Accepted for publication in the Journal of Automata, Languages and Combinatorics
Final version created on January 14, 2022

PRIMITIVITY OF ATOM WATSON-CRICK FIBONACCI WORDS

LILA KARI[(A,B)] KALPANA MAHALINGAM[(C)] PALAK PANDOH[(D)] ZIHAO WANG[(A)]

[(A)]*School of Computer Science, University of Waterloo*
200 University Ave W, Waterloo, ON, N2L 3G1, Waterloo, Canada
lila.kari@uwaterloo.ca z465wang@uwaterloo.ca

[(C)]*Department of Mathematics, IIT Madras,*
Chennai, Tamil Nadu, 600036, India
kmahalingam@iitm.ac.in

[(D)]*School of Mathematics, Shri Mata Vaishno Devi University,*
Katra, Jammu and Kashmir, 182320, India
palakpandohiitmadras@gmail.com

ABSTRACT

"Fibonacci strings" were first defined by Knuth in his 1968 "The Art of Computer Programming," as being an infinite sequence of strings obtained from two initial letters $f_1 = a$ and $f_2 = b$, by the recursive definition $f_{n+2} = f_{n+1} \cdot f_n$, for all positive integers $n \geq 1$, where "$\cdot$" denotes word concatenation. Motivated by theoretical studies of DNA computing, several generalizations of Fibonnaci words have been proposed under the umbrella term involutive Fibonacci words. These include ϕ-Fibonacci words and indexed ϕ-Fibonacci words, where ϕ denotes either a morphic or an antimorphic involution. (In the particular case of the DNA alphabet $\Delta = \{A, C, G, T\}$, where ϕ is the Watson-Crick complementarity (antimorphic) involution on Δ^* that maps A to T, G to C, and vice versa, the ϕ-Fibonacci words are termed atom Watson-Crick Fibonacci words.) In this paper, we investigate the properties of atom ϕ-Fibonacci words over a four-letter alphabet, whereby "atom" indicates that the two initial words are singleton letters. The results are different from the case of the classical Fibonacci words over a two-letter alphabet, which are all primitive, in that for some (anti)morphic involutions, some initial letters, and some indices n, we have that the n-th atom ϕ-Fibonacci word is primitive, while for some others it is not. In the particular case of the Watson-Crick complementarity antimorphic involution, regardless of the initial two letters in the Fibonacci recursion (different, or the same), for all $n > 3$, the n-th atom Watson-Crick Fibonacci word is primitive.

Keywords: DNA computing, Watson-Crick complementarity, antimorphic involution, atom Fibonacci words, involutive Fibonacci words

---

[(B)]This work was supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC) Discovery Grant R2824A01 to L.K.

1. Introduction

Fibonacci words or Fibonacci strings were introduced as word counterparts of the Fibonacci numbers defined by $F_0 = 0$, $F_1 = 1$, and the recursion $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Fibonacci words form an infinite sequence of strings obtained from two initial letters $f_1 = a$ and $f_2 = b$, by the recursive definition $f_{n+2} = f_{n+1} \cdot f_n$, for all positive integers $n \geq 1$, where "$\cdot$" denotes word concatenation. A natural generalization is to replace the two initial letters a and b by non-empty words u and v, and many other generalizations of Fibonacci words have also been proposed, see [3,5,6,9,11,24,25,26,29,32], to name just a few. In particular, involutive Fibonacci words were introduced and investigated in [15], motivated by theoretical studies of DNA computing. Involutive Fibonacci words over an alphabet Σ were formally defined as ϕ-Fibonacci words, where ϕ is a morphic or an antimorphic involution on Σ^*. Their connection to DNA computing stems from the Watson-Crick DNA complementarity of DNA strands (words over the four-letter DNA alphabet $\Delta = \{A, C, G, T\}$) whereby two such Watson-Crick (W/C) complementary DNA single strands of opposite orientations bind to each other to form a helical DNA double strand. The Watson-Crick complementarity has been mathematically formalized as an antimorphic involution, see [13, 22]. More precisely, the antimorphic involution θ_{DNA} that models Watson-Crick complementarity is defined by $\theta_{DNA}(A) = T$ and $\theta_{DNA}(C) = G$, and by the additional requirements that it models the biochemical properties of Watson-Crick complementarity by being both an involution on Δ (whereby $\theta_{DNA}(\theta_{DNA}(w)) = w$ for all $w \in \Delta^*$), and an antimorphism on Δ^* (whereby $\theta_{DNA}(uw) = \theta_{DNA}(w)\theta_{DNA}(u)$ for all $u, w \in \Delta^*$). In light of this formalization, when $\phi = \theta_{DNA}$, these particular θ_{DNA}-Fibonacci words are termed *Watson-Crick Fibonacci words*.

Most of the bio-operations involved in DNA computations rely on the capability of controlling the bonds that can be formed between (single-stranded) DNA molecules, via the Watson-Crick complementarity, θ_{DNA}. It is important to note that bonds can also form between complementary *parts* of two DNA molecules and that, moreover, a DNA molecule containing two complementary parts can even bind to *itself*. The success of a DNA bio-operation relies on the assumption that no undesired bonds form between DNA molecules in the test tube before the bio-operation is initiated. With this motivation, one of the foremost problems in DNA computing is to design DNA strands that are not (partially) W/C complementary to each other, and that have no W/C complementary parts within themselves, [10,12,14,19,20,22,23,30]. This has led to the concept of ϕ-primitivity, [7], whereby a word w is called ϕ-primitive if there is no shorter word u such that w can be written as repetitions of u and $\phi(u)$. A word over the DNA alphabet that is suitable for computations should thus be θ_{DNA}-primitive, and the need also arises for methods to generate sufficiently many, and sufficiently long, θ_{DNA}-primitive DNA words. One such method is the simple iterative process that gives rise to Watson-Crick Fibonacci words and, for such words to be useful for DNA computations, they have to be θ_{DNA}-primitive. The first step towards a comprehensive study of θ_{DNA}-primitivity is the study of primitivity, and this paper studies primitivity properties of atom Watson-Crick Fibonacci words.

It is well known that for all $n \geq 1$, the n-th atom Fibonacci word is primitive [8].

The qualifier "atom" indicates that the two initial words of the Fibonacci recursion are singleton letters and, as a consequence, all atom Fibonacci words are over a two-letter alphabet. In this paper, we present an exhaustive study of the primitivity properties of atom ϕ-Fibonacci words for all morphic and antimorphic involutions ϕ over a four-letter alphabet. Section 3 studies the primitivity of ϕ-Fibonacci words when the two initial letters are different from each other, while Section 4 studies the primitivity of ϕ-Fibonacci words when the two initial letters coincide. As it turns out, the situation is different from the classical case of atom Fibonacci words, in that for some morphic or antimorphic involutions ϕ, some initial Fibonacci letters, and some indices n, we have that the n-th ϕ-Fibonacci word is primitive, while in other situations it is not primitive (see Tables 8 and 12 for a summary). In the particular case of the Watson-Crick complementarity antimorphic involution θ_{DNA} over the DNA alphabet Δ, our results imply that regardless of the two initial letters of the Fibonacci recursion (different, or the same), for all $n > 3$, the n-th atom Watson-Crick Fibonacci word is primitive.

