpha = \lambda$ then $uv = x_1x_2$ and $vu = x_2x_1$ which, along with the fact $\rho_\theta(uv) = \rho_\theta(vu) = x$, imply $x_1x_2 = x_2x_1$. This further implies $x_1, x_2 \in p^+$ for $p \in \Sigma^+$, i.e., $x \notin Q_\theta$, a contradiction. If $\alpha \neq \lambda$, that is, $\alpha = x\gamma$ with $\gamma \in \{x, \theta(x)\}^*$, then $uv = \alpha x = x_1x_2\gamma x$, $vu = x_2x_1x_2\gamma x_1$. Since $\rho_\theta(uv) = \rho_\theta(vu) = x$, $x_1x_2 = x_2x_1$ which implies $x_1, x_2 \in p^+$ for $p \in \Sigma^+$ which further implies $x \notin Q_\theta$, a contradiction.

In the second situation, if $\alpha = \lambda$ then $uv = \theta(x_1)\theta(x_2)$ and $vu = \theta(x_2)\theta(x_1)$ which, along with the fact that $\rho_\theta(uv) = \rho_\theta(vu) = x$ implies $\theta(x_1x_2) = \theta(x_2x_1) = x_1x_2$. This further implies $x_1x_2 = x_2x_1$ which leads to $x_1, x_2 \in p^+$ for $p \in \Sigma^+$, i.e., $x \notin Q_\theta$, a contradiction. If $\alpha \neq \lambda$, i.e., $\alpha = x\gamma$ with $\gamma \in \{x, \theta(x)\}^*$, then $uv = \alpha\theta(x) = x_1x_2\gamma\theta(x)$, $vu = \theta(x_2)x_1x_2\gamma\theta(x_1)$. Since $\rho_\theta(uv) = \rho_\theta(vu) = x$, $x_1x_2 = \theta(x_2)x_1$, that is, x_1 θ-commutes with x_2. We arrive at a similar contradiction as that of Case (2).

Thus all possible cases where $u, v \notin x\{x, \theta(x)\}^*$ led to contradictions. The only remaining possibility is $u, v \in x\{x, \theta(x)\}^*$, which implies that $\rho_\theta(u) = \rho_\theta(v) = x$.

The converse is straightforward. □

Recall that, given a word $w \in \Sigma^+$, $w = a_1a_2 \dots a_k$, $k \geq 1$, and numbers $1 \leq i \leq j \leq k$, we denote by $w[i..j]$ the subword of w that starts with a_i and ends with a_j, that is, the word $a_i \dots a_j$. Also, a word y is said to be a factor of word w if there exists $x, z \in \Sigma^*$ such that $w = xyz$. Moreover, y is said to be a proper factor of w if at least one of x or z is in Σ^+. The following result [5] provides an algorithm to determine whether or not a given word $w \in \Sigma^+$ is θ-primitive, and finds all the θ-primitive factors of w.

Proposition 19. *[5] Let $\theta : \Sigma^* \to \Sigma^*$ be (anti)morphism and $w \in \Sigma^*$ be a given word with $|w| = n$.*

(I) *One can identify in time $\mathcal{O}(n^{3.5})$ the triplets (i, j, k) with $w[i..j] \in \{t, \theta(t)\}^k$, for a proper factor t of $w[i..j]$.*

(II) *One can identify in time $\mathcal{O}(n^2k)$ the pairs (i, j) such that $w[i..j] \in \{t, \theta(t)\}^k$ for a proper factor t of $w[i..j]$, when k is also given as input.*

For a non-erasing θ we solve (I) in $\Theta(n^3)$ time and (II) in $\Theta(n^2)$ time. For a literal θ we solve (I) in $\Theta(n^2 \, lg\, n)$ time and (II) in $\Theta(n^2)$ time.

The following proposition describes an algorithm that, given an (anti)morphic involution θ of Σ^* and two different words $u, v \in \Sigma^+$, decides whether or not $u \perp_\theta v$.