2. Preliminaries

An alphabet Σ is a finite non-empty set of symbols or letters, and Σ^* denotes the set of all words over Σ including the empty word λ, while Σ^+ is the set of all non-empty words over Σ. The length of a word $u \in \Sigma^*$ (i.e., the number of symbols in a word) is denoted by $|u|$. We denote by $|u|_a$, the number of occurrences of the letter a in u and by $\mathrm{Alph}(u)$, the set of all symbols occurring in u. Throughout the paper, we either use the convention that the set Σ_4 denotes an alphabet consisting of exactly 4 (distinct) letters, or use the DNA alphabet $\Delta = \{A, C, G, T\}$.

A word $w \in \Sigma^+$ is said to be *primitive* if $w = u^i$ implies $w = u$ and $i = 1$. Let Q denote the set of all primitive words. For every word $w \in \Sigma^+$, there exists a unique word $\rho(w) \in \Sigma^+$, called the *primitive root* of w, such that $\rho(w) \in Q$ and $w = \rho(w)^n$ for some $n \geq 1$.

The left derivative of a language L with respect to a word w is defined as $\partial^l_w L = \{u \in \Sigma^* | wu \in L\}$, and the right derivative is defined analogously, see [4]. Since in this paper we only use the left derivative, the superscript l will be omitted, and we will denote the left derivative of a language L with respect to a word w simply by $\partial_w L$.

A function $h : \Sigma^* \to \Sigma^*$ is called a *morphism* on Σ^* if for all words $u, v \in \Sigma^*$ we have that $\phi(uv) = \phi(u)\phi(v)$, and an *antimorphism* on Σ^* if $\phi(uv) = \phi(v)\phi(u)$. A function f is called an *involution* if $f(f(x)) = x$ for all x in the domain of f. A function $\phi : \Sigma^* \to \Sigma^*$ is called a *morphic involution on* Σ^* (respectively, an *antimorphic involution on* Σ^*) if it is an involution on Σ extended to a morphism (respectively, to an antimorphism) on Σ^*. For convenience, in the remainder of this paper, we use the convention that the letter ϕ denotes an involution that is either morphic or antimorphic (such a function will be termed an *(anti)morphic involution*), that the letter θ denotes an antimorphic involution, and that the letter μ denotes a morphic involution.

A word $w \in \Sigma^*$ is called a palindrome if $w = w^R$, where the reverse, or mirror image operator is defined as $\lambda = \lambda^R$ and $(a_1a_2 \ldots a_n)^R = a_n \ldots a_2a_1$, when $a_i \in \Sigma$ for all $1 \leq i \leq n$. A word $w \in \Sigma^*$ is called a ϕ-palindrome if $w = \phi(w)$, and the set of all ϕ-palindromes is denoted by P_ϕ. If $\phi = \mu$ is a morphic involution on Σ^* then the only μ-palindromes are the words over Σ', where $\Sigma' \subseteq \Sigma$, and μ is the identity on Σ'. Lastly, if $\phi = \theta$ is the identity function on Σ extended to an antimorphism on Σ^*, then a θ-palindrome is a classical palindrome, while if $\phi = \mu$ is the identity function on Σ extended to a morphism on Σ^*, then every word is a μ-palindrome.

The standard Fibonacci words $f_n(u,v)$, for the particular case when $u,v \in \Sigma$, were first introduced in [16, 17] and studied in, e.g. [1, 2, 8, 21, 27, 28].

Definition 1. Let Σ be an alphabet with $|\Sigma| \geq 2$ and let $u,v \in \Sigma^+$ with $u \neq v$. The n-th standard Fibonacci words are defined recursively as:

$$f_1(u,v) = u,\ f_2(u,v) = v,$$

$$f_n(u,v) = f_{n-1}(u,v) \cdot f_{n-2}(u,v),\ n \geq 3.$$

The sequence of standard Fibonacci words is defined as $F(u,v) = \{f_n(u,v)\}_{n\geq 1}$, i.e., $F(u,v) = \{u, v, vu, vuv, vuvvu, vuvvuvuv, vuvvuvuvvuvvu, \ldots\}$. Similarly, the n-th reverse Fibonacci words are defined recursively as:

$$f'_1(u,v) = u,\ f'_2(u,v) = v,$$

$$f'_n(u,v) = f'_{n-2}(u,v) \cdot f'_{n-1}(u,v),\ n \geq 3,$$

and the sequence of the reverse Fibonacci words is defined as $F'(u,v) = \{f'_n(u,v)\}_{n\geq 1}$, that is, $F'(u,v) = \{u, v, uv, vuv, uvvuv, vuvuvvuv, uvvuvvuvuvvuv, \ldots\}$.

If the initial words u and v are singleton letters, the resulting words are called atom standard Fibonacci words and, respectively, atom reverse Fibonacci words.

Note that the length of the n-th atom Fibonacci word f_n is in fact the Fibonacci number F_n for $n \geq 1$. The following observations will be used in the remainder of the paper.

Lemma 2. *For n, $m \geq 1$, the following hold.*

(I) $\gcd(n, n+1) = 1$.
(II) *For m even, if* $\gcd(n,m) = 1$, *then* $\gcd(n, \frac{m}{2}) = 1$.
(III) $\gcd(n, n+2)$ *is* 1 *if n is odd and* 2 *if n is even.*

We will also make use of the following identities on Fibonacci number F_n, that can be proved using Lemma 2 and induction.

Lemma 3. *For all $n \geq 1$, the following identities hold.*

(I) $\gcd(F_n, F_{n+1}) = 1$.
(II) $\gcd(F_n, \frac{F_{n+1}}{2}) = 1$ *for F_{n+1} even.*
(III) $\gcd(\frac{F_n}{2}, F_{n+1}) = 1$ *for F_n even.*
(IV) $\gcd(F_n - 1, F_n + 1)$ *is* 1 *if F_n is even and* 2 *if F_n is odd.*
(V) $\gcd(\frac{F_n-1}{2}, \frac{F_n+1}{2}) = 1$ *if F_n is odd.*

We recall the definition of involutive Fibonacci words, recently introduced in [15].

Definition 4. [15] Let Σ be an alphabet with $|\Sigma| \geq 2$, let ϕ be an (anti)morphic involution on Σ^*, and $u,v \in \Sigma^+$. If the first two ϕ-Fibonacci words are u, respectively, v, the three types of n-th standard ϕ-Fibonacci words, $g_n^{\phi}(u,v)$, $w_n^{\phi}(u,v)$, $z_n^{\phi}(u,v)$, $n \geq 3$, are defined recursively as follows:

$$g_n^{\phi}(u,v) = \phi(g_{n-1}^{\phi}(u,v)) \cdot g_{n-2}^{\phi}(u,v) \qquad \text{(standard alternating } \phi\text{-Fibonacci words)},$$
$$w_n^{\phi}(u,v) = \phi(w_{n-1}^{\phi}(u,v)) \cdot \phi(w_{n-2}^{\phi}(u,v)) \qquad \text{(standard palindromic } \phi\text{-Fibonacci words)},$$
$$z_n^{\phi}(u,v) = z_{n-1}^{\phi}(u,v) \cdot \phi(z_{n-2}^{\phi}(u,v)) \qquad \text{(standard hairpin } \phi\text{-Fibonacci words)}.$$

Similarly, the three types of n-th reverse ϕ-Fibonacci words, $[g_n^{\phi}(u,v)]'$, $[w_n^{\phi}(u,v)]'$, $[z_n^{\phi}(u,v)]'$, $n \geq 3$, are defined recursively as follows:

$$[g_n^{\phi}(u,v)]' = [g_{n-2}^{\phi}(u,v)]' \cdot \phi([g_{n-1}^{\phi}(u,v)]') \qquad \text{(reverse alternating } \phi\text{-Fibonacci words)},$$
$$[w_n^{\phi}(u,v)]' = \phi([w_{n-2}^{\phi}(u,v)]') \cdot \phi([w_{n-1}^{\phi}(u,v)]') \qquad \text{(reverse palindromic } \phi\text{-Fibonacci words)},$$
$$[z_n^{\phi}(u,v)]' = \phi([z_{n-2}^{\phi}(u,v)]') \cdot [z_{n-1}^{\phi}(u,v)]' \qquad \text{(reverse hairpin } \phi\text{-Fibonacci words)}.$$

If the first two words of the sequence are singleton letters in Σ, the ϕ-Fibonacci words will be called atom ϕ-Fibonacci words.

In the remainder of this paper, when the particular (anti)morphic involution ϕ involved in the Fibonacci recursion needs to be emphasized, we will use the notation $g_n^{\phi}(u,v)$, or $w_n^{\phi}(u,v)$, or $z_n^{\phi}(u,v)$ to denote the corresponding ϕ-Fibonacci words for $n \geq 1$. However, if either the initial words u and v or the mapping ϕ are clear from the context (as is the case in Definition 4), they will sometimes be omitted.

In the sequel, we will often have to make statements that hold for several types of ϕ-Fibonacci words. For brevity, we will use the notational convention that a statement of the type "$\alpha_n \in \{g_n, w_n, z_n\}$ for all $n \geq 1$" means either we have that $\alpha_n = g_n$ for all $n \geq 1$, or that $\alpha_n = w_n$ for all $n \geq 1$, or that $\alpha_n = z_n$ for all $n \geq 1$.

Similar to the case of the atom Fibonacci words defined in Definition 1, the length of the n-th atom ϕ-Fibonacci word $\alpha_n^{\phi}(a,b)$ is the Fibonacci number F_n for $n \geq 1$, where $\alpha_n \in \{g_n, w_n, z_n\}$ for all $n \geq 1$.

Given a set $\Sigma_4 = \{x_1, x_2, x_3, x_4\}$, out of all possible permutations of Σ_4 of the form

$$\phi = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ \phi(x_1) & \phi(x_2) & \phi(x_3) & \phi(x_4) \end{pmatrix},$$

there are 10 mappings that are involutions on Σ_4. We denote them by ϕ_i, $1 \leq i \leq 10$, and they are listed in Table 1.

Note that the mapping ϕ_1 is the involution whereby all letters are mapped to themselves (the identity on Σ_4). The mappings ϕ_i, $2 \leq i \leq 7$, are the involutions whereby two of the letters are mapped to each other, and the other two are mapped to themselves. The mappings ϕ_8, ϕ_9, ϕ_{10}, are the only involutions whereby two of the letters are mapped to each other, and the other two letters are also mapped to each other.

Throughout this paper, we use the convention that for a sequence (x_1, x_2, x_3, x_4) of letters from Σ_4, we have $x_i \neq x_j$ whenever $i \neq j$. In the particular case of the DNA

	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	ϕ_9	ϕ_{10}
x_1	x_1	x_2	x_3	x_4	x_1	x_1	x_1	x_2	x_3	x_4
x_2	x_2	x_1	x_2	x_2	x_3	x_4	x_2	x_1	x_4	x_3
x_3	x_3	x_3	x_1	x_3	x_2	x_3	x_4	x_4	x_1	x_2
x_4	x_4	x_4	x_4	x_1	x_4	x_2	x_3	x_3	x_2	x_1

Table 1: List of all possible involutions over the set $\Sigma_4 = \{x_1, x_2, x_3, x_4\}$. If the letter in the cell of the column of ϕ_i and row of x_j is x_k, where $1 \le i \le 10$ and $1 \le j, k \le 4$, this denotes that $\phi_i(x_j) = x_k$. For example, $\phi_3(x_3) = x_1$.

alphabet, that is, where $\Sigma_4 = \Delta = \{A, C, G, T\}$, there are a total of $4! = 24$ possibilities for the choice of (x_1, x_2, x_3, x_4) for $x_i \ne x_j$, $i \ne j$. For each such choice of (x_1, x_2, x_3, x_4), the Watson-Crick involution θ_{DNA} will coincide, on Δ^*, with one of $\phi_8, \phi_9, \phi_{10}$. For example, if we fix $(x_1, x_2, x_3, x_4) = (A, C, G, T)$, then ϕ_{10} coincides with θ_{DNA} on Δ^*, whereas if we fix $(x_1, x_2, x_3, x_4) = (C, G, A, T)$, then ϕ_8 coincides with θ_{DNA} on Δ^*, and if we fix $(x_1, x_2, x_3, x_4) = (A, C, T, G)$, then ϕ_9 coincides with θ_{DNA} on Δ^*.

Table 2 illustrates a particular case of Table 1, where $(x_1, x_2, x_3, x_4) = (A, C, G, T)$, and for this example we list all possible mappings ϕ on Δ that can be extended to an (anti)morphic involution on Δ^*.

	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	ϕ_9	ϕ_{10}
A	A	C	G	T	A	A	A	C	G	T
C	C	A	C	C	G	T	C	A	T	G
G	G	G	A	G	C	G	T	T	A	C
T	T	T	T	A	T	C	G	G	C	A

Table 2: List of all possible involutions over the DNA alphabet $\Delta = \{A, C, G, T\}$.

In the remainder of the paper, the mapping ϕ_i on Δ extended to a morphic involution on Δ^* will be denoted by μ_i, for $1 \le i \le 10$, and similarly, the mapping ϕ_i on Δ extended to an antimorphic involution on Δ^* will be denoted by θ_i, for $1 \le i \le 10$. Note that for $(x_1, x_2, x_3, x_4) = (A, C, G, T)$, the morphic involution μ_1 is the identity on Δ^*, the antimorphic involution θ_1 is the mirror image, and the antimorphic involution $\theta_{10} = \theta_{DNA}$ formalizes the Watson-Crick complementarity of DNA strings in Δ^*, see [13,22].

It is well known that atom Fibonacci words are primitive [8]. In this paper, we study the primitivity of n-th atom ϕ-Fibonacci words for all possible involution mappings ϕ on a four-letter alphabet (see Table 1), and all $n \ge 1$. Note that the first and second atom ϕ-Fibonacci words are singleton letters and hence primitive, and therefore we only need to prove primitivity results for $n \ge 3$. We recall the following from [8,15,18,31].

Theorem 5. *[8] For $n \ge 1$, the atom Fibonacci word f_n is primitive.*

Lemma 6. *[18] Let $x, y \in \Sigma^+$ be two non-empty words.*

(I) *If $xy = p^i$, $p \in Q$, $i \ge 1$, then, $yx = q^i$ for some $q \in Q$.*

(II) *If $xy = yx$, then $\rho(x) = \rho(y)$.*

Proposition 7. *[31] Let p and q be primitive and $d = \gcd(|p|,|q|)$. If $p^m = qx$, for $m \geq 2$ with $q = xy$, for $y \in \Sigma^+$ and $|x| \geq |p| - d$, then $p = q$.*

Proposition 8. *[15] Let $u,v \in \Sigma^+$ and ϕ be an (anti)morphic involution on Σ^* such that $\phi(u) = v$. Then, for all $n \geq 3$, we have:*

(I) *If n is odd, $g_n^\phi(u,v) = u^{F_n}$, and if n is even, $g_n^\phi(u,v) = v^{F_n}$.*

(II) *If $\phi = \theta$ is an antimorphic involution and u and v are palindromes, then we have $w_n^\theta(u,v) = (uv)^i$, where $n \bmod 3 = 0, i = \frac{|F_n|}{2}$.*