Proposition 20. *Let θ be an (anti)morphic involution over Σ^* and $u, v \in \Sigma^+$ be two words with $u \neq v$. It is decidable, in $\Theta(n^2 lg\, n)$ time, whether $u \perp_\theta v$, where $n = max\{|u|, |v|\}$.*

Proof. Let θ be an (anti)morphic involution and $u, v \in \Sigma^+$, $u \neq v$, with $n = \max\{|u|, |v|\}$. Using Proposition 19 (I), we identify triplets (i, j, k) and (i', j', k') with $u[i..j] \in \{t, \theta(t)\}^k$ and $v[i'..j'] \in \{t', \theta(t')\}^{k'}$. Since an (anti)morphic involution is a literal morphism, this algorithm takes $\Theta(n^2 lg\, n)$ time. We have the following cases:

Case (1): There do not exist triplets $(1, |u|, k)$ and $(1, |v|, k')$. Then $u \perp_\theta v$.

Case (2): There exists a triplet $(1, |u|, k)$ such that $u[1..|u|] \in \{t, \theta(t)\}^k$, but a similar triplet $(1, |v|, k')$ does not exist. Let $u = t_1t_2 \cdots t_k$ where $t_l \in \{t, \theta(t)\}$ for $1 \leq l \leq k$. If $t_1 \neq v$ then $u \perp_\theta v$, else $u \not\perp_\theta v$.

Case (3): There exists a triplet $(1, |v|, k)$ such that $v[1..|v|] \in \{t', \theta(t')\}^{k'}$, but a similar triplet $(1, |u|, k)$ does not exist. Let $v = t'_1t'_2 \cdots t'_k$ where $t'_l \in \{t', \theta(t')\}$ for $1 \leq l \leq k'$. If $t'_1 \neq u$ then $u \perp_\theta v$, else $u \not\perp_\theta v$.

Case (4): There exist triplets $(1, |u|, k)$ and $(1, |v|, k')$ such that $u[1..|u|] \in \{t, \theta(t)\}^k$ and $v[1..|v|] \in \{t', \theta(t')\}^{k'}$. Let $u = t_1 t_2 \cdots t_k$ and $v = t'_1 t'_2 \cdots t'_{k'}$ where $t_l \in \{t, \theta(t)\}$ for $1 \le l \le k$, and $t'_m \in \{t', \theta(t')\}$ for $1 \le m \le k'$. If $t_1 \neq t'_1$ then $u \perp_\theta v$, else $u \not\perp_\theta v$.

Hence given $u, v \in \Sigma^+$, $u \neq v$, it is decidable whether $u \perp_\theta v$, in $\Theta(n^2 lg\, n)$ time, where $n = \max\{|u|, |v|\}$. □

4. Binary word operations and relatively θ-primitive words

In the previous section, we have seen various properties of two binary relations on words : relative θ-primitivity, $u \perp_\theta v$, vs. having common θ-primitive root $u \not\perp_\theta v$. Given two relatively θ-primitive words $u, v \in \Sigma^+$, we now investigate the relationship between the words u, v, and the result of the application of a word operation to u and v. In this setting we consider various binary word operations such as perfect shuffle, shuffle and θ-catenation. Given $u, v \in \Sigma^+$, define their shuffle $u \sqcup\!\sqcup v$ as

$$u \sqcup\!\sqcup v = \{u_1 v_1 \cdots u_k v_k |\ k \ge 1, u = u_1 \cdots u_k, v = v_1 \cdots v_k, u_i, v_i \in \Sigma^* \text{ for } 1 \le i \le k\}.$$

Similarly, the perfect shuffle of two words u and v of the same length, $|u| = |v| = k$, $k \ge 1$, is defined as

$$u \sqcup\!\sqcup_p v = a_1 b_1 \cdots a_k b_k \text{ where } u = a_1 a_2 \cdots a_k, v = b_1 b_2 \cdots b_k, a_i, b_i \in \Sigma \text{ for } 1 \le i \le k.$$

In the following proposition we show that there cannot exist two θ-palindromes of equal length that are relatively θ-primitive, with the property that their perfect shuffle is a θ-palindrome.

Proposition 21. *Let θ be an antimorphic involution over Σ^* and let $u, v \in \Sigma^+$ be two equi-length θ-palindromes, that is, $u, v \in P_\theta$ and $|u| = |v|$. If $u \perp_\theta v$, then $u \sqcup\!\sqcup_p v \notin P_\theta$.*

Proof. Assume that $u \sqcup\!\sqcup_p v \in P_\theta$ such that $|u| = |v| = m$.