Theorem 9. *[15] Let $\phi = \mu$ be a morphic involution on Σ^*, $u,v \in \Sigma^+$, and $(\alpha_n, \beta_n) \in \{(f_n,g_n),(g_n,f_n),(z_n,w_n),(w_n,z_n)\}$, for all $n \geq 1$. Then, for all $n \geq 1$, we have $\alpha_n^\mu(u,v) = \beta_n^\mu(u,\mu(v))$ if n is odd, and $\alpha_n^\mu(u,v) = \beta_n^\mu(\mu(u),v)$ otherwise.*

Theorem 10. *[15] Let ϕ be an (anti)morphic involution on Σ^*, $\alpha_n \in \{g_n,w_n,z_n\}$ for all $n \geq 1$, and $u,v \in \Sigma^+$ be two palindromes. Then, for all $n \geq 1$, we have $[\alpha_n^\phi(u,v)]' = [\alpha_n^\phi(u,v)]^R$.*

From Theorem 10 and the fact that words of length 1 are palindromes, we have the following observation.

Lemma 11. *Let ϕ be an (anti)morphic involution on Σ^*, $\alpha_n \in \{g_n,w_n,z_n\}$ for all $n \geq 1$, and $a,b \in \Sigma$. For $n \geq 1$, the atom standard ϕ-Fibonacci word $\alpha_n^\phi(a,b)$ is primitive iff the atom reverse ϕ-Fibonacci word $[\alpha_n^\phi(a,b)]'$ is primitive.*

By Lemma 11 it is sufficient to discuss the primitivity of atom standard ϕ-Fibonacci words.

Lastly, in the remainder of this paper, we will make an extensive use of the following result.

Lemma 12. *For a word x over an alphabet Σ, we have that if $\gcd(|x|_a,|x|_b) = 1$ for any two letters $a,b \in \Sigma$, then x is primitive.*

Proof. We prove the contrapositive. By definition, if x is not primitive, then it can be written as $x = p^i$, where $p \in Q$ and $i > 1$. For all pairs of letters $a,b \in \mathrm{Alph}(x)$, we have $|x|_a = i \cdot |p|_a$ and $|x|_b = i \cdot |p|_b$. Therefore, we have, $\gcd(|x|_a,|x|_b)$ is a multiple of i, thus, $\gcd(|x|_a,|x|_b) \neq 1$. For all pairs of letters $a,b \in \Sigma$, if one of the letters is not in $\mathrm{Alph}(x)$, then $\gcd(|x|_a,|x|_b) \neq 1$. □

3. Primitivity of atom ϕ-Fibonacci words with different initial letters

In this section, we discuss the primitivity of atom ϕ-Fibonacci words $\alpha_n^\phi(a,b)$ with different initial letters $a,b \in \Sigma_4$, for all $n \geq 1$, where ϕ is an (anti)morphic involution on Σ_4^*, and $\alpha_n \in \{g_n,w_n,z_n\}$ for all $n \geq 1$. We first show that if we have an alphabet $\Sigma_4 = \{x_1,x_2,x_3,x_4\}$ and a sequence (x_1,x_2,x_3,x_4), if we choose x_1 and x_2 as the two initial letters of the ϕ-Fibonacci sequence $\{\alpha_n^{\phi}(x_1,x_2)\}_{n\ge1}$, it is enough to discuss the primitivity properties of atom ϕ-Fibonacci words for the mappings $\phi_1,\phi_2,\phi_4,\phi_5,\phi_{10}$.

Note that if $\phi=\mu_1$, the identity function on Σ_4^*, and $\alpha_n\in\{g_n,w_n,z_n\}$ for all $n\ge1$, we have that the atom ϕ-Fibonacci words $\alpha_n^{\mu_1}(x_1,x_2)$ coincide with the classical Fibonacci words $f_n(x_1,x_2)$, which are primitive when $x_1,x_2\in\Sigma_4$, for all $n\ge1$, see [8].

Lastly, note that the proofs for the results of this section hold for any choice of (x_1,x_2,x_3,x_4) and with x_1 and x_2 as the initial letters of the ϕ-Fibonacci sequence: This justifies stating/proving the subsequent results for only one of the cases (usually the sequence (A,C,G,T) with A and C as the two initial letters).

In the sequel, we denote by $[x]_{a\to b}$ the word obtained from x by replacing all occurrences of a in x by b, and denote by $[x]_{a\rightleftarrows b}$ the word obtained from x by replacing all occurrences of a by b and all occurrences of b by a. For example, if $x=abbab$ then $[x]_{a\to b}=[abbab]_{a\to b}=bbbbb$ and $[x]_{a\rightleftarrows b}=[abbab]_{a\rightleftarrows b}=baaba$. We first observe the following.

Lemma 13. *Let $a,b\in\Sigma$ and $x\in\Sigma^*$. We have that:*

- (I) $[x]_{a\rightleftarrows b}=[x]_{b\rightleftarrows a}$.
- (II) *If $b\notin Alph(x)$, we have that $[x]_{a\to b}=[x]_{a\rightleftarrows b}$.*
- (III) *If $b\notin Alph(x)$, we have $[[x]_{a\to b}]_{b\to a}=x$.*

Lemma 14. *Let $a,b\in\Sigma$, $x\in\Sigma^*$ and $i\ge0$. We have that $[x^i]_{a\to b}=([x]_{a\to b})^i$ and $[x^i]_{a\rightleftarrows b}=([x]_{a\rightleftarrows b})^i$.*

Lemma 15. *Let $a,b\in\Sigma$ and $x\in\Sigma^*$. We have that:*

- (I) *If $b\notin Alph(x)$, then x is primitive iff $[x]_{a\to b}$ is primitive.*
- (II) *If $a,b\in Alph(x)$, then x is primitive iff $[x]_{a\rightleftarrows b}$ is primitive.*

Proof. For statement (I), if $a\notin \mathrm{Alph}(x)$, then $x=[x]_{a\to b}$ and hence x is primitive iff $[x]_{a\to b}$ is primitive. Let, $a\in\mathrm{Alph}(x)$. Assume x is not primitive but $[x]_{a\to b}$ is primitive. Since x is not primitive, we have $x=q^i$, where $q\in Q$ and $i\ge2$. By Lemma 14, we have $[x]_{a\to b}=[q^i]_{a\to b}=([q]_{a\to b})^i$, which is a contradiction. The case where x is primitive but $[x]_{a\to b}$ is not primitive can be proved similarly. Therefore, x is primitive iff $[x]_{a\to b}$ is primitive. The statement (II) can be proved similarly using Lemma 14. □

Theorem 16. *Let $\phi_i\in\{\mu_i,\theta_i\}$, $1\le i\le10$, be an (anti)morphic involution on Σ_4^* and $\alpha_n\in\{g_n,w_n,z_n\}$ for $n\ge1$. For $n\ge1$, the following statements hold for ϕ_i-Fibonacci words $\alpha_n^{\phi_i}(x_1,x_2)$:*

- (I) $[\alpha_n^{\phi_4}(x_1,x_2)]_{x_4\to x_3}$ *is primitive iff* $\alpha_n^{\phi_4}(x_1,x_2)$ *is primitive.*
- (II) $[\alpha_n^{\phi_5}(x_1,x_2)]_{x_3\to x_4}$ *is primitive iff* $\alpha_n^{\phi_5}(x_1,x_2)$ *is primitive.*
- (III) $[\alpha_n^{\phi_{10}}(x_1,x_2)]_{x_4\rightleftarrows x_3}$ *is primitive iff* $\alpha_n^{\phi_{10}}(x_1,x_2)$ *is primitive.*

Proof.

- (I) Let $\alpha_n\in\{g_n,w_n,z_n\}$ for all $n\ge1$. One can easily prove by induction that, for all $n\ge1$, we have $x_3\notin\mathrm{Alph}(\alpha_n^{\phi_4}(x_1,x_2))$. Hence by Lemma 15, the statement holds.

(II) Let $\alpha_n \in \{g_n, w_n, z_n\}$ for all $n \geq 1$. One can prove by induction that $x_4 \notin \mathrm{Alph}(\alpha_n^{\phi_5}(x_1,x_2))$ for all $n \geq 1$. Hence by Lemma 15, the statement holds.

(III) We have two cases. The first one is when $\alpha_n \in \{w_n, z_n\}$ for all $n \geq 1$. In this case, if $1 \leq n \leq 4$, the statement can be easily verified. If $n \geq 5$, then one can easily prove by induction that $x_3, x_4 \in \mathrm{Alph}(\alpha_n^{\phi_{10}}(x_1,x_2))$. Thus, the statement holds by Lemma 15.

The second case is when $\alpha_n = g_n$ for all $n \geq 1$. In this case, one can first prove by induction that, for all $n \geq 1$, we have $x_3 \notin \mathrm{Alph}(g_n^{\phi_{10}}(x_1,x_2))$ if n is even, and $x_4 \notin \mathrm{Alph}(g_n^{\phi_{10}}(x_1,x_2))$ if n is odd. Using this fact, if n is even then by Lemma 13, we have $[g_n^{\phi_{10}}(x_1,x_2)]_{x_4 \rightleftarrows x_3} = [g_n^{\phi_{10}}(x_1,x_2)]_{x_3 \rightleftarrows x_4} = [g_n^{\phi_{10}}(x_1,x_2)]_{x_3 \to x_4}$. Also if n is even, then by Lemma 15, we have that $[g_n^{\phi_{10}}(x_1,x_2)]_{x_4 \rightleftarrows x_3} = [g_n^{\phi_{10}}(x_1,x_2)]_{x_3 \to x_4}$ is primitive iff $g_n^{\phi_{10}}(x_1,x_2)$ is primitive. The case where n is odd can be proved similarly.

□

We now show the equivalence of various ϕ-Fibonacci words $\alpha_n^{\phi}(x_1,x_2)$ for certain values of $\phi \in \{\phi_i \mid 1 \leq i \leq 10\}$. We use the following lemma, which can be proved easily by induction on n.

Lemma 17. *Let $\phi_i \in \{\mu_i, \theta_i\}$, $1 \leq i \leq 10$, be an (anti)morphic involution on Σ_4^* and $\alpha_n \in \{g_n, w_n, z_n\}$ for $n \geq 1$. The following equalities hold, for ϕ_i-Fibonacci words $\alpha_n^{\phi_i}(x_1,x_2)$ and for all $n \geq 1$:*

(I) $\phi_1(\alpha_n^{\phi_1}(x_1,x_2)) = \phi_7(\alpha_n^{\phi_7}(x_1,x_2))$ *iff* $\alpha_n^{\phi_1}(x_1,x_2) = \alpha_n^{\phi_7}(x_1,x_2)$.

(II) $\phi_2(\alpha_n^{\phi_2}(x_1,x_2)) = \phi_8(\alpha_n^{\phi_8}(x_1,x_2))$ *iff* $\alpha_n^{\phi_2}(x_1,x_2) = \alpha_n^{\phi_8}(x_1,x_2)$.

(III) $[\phi_4(\alpha_n^{\phi_4}(x_1,x_2))]_{x_4 \to x_3} = \phi_3(\alpha_n^{\phi_3}(x_1,x_2))$ *iff* $[\alpha_n^{\phi_4}(x_1,x_2)]_{x_4 \to x_3} = \alpha_n^{\phi_3}(x_1,x_2)$.

(IV) $[\phi_5(\alpha_n^{\phi_5}(x_1,x_2))]_{x_3 \to x_4} = \phi_6(\alpha_n^{\phi_6}(x_1,x_2))$ *iff* $[\alpha_n^{\phi_5}(x_1,x_2)]_{x_3 \to x_4} = \alpha_n^{\phi_6}(x_1,x_2)$.

(V) $[\phi_{10}(\alpha_n^{\phi_{10}}(x_1,x_2))]_{x_4 \rightleftarrows x_3} = \phi_9(\alpha_n^{\phi_9}(x_1,x_2))$ *iff* $[\alpha_n^{\phi_{10}}(x_1,x_2)]_{x_4 \rightleftarrows x_3} = \alpha_n^{\phi_9}(x_1,x_2)$.

Proof.

We only prove statement (I), by induction, and it is sufficient to prove it for one of $\alpha_n \in \{g_n, w_n, z_n\}$ for all $n \geq 1$ since the other cases are similar. Let ϕ_1 be a morphic involution, and, without loss of generality, let $\alpha_n = z_n$ for all $n \geq 1$. By Definition 4 and Table 1, the result holds for $n = 1$ and $n = 2$. For the inductive step, assume that $\phi_1(z_i^{\phi_1}(x_1,x_2)) = \phi_7(z_i^{\phi_7}(x_1,x_2))$ iff $z_i^{\phi_1}(x_1,x_2) = z_i^{\phi_7}(x_1,x_2)$ for all $3 \leq i < k$. We now have to prove that the equivalence holds for k.

For the direct implication, assume $\phi_1(z_k^{\phi_1}(x_1,x_2)) = \phi_7(z_k^{\phi_7}(x_1,x_2))$. Therefore, we have $\phi_1(z_k^{\phi_1}(x_1,x_2)) = \phi_1(z_{k-1}^{\phi_1}(x_1,x_2)\phi_1(z_{k-2}^{\phi_1}(x_1,x_2))) = \phi_1(z_{k-1}^{\phi_1}(x_1,x_2))z_{k-2}^{\phi_1}(x_1,x_2)$, and similarly we have $\phi_7(z_k^{\phi_7}(x_1,x_2)) = \phi_7(z_{k-1}^{\phi_7}(x_1,x_2))z_{k-2}^{\phi_7}(x_1,x_2)$. By $\phi_1(z_k^{\phi_1}(x_1,x_2)) = \phi_7(z_k^{\phi_7}(x_1,x_2))$, and the fact that $|z_{k-2}^{\phi_7}(x_1,x_2)| = |z_{k-2}^{\phi_1}(x_1,x_2)|$, we have $z_{k-2}^{\phi_7}(x_1,x_2) = z_{k-2}^{\phi_1}(x_1,x_2)$ and $\phi_7(z_{k-1}^{\phi_7}(x_1,x_2)) = \phi_1(z_{k-1}^{\phi_1}(x_1,x_2))$. Therefore, we have $z_k^{\phi_1}(x_1,x_2) = z_{k-1}^{\phi_1}(x_1,x_2)\phi_1(z_{k-2}^{\phi_1}(x_1,x_2)) = z_{k-1}^{\phi_7}(x_1,x_2)\phi_7(z_{k-2}^{\phi_7}(x_1,x_2)) = z_k^{\phi_7}(x_1,x_2)$ by inductive hypothesis.