Case (1): m is even, i.e., $m = 2k$ for some $k \ge 1$. Let $u = a_1 a_2 \cdots a_k \cdots a_{2k}$, implying $\theta(u) = \theta(a_{2k}) \cdots \theta(a_k) \cdots \theta(a_2)\theta(a_1)$. Since u is a θ-palindrome, we have that $u = a_1 a_2 \cdots a_k \theta(a_k) \cdots \theta(a_1)$ where $a_i \in \Sigma$ for $1 \le i \le k$. Similarly, $v = b_1 b_2 \cdots b_k \theta(b_k) \cdots \theta(b_1)$ where $b_j \in \Sigma$ for $1 \le j \le k$. Then,

$$u \sqcup\!\sqcup_p v = a_1 b_1 \cdots a_k b_k \theta(a_k)\theta(b_k) \cdots \theta(a_1)\theta(b_1).$$

Since $u \sqcup\!\sqcup_p v \in P_\theta$, $\theta(b_k) = \theta(a_k), \ldots, \theta(b_1) = \theta(a_1)$ which implies $b_i = a_i$ for $1 \le i \le k$ which further implies $u = v$, a contradiction since $u \perp_\theta v$.

Case (2): m is odd, i.e., $m = 2k + 1$ for some $k \ge 1$. Since u is a θ-palindrome, we have that u is of the form $u = a_1 a_2 \cdots a_k a' a_{k+1} \cdots a_{2k} = \theta(u) = \theta(a_{2k}) \cdots \theta(a') \cdots \theta(a_2)\theta(a_1)$. Thus $u = a_1 a_2 \cdots a_k a' \theta(a_k) \cdots \theta(a_1)$ where $a_i, a' \in \Sigma$ for $1 \le i \le k$ and $\theta(a') = a'$. Similarly, $v = b_1 b_2 \cdots b_k b' \theta(b_k) \cdots \theta(b_1)$. where $b_j, b' \in \Sigma$ for $1 \le j \le k$ and $\theta(b') = b'$. Then,

$$u \sqcup\!\sqcup_p v = a_1 b_1 \cdots a_k b_k a' b' \theta(a_k)\theta(b_k) \cdots \theta(a_1)\theta(b_1).$$

Since $u \sqcup\!\sqcup_p v \in P_\theta$, we should have $b' = \theta(a') = a', \theta(b_k) = \theta(a_k)...\theta(b_1) = \theta(a_1)$ which imply $b' = a', b_i = a_i$ for $1 \leq i \leq k$ which further implies $u = v$, a contradiction since $u \perp_\theta v$.

Since both the cases lead to a contradiction, $u \sqcup\!\sqcup_p v \notin P_\theta$. □

In Proposition 23, we will prove that, under certain conditions, if two equi-length words u and v are relatively θ-primitive then u and any word in the shuffle $(u \sqcup\!\sqcup v)$ are relatively θ-primitive, and the same holds for v. For a letter $a \in \Sigma$ we denote by $|u|_{a,\theta(a)}$ the number of occurrences of a's and $\theta(a)$'s in u (see [4]). Note that, for a word $u \in \Sigma^*$, a letter $a \in \Sigma$, and an (anti)morphic involution θ, we have that $|u|_{a,\theta(a)} = |\theta(u)|_{a,\theta(a)}$.

Lemma 22. *Let θ be an (anti)morphic involution over Σ^*, and $u, v \in \Sigma^+$ be such that $|u| = |v|$ and there exists $a \in \Sigma$ such that $|u|_{a,\theta(a)} \neq |v|_{a,\theta(a)}$. Then $u \perp_\theta v$.*

Proof. Assume that $u \not\perp_\theta v$, i.e., $\rho_\theta(u) = \rho_\theta(v) = x$, $x \in \Sigma^+$. Then $u, v \in x\{x, \theta(x)\}^*$ and, since $|u| = |v|$, we have that $u = xx_1x_2 \dots x_{m-1}$ and $v = xy_1y_2 \dots y_{m-1}$, where $m \geq 1$ and $x_i, y_i \in \{x, \theta(x)\}$, $1 \leq i \leq m-1$. For all $a \in \Sigma$, since $|x|_{a,\theta(a)} = |\theta(x)|_{a,\theta(a)}$, we have that $|u|_{a,\theta(a)} = m|x|_{a,\theta(a)} = |v|_{a,\theta(a)}$. This contradicts the hypothesis that there exists $a \in \Sigma$ such that $|u|_{a,\theta(a)} \neq |v|_{a,\theta(a)}$. Thus $u \perp_\theta v$. □