For the converse implication, assume $z_k^{\phi_1}(x_1,x_2) = z_k^{\phi_7}(x_1,x_2)$. Therefore, we have $z_k^{\phi_1}(x_1,x_2) = z_{k-1}^{\phi_1}(x_1,x_2)\phi_1(z_{k-2}^{\phi_1}(x_1,x_2))$, and similarly we have $z_k^{\phi_7}(x_1,x_2) = z_{k-1}^{\phi_7}(x_1,x_2)\phi_7(z_{k-2}^{\phi_7}(x_1,x_2))$. By $z_k^{\phi_1}(x_1,x_2) = z_k^{\phi_7}(x_1,x_2)$, and the fact that $|z_{k-1}^{\phi_7}(x_1,x_2)| = |z_{k-1}^{\phi_1}(x_1,x_2)|$, we have $z_{k-1}^{\phi_7}(x_1,x_2) = z_{k-1}^{\phi_1}(x_1,x_2)$ and $\phi_7(z_{k-2}^{\phi_7}(x_1,x_2)) = \phi_1(z_{k-2}^{\phi_1}(x_1,x_2))$. Therefore, we have $\phi_1(z_k^{\phi_1}(x_1,x_2)) = \phi_1(z_{k-1}^{\phi_1}(x_1,x_2)\phi_1(z_{k-2}^{\phi_1}(x_1,x_2))) = \phi_1(z_{k-1}^{\phi_1}(x_1,x_2))z_{k-2}^{\phi_1}(x_1,x_2) = \phi_7(z_{k-1}^{\phi_7}(x_1,x_2))z_{k-2}^{\phi_7}(x_1,x_2) = \phi_7(z_{k-1}^{\phi_7}(x_1,x_2)\phi_7(z_{k-2}^{\phi_1}(x_1,x_2))) = \phi_7(z_k^{\phi_7}(x_1,x_2))$ by inductive hypothesis. The case when ϕ_1 is an antimorphic involution is similar. □

We now prove the following.

Lemma 18. *Let $\phi_i \in \{\mu_i,\theta_i\}$, $1 \le i \le 10$, be an (anti)morphic involution on Σ_4^* and $\alpha_n \in \{g_n,w_n,z_n\}$ for $n \ge 1$. For all $n \ge 1$, the following equalities hold for ϕ_i-Fibonacci words $\alpha_n^{\phi_i}(x_1,x_2)$:*

(I) $\alpha_n^{\phi_1}(x_1,x_2) = \alpha_n^{\phi_7}(x_1,x_2)$.

(II) $\alpha_n^{\phi_2}(x_1,x_2) = \alpha_n^{\phi_8}(x_1,x_2)$.

(III) $[\alpha_n^{\phi_4}(x_1,x_2)]_{x_4\to x_3} = \alpha_n^{\phi_3}(x_1,x_2)$.

(IV) $[\alpha_n^{\phi_5}(x_1,x_2)]_{x_3\to x_4} = \alpha_n^{\phi_6}(x_1,x_2)$.

(V) $[\alpha_n^{\phi_{10}}(x_1,x_2)]_{x_4\rightleftarrows x_3} = \alpha_n^{\phi_9}(x_1,x_2)$.

Proof. We only prove statement (I), by induction, as the other cases are similar. By Definition 4, the result holds for $n = 1$ and $n = 2$. We assume $\alpha_i^{\phi_1}(x_1,x_2) = \alpha_i^{\phi_7}(x_1,x_2)$ holds for $3 \le i < k$. It is enough to prove for one of $\alpha_n \in \{g_n,w_n,z_n\}$ for all $n \ge 1$. Without loss of generality, let $\alpha_n = z_n$ for all $n \ge 1$. Then, $\alpha_k^{\phi_1}(x_1,x_2) = z_k^{\phi_1}(x_1,x_2) = z_{k-1}^{\phi_1}(x_1,x_2)\phi_1(z_{k-2}^{\phi_1}(x_1,x_2))$. By inductive hypothesis and by Lemma 17 we have,

$$z_k^{\phi_1}(x_1,x_2) = z_{k-1}^{\phi_7}(x_1,x_2)\phi_7(z_{k-2}^{\phi_7}(x_1,x_2)) = z_k^{\phi_7}(x_1,x_2).$$

Hence, the result. □

As a consequence of Theorem 16 and Lemma 18, we only need to study the primitivity of atom ϕ-Fibonacci words $\alpha_n^{\phi}(x_1,x_2)$ for all $n \ge 1$, when $\phi \in \{\theta_1,\theta_2,\theta_4,\theta_5,\theta_{10},\mu_2,\mu_4,\mu_5,\mu_{10}\}$, and $\alpha_n \in \{g_n,w_n,z_n\}$ for all $n \ge 1$.

Note that the results obtained above hold for any choice of (x_1,x_2,x_3,x_4) and with x_1 and x_2 as the initial letters of the ϕ-Fibonacci sequence. Therefore, in the remainder of this section, we will only prove primitivity results about one of the cases, namely the sequence (A,C,G,T) over the DNA alphabet Δ, with A and C as the two initial letters.

3.1. Atom alternating ϕ-Fibonacci words

We first discuss the primitivity of atom alternating ϕ-Fibonacci words g_n for $n \ge 1$.

In Table 3, we give the first few values of the sequences $\{g_n^{\phi}(A,C)\}_{n\ge1}$ for (anti)morphic involutions $\phi \in \{\mu_1,\mu_2,\mu_4,\mu_5,\mu_{10},\theta_1,\theta_2,\theta_4,\theta_5,\theta_{10}\}$. We recall the following lemma from [15].

ϕ	$g_3^{\phi}(A,C)$	$g_4^{\phi}(A,C)$	$g_5^{\phi}(A,C)$	$g_6^{\phi}(A,C)$	$g_7^{\phi}(A,C)$
μ_1	CA	CAC	$CACCA$	$CACCACAC$	$CACCACACCACCA$
μ_2	AA	CCC	$AAAAA$	$CCCCCCCC$	$AAAAAAAAAAAAA$
μ_4	CA	CTC	$CACCA$	$CTCCTCTC$	$CACCACACCACCA$
μ_5	GA	CAC	$GAGGA$	$CACCACAC$	$GAGGAGAGGAGGA$
μ_{10}	GA	CTC	$GAGGA$	$CTCCTCTC$	$GAGGAGAGGAGGA$
θ_1	CA	ACC	$CCACA$	$ACACCACC$	$CCACCACACCACA$
θ_2	AA	CCC	$AAAAA$	$CCCCCCCC$	$AAAAAAAAAAAAA$
θ_4	CA	TCC	$CCACA$	$TCTCCTCC$	$CCACCACACCACA$
θ_5	GA	ACC	$GGAGA$	$ACACCACC$	$GGAGGAGAGGAGA$
θ_{10}	GA	TCC	$GGAGA$	$TCTCCTCC$	$GGAGGAGAGGAGA$

Table 3: List of words $g_n^{\phi}(A,C)$, where $3 \le n \le 7$ and $\phi \in \{\mu_1,\mu_2,\mu_4,\mu_5,\mu_{10},\theta_1,\theta_2,\theta_4,\theta_5,\theta_{10}\}$.

Lemma 19. *[15] Let $\Sigma = \{A,C\}$, and $f_1 = A$ and $f_2 = C$. Then, for $n \ge 3$, we have that $f_n = s_n d_n$ and $f'_n = d'_n s_n$ where $d'_n = d_n^R$ such that s_n is a palindrome, and $d_n = AC$ if n is even, while $d_n = CA$ if n is odd.*

Using Lemma 19, we prove the following theorem. Note that the cyclic shift by 1 position from the front of a word $x \in \Sigma^*$ to the end of it, can be represented by a composition of (left) derivative, concatenation and finite union, that is, by $\bigcup_{a\in\Sigma}(\partial_a x)a$.

Theorem 20. *Let $\phi = \theta_1$. For all $n \ge 1$, the atom ϕ-Fibonacci word $g_n^{\theta_1}(A,C)$ is a conjugate of $f'_n(A,C)$. More precisely,*

$$g_n^{\theta_1} = \begin{cases} (\partial_C f'_n)C \ : \ n \bmod 2 = 0, \\ (\partial_A f'_n)A \ : \ n \bmod 2 = 1. \end{cases}$$

Proof. For $1 \le n \le 7$, this can be easily checked from Table 3. Assume the statement holds for $g_i^{\theta_1}$, where $7 \le i < k$. We now prove this for g_k. Without loss of generality, let k be even. Then, by definition and Lemma 19, we have

$$\begin{aligned} g_k^{\theta_1} &= \theta_1(g_{k-1}^{\theta_1})g_{k-2}^{\theta_1} = (g_{k-1}^{\theta_1})^R g_{k-2}^{\theta_1} = ((\partial_A f'_{k-1})A)^R \cdot ((\partial_C f'_{k-2})C) \\ &= (((\partial_A f'_{k-3} f'_{k-2}))A)^R \cdot ((\partial_C f'_{k-2})C) = A(Cs_{k-3}CAs_{k-2})^R As_{k-2}C \\ &= As_{k-2}ACs_{k-3}CAs_{k-2}C = As_{k-2}(f'_{k-3}f'_{k-2})C = As_{k-2}f'_{k-1}C \\ &= (\partial_C CAs_{k-2}f'_{k-1})C = (\partial_C f'_{k-2}f'_{k-1})C = (\partial_C f'_k)C. \end{aligned}$$

□

By Theorem 5 and Lemma 6, we have the following corollary.

Corollary 21. *Let $\phi = \theta_1$. For all $n \ge 1$, the atom ϕ-Fibonacci word $g_n^{\theta_1}(A,C)$ is primitive.*

We now use the following lemma which is a generalized version of a result proved in [15]. The result was proved in [15] when ϕ is a morphic involution. We show that the result also holds when ϕ is an antimorphic involution.

Lemma 22. *Let ϕ be an (anti)-morphic involution on Σ^*, let μ_1 be a morphic involution on Σ^* such that $\mu_1\phi = \phi\mu_1$, and let $u, v \in \Sigma^+$. If $\alpha_n \in \{f_n, g_n, w_n, z_n\}$ for all $n \geq 1$, then for all $n \geq 1$, we have $\mu_1(\alpha_n^\phi(u,v)) = \alpha_n^\phi(\mu_1(u), \mu_1(v))$, and $\mu_1([\alpha_n^\phi(u,v)]') = [\alpha_n^\phi(\mu_1(u), \mu_1(v))]'$.*

Proof. We consider the standard ϕ-Fibonacci words g_n. The proof is by induction on n. By definition of g_n (Definition 4), we have $\mu_1(g_1(u,v)) = \mu_1(u) = g_1(\mu_1(u), \mu_1(v))$, and $\mu_1(g_2(u,v)) = \mu_1(v) = g_2(\mu_1(u), \mu_1(v))$, so the base case holds. Assume that $\mu_1(g_i(u,v)) = g_i(\mu_1(u), \mu_1(v))$, for all $1 \leq i \leq k$. Using the definition of g_n (Definition 4), the fact that μ_1 is a morphism, and the induction hypothesis, we have,

$$\begin{aligned}
\mu_1(g_{k+1}(u,v)) &= \mu_1(\phi(g_k(u,v)) \cdot g_{k-1}(u,v)) \\
&= \phi(\mu_1(g_k(u,v))) \cdot \mu_1(g_{k-1}(u,v)) \\
&= \phi(g_k(\mu_1(u), \mu_1(v))) \cdot g_{k-1}(\mu_1(u), \mu_1(v)) \\
&= g_{k+1}(\mu_1(u), \mu_1(v)).
\end{aligned}$$

The proofs for other ϕ-Fibonacci words are similar. $\square$

We have the following result which can be proved by induction and Lemma 22. We first observe that $\theta_i = \theta_1\mu_i = \mu_i\theta_1$ for $1 \leq i \leq 10$.

Lemma 23. *For $i \in \{4, 5, 10\}$ and for all $n \geq 1$, the following relations between the atom alternating ϕ-Fibonacci words $g_n^{\theta_1}$ and the atom alternating ϕ-Fibonacci words $g_n^{\theta_i}(A,C)$ hold.*

(I) *If $i = 10$, for $n \geq 1$, we have that*

$$g_n^{\theta_{10}}(A,C) = \begin{cases} g_n^{\theta_1}(T,C) : n \bmod 2 = 0, \\ g_n^{\theta_1}(A,G) : n \bmod 2 = 1. \end{cases}$$

(II) *If $i = 5$, for $n \geq 1$, we have that*

$$g_n^{\theta_5}(A,C) = \begin{cases} g_n^{\theta_1}(A,C) : n \bmod 2 = 0 \\ g_n^{\theta_{10}}(A,C) : n \bmod 2 = 1 \end{cases} = \begin{cases} g_n^{\theta_1}(A,C) : n \bmod 2 = 0, \\ g_n^{\theta_1}(A,G) : n \bmod 2 = 1. \end{cases}$$

(III) *If $i = 4$, for $n \geq 1$, we have that*

$$g_n^{\theta_4}(A,C) = \begin{cases} g_n^{\theta_{10}}(A,C) : n \bmod 2 = 0 \\ g_n^{\theta_1}(A,C) : n \bmod 2 = 1 \end{cases} = \begin{cases} g_n^{\theta_1}(T,C) : n \bmod 2 = 0, \\ g_n^{\theta_1}(A,C) : n \bmod 2 = 1. \end{cases}$$

Proof. We only prove statement (I), by induction on n, as the other cases are similar. By definition of g_n (Definition 4), the result holds for $n = 1$ and $n = 2$. Assume the statement to be true for $g_i^{\theta_{10}}(A,C)$, where $3 \leq i < k$. If k is even, then $k \bmod 2 = 0$ and by inductive hypothesis,

$$g_k^{\theta_{10}}(A,C) = \theta_{10}(g_{k-1}^{\theta_{10}}(A,C))\, g_{k-2}^{\theta_{10}}(A,C) = \theta_{10}(g_{k-1}^{\theta_1}(A,G))\, g_{k-2}^{\theta_1}(T,C)$$

$$= \theta_1(\mu_{10}(g^{\theta_1}_{k-1}(A,G)))\, g^{\theta_1}_{k-2}(T,C).$$

Then by Lemma 22, we have,

$$g^{\theta_{10}}_k(A,C) = \theta_1(g^{\theta_1}_{k-1}(\mu_{10}(A),\mu_{10}(G)))\, g^{\theta_1}_{k-2}(T,C) = \theta_1(g^{\theta_1}_{k-1}(T,C))\, g^{\theta_1}_{k-2}(T,C) = g^{\theta_1}_k(T,C).$$

The case when k is odd is similar. □

Hence, we conclude the following.