Proposition 23. *Let θ be an (anti)morphic involution over Σ^*, and let $u, v \in \Sigma^+$ such that $u \perp_\theta v$, $|u| = |v|$, and there exists $a \in \Sigma$ such that $|u|_{a,\theta(a)} \neq |v|_{a,\theta(a)}$. Then $(u \sqcup\!\sqcup v) \perp_\theta \{u, v\}$.*

Proof. Assume for the sake of contradiction that there exists $w \in u \sqcup\!\sqcup v$ such that either $w \not\perp_\theta u$ or $w \not\perp_\theta v$. Without loss of generality, assume $w \not\perp_\theta u$. Then $\rho_\theta(w) = \rho_\theta(u) = x$, $x \in \Sigma^+$. Since $\rho_\theta(u) = x$, we have that $u = xx_1x_2 \cdots x_{m-1}$ for some $m \geq 1$ and $x_i \in \{x, \theta(x)\}$ for $1 \leq i \leq m-1$. Since $\rho_\theta(w) = x$, we have that $w = xx_1x_2 \cdots x_{k-1}$ for some $k \geq 1$ and $x_j \in \{x, \theta(x)\}$ for $1 \leq j \leq k-1$. Note that, since $w \in u \sqcup\!\sqcup v$ and $|u| = |v|$, we have that $k = 2m$.

Let a be an arbitrary letter in Σ. We have that $|u|_{a,\theta(a)} = m|x|_{a,\theta(a)}$, and $|w|_{a,\theta(a)} = 2m|x|_{a,\theta(a)}$. Moreover, since $w \in u \sqcup\!\sqcup v$ we have that $|w|_{a,\theta(a)} = |u|_{a,\theta(a)} + |v|_{a,\theta(a)}$. This, further implies that $|v|_{a,\theta(a)} = |w|_{a,\theta(a)} - |u|_{a,\theta(a)} = 2m|x|_{a,\theta(a)} - m|x|_{a,\theta(a)} = m|x|_{a,\theta(a)} = |u|_{a,\theta(a)}$, for all $a \in \Sigma$. This contradicts the hypothesis that there exists a letter $a \in \Sigma$ such that $|u|_{a,\theta(a)} \neq |v|_{a,\theta(a)}$. Hence $(u \sqcup\!\sqcup v) \perp_\theta u$. □

Definition 24. [7] For (anti)morphic involution θ on Σ^* and two words $u, v \in \Sigma^*$, the binary word operation of θ-catenation is defined as

$$u \odot v = \{uv, u\theta(v)\}.$$

For an (anti)morphic involution θ, if $u \perp_\theta v$, this does not necessarily imply that for any $x \in u \odot v$, $x \perp_\theta u$, as seen in the following example.

Example 25. Let $\Sigma = \{a, b, c, d\}$ such that $\theta(a) = b$, $\theta(c) = d$ and vice versa for an antimorphic involution θ. Then, for $u = ac$, $v = db$ and $u \odot v = \{acdb, acac\}$. We have that $u \perp_\theta v$ but $\rho_\theta(acdb) = \rho_\theta(acac) = ac$ and thus $acdb \not\perp_\theta u$ and $acac \not\perp_\theta u$.

We observe in the above example that $u \not\perp_\theta \theta(v)$, and hence for $u \odot v$ to be relatively θ-primitive with u, it is necessary to have the condition that $u \perp \{v, \theta(v)\}$.

Proposition 26. *Let θ be an (anti)morphic involution over Σ^* and $u, v \in \Sigma^+$ be such that $u \perp_\theta \{v, \theta(v)\}$. Then, for all $x \in u \odot v$, we have that $x \perp_\theta u$.*

Proof. Assume that for some $x \in u \odot v$, $x \not\perp_\theta u$, i.e., $\rho_\theta(x) = \rho_\theta(u) = y$. This implies that either $uv \in y\{y, \theta(y)\}^*$ or $u\theta(v) \in y\{y, \theta(y)\}^*$. Since $u \in y\{y, \theta(y)\}^*$, this further implies that either $v \in \{y, \theta(y)\}^*$ or $\theta(v) \in \{y, \theta(y)\}^*$. Thus, either $v \not\perp_\theta u$ or $\theta(v) \not\perp_\theta u$, a contradiction. Hence, for all $x \in u \odot v$, we have that $x \perp_\theta u$. □

Proposition 27. *Let θ be an (anti)morphic involution over Σ^* and $u, v \in \Sigma^+$ be such that $u \perp_\theta v$. Then, for all $x \in u \odot v$, we have that $x \perp_\theta v$.*

Proof. Assume that for some $x \in u \odot v$, $x \not\perp_\theta v$, i.e., $\rho_\theta(x) = \rho_\theta(v) = y$. Then either $uv \in y\{y, \theta(y)\}^*$ or $u\theta(v) \in y\{y, \theta(y)\}^*$ which further implies that $u \in y\{y, \theta(y)\}^*$. Thus $u \not\perp_\theta v$, a contradiction. Hence, for all $x \in u \odot v$, we have that $x \perp_\theta v$. □

For a given (anti)morphic involution θ and a word $x \in \Sigma^+$, consider the language of all words w that are relatively θ-primitive with x,

$$L_{\theta,\lambda}(x) = \{w \in \Sigma^* | w \perp_\theta x\}$$

as well as its complement, the language of words that have a common θ-primitive root with x

$$L_\theta(x) = \{w \in \Sigma^* | w \not\perp_\theta x\}.$$

Note that, for a given (anti)morphic involution θ and $x \in \Sigma^+$ we have that $L_{\theta,\lambda}(x) \cup L_\theta(x) = \Sigma^*$. In addition, the language $L_\theta(x)$ can be described using the θ-catenation power. Indeed, define [7] the iterated θ-catenation $\odot^i$, for $i \geq 0$, as $L_1 \odot^0 L_2 = L_1$ and $L_1 \odot^i L_2 = (L_1 \odot^{i-1} L_2) \odot L_2$. Then the i-th $\odot$-power of a non-empty language L is defined as $L^{\odot(0)} = \lambda$ and $L^{\odot(i)} = L \odot^{i-1} L$ for $i \geq 1$. We can now describe the language of all words that have a common θ-primitive root with $x \in \Sigma^+$ as

$$L_\theta(x) = \{w \in \Sigma^+ | w \in \rho_\theta(x)^{\odot(n)}, n \geq 1\}.$$

Proposition 28. *For a given (anti)morphic involution θ and $x \in \Sigma^+$, the languages $L_\theta(x), L_{\theta,\lambda}(x)$ are regular.*

Proof. For a given $x \in \Sigma^+$, the language $L_\theta(x)$ is generated by the right-linear grammar $G = (N, \Sigma, S, P)$ with the set of nonterminals $N = \{S, S_1\}$ and the set of productions $P = \{S \to \rho_\theta(x)S_1, S_1 \to \rho_\theta(x)S_1 | \theta(\rho_\theta(x))S_1 | \lambda\}$. Since the family of regular languages is closed under complementation, $L_{\theta,\lambda}(x)$ is regular as well. □

Finally, note that there does not exist any language L that is independent with respect to $\perp_\theta$, and that the language of all θ-primitive words, Q_θ, is independent with respect to $\not\perp_\theta$.

5. Conclusions and future work

In this paper we introduced and investigated the notion of relative θ-primitivity of words, for any (anti)morphic involution θ over Σ^*: Two words are relatively θ-primitive if they do not have a common θ-primitive root. If θ is the identity on Σ, extended to a morphism of Σ^*, this becomes relative primitivity, and if θ_Δ is the Watson-Crick reverse-complementarity over the DNA alphabet $\Delta = \{A, C, G, T\}$, this becomes the relative Watson-Crick primitivity of words. Note that, due to the way the θ-primitive root of a word w was defined, to ensure its uniqueness, the situation occurs where two words which obviously have similarities, such as $u = xxx\theta(x)\theta(x)$ and $v = \theta(x)xx\theta(x)\theta(x)$, are nevertheless relatively θ-primitive (the θ-primitive root of u is x, which is usually distinct from $\theta(x)$, the θ-primitive root of v). A stronger definition, perhaps more natural from a biological perspective, would be the notion of (strong) relative θ-primitivity: Two words are (strong) relatively θ-primitive if their θ-primitive roots are neither identical nor θ-images of each other.

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