Theorem 24. *For all $n \geq 1$ and $\phi \in \{\theta_4,\theta_5,\theta_{10}\}$, the atom ϕ-Fibonacci word $g^{\phi}_n(A,C)$ is primitive.*

Proof. It is clear from Corollary 21, that the atom ϕ-Fibonacci words $g^{\theta_1}_n(T,C)$, $g^{\theta_1}_n(A,G)$ and $g^{\theta_1}_n(A,C)$ are primitive for $n \geq 1$. Hence, from Lemma 23, we conclude that $g^{\phi}_n(A,C)$ are primitive for all $n \geq 1$ and $\phi \in \{\theta_4,\theta_5,\theta_{10}\}$. □

We now show (Theorem 25), that for all $\mu = \mu_i$, $i \in \{4,5,10\}$, the atom μ-Fibonacci words g_n are primitive for all $n \geq 1$.

Theorem 25. *Let $\phi \in \{\mu_4,\mu_5,\mu_{10}\}$. For all $n \geq 1$, the atom ϕ-Fibonacci word $g^{\phi}_n(A,C)$ is primitive.*

Proof. By Theorem 9, we have $g^{\phi}_n(A,C) = f_n(A,\phi(C))$ if n is odd, and $g^{\phi}_n(A,C) = f_n(\phi(A),C)$, otherwise. Note that $\phi(A) \neq C$ and $\phi(C) \neq A$ for $\phi \in \{\mu_4,\mu_5,\mu_{10}\}$. Then, by Theorem 5, the word $g^{\phi}_n(A,C)$ is primitive for $\phi \in \{\mu_4,\mu_5,\mu_{10}\}$ and $n \geq 1$. □

We have the following theorem.

Theorem 26. *Let $\phi \in \{\mu_2,\theta_2\}$. For all $n \geq 3$, the atom ϕ-Fibonacci word $g^{\phi}_n(A,C)$ is not primitive.*

Proof. Note that for $\phi \in \{\mu_2,\theta_2\}$, ϕ maps A to C and vice versa, and hence, by Proposition 8, we have, $g^{\phi}_n = A^{F_n}$ when n is odd and $g^{\phi}_n = C^{F_n}$ when n is even. Thus, g^{ϕ}_n is not primitive for $\phi \in \{\mu_2,\theta_2\}$ for $n \geq 3$. □

3.2. Atom palindromic ϕ-Fibonacci words

In this subsection, we discuss the primitivity of atom palindromic ϕ-Fibonacci words. In Table 4, we give the first few values of the sequences $\{w^{\phi}_n(A,C)\}_{n\geq 1}$ for (anti)morphic involutions $\phi \in \{\mu_1,\mu_2,\mu_4,\mu_5,\mu_{10},\theta_1,\theta_2,\theta_4,\theta_5,\theta_{10}\}$.

By definition of ϕ and w^{ϕ}_n and using induction, we calculate the number of occurrences of letters in the words $w^{\phi}_n(A,C)$, for $n \geq 3$ and an (anti)morphic involution $\phi \in \{\phi_1,\phi_4,\phi_5,\phi_{10}\}$, and these values are summarized in Table 5.

We now discuss the primitivity of $w^{\mu_2}_n(A,C)$ for all $n \geq 1$, and we use the following lemma.

ϕ	$w_3^{\phi}(A,C)$	$w_4^{\phi}(A,C)$	$w_5^{\phi}(A,C)$	$w_6^{\phi}(A,C)$	$w_7^{\phi}(A,C)$
μ_1	AC	CAC	$CACCA$	$CACCACAC$	$CACCACACCACCA$
μ_2	AC	CAA	$ACCCA$	$CAAACACC$	$ACCCACAACAAAC$
μ_4	CT	CAC	$CTCCA$	$CACCTCTC$	$CTCCACACCACCT$
μ_5	GA	CAG	$GACCA$	$CAGGAGAC$	$GACCACAGCAGGA$
μ_{10}	GT	CAG	$GTCCA$	$CAGGTGTC$	$GTCCACAGCAGGT$
θ_1	CA	ACC	$CCAAC$	$CAACCCCA$	$ACCCCAACCAACC$
θ_2	AC	ACA	$CACAC$	$ACACACAC$	$ACACACACACACA$
θ_4	CT	ACC	$CCTAC$	$CTACCCCT$	$ACCCCTACCTACC$
θ_5	GA	ACG	$CGAAC$	$GAACGCGA$	$ACGCGAACGAACG$
θ_{10}	GT	ACG	$CGTAC$	$GTACGCGT$	$ACGCGTACGTACG$

Table 4: List of words $w_n^{\phi}(A,C)$, where $3 \le n \le 7$ and $\phi \in \{\mu_1,\mu_2,\mu_4,\mu_5,\mu_{10},\theta_1,\theta_2,\theta_4,\theta_5,\theta_{10}\}$.

<table>
<tr><td></td><td>$i=1$</td><td colspan="3">$i=2$</td><td colspan="3">$i=5$</td></tr>
<tr><td>n mod 3</td><td>≥ 0</td><td>0</td><td>1</td><td>2</td><td>0</td><td>1</td><td>2</td></tr>
<tr><td>$|w_n^{\phi_i}(A,C)|_A$</td><td>F_{n-2}</td><td rowspan="2">$\frac{F_n}{2}$</td><td>$\frac{F_n+1}{2}$</td><td>$\frac{F_n-1}{2}$</td><td colspan="3">F_{n-2}</td></tr>
<tr><td>$|w_n^{\phi_i}(A,C)|_C$</td><td>F_{n-1}</td><td>$\frac{F_n-1}{2}$</td><td>$\frac{F_n+1}{2}$</td><td>$\frac{F_{n-1}-1}{2}$</td><td rowspan="2">$\frac{F_{n-1}}{2}$</td><td>$\frac{F_{n-1}+1}{2}$</td></tr>
<tr><td>$|w_n^{\phi_i}(A,C)|_G$</td><td>–</td><td colspan="3">–</td><td>$\frac{F_{n-1}+1}{2}$</td><td>$\frac{F_{n-1}-1}{2}$</td></tr>
</table>

<table>
<tr><td></td><td colspan="3">$i=4$</td><td colspan="3">$i=10$</td></tr>
<tr><td>n mod 3</td><td>0</td><td>1</td><td>2</td><td>0</td><td>1</td><td>2</td></tr>
<tr><td>$|w_n^{\phi_i}(A,C)|_A$</td><td>$\frac{F_{n-2}-1}{2}$</td><td>$\frac{F_{n-2}+1}{2}$</td><td>$\frac{F_{n-2}}{2}$</td><td>$\frac{F_{n-2}-1}{2}$</td><td>$\frac{F_{n-2}+1}{2}$</td><td>$\frac{F_{n-2}}{2}$</td></tr>
<tr><td>$|w_n^{\phi_i}(A,C)|_C$</td><td colspan="3">F_{n-1}</td><td>$\frac{F_{n-1}-1}{2}$</td><td rowspan="2">$\frac{F_{n-1}}{2}$</td><td>$\frac{F_{n-1}+1}{2}$</td></tr>
<tr><td>$|w_n^{\phi_i}(A,C)|_G$</td><td colspan="3">–</td><td>$\frac{F_{n-1}+1}{2}$</td><td>$\frac{F_{n-1}-1}{2}$</td></tr>
<tr><td>$|w_n^{\phi_i}(A,C)|_T$</td><td>$\frac{F_{n-2}+1}{2}$</td><td>$\frac{F_{n-2}-1}{2}$</td><td>$\frac{F_{n-2}}{2}$</td><td>$\frac{F_{n-2}+1}{2}$</td><td>$\frac{F_{n-2}-1}{2}$</td><td>$\frac{F_{n-2}}{2}$</td></tr>
</table>

Table 5: The numbers of occurrences of letters in the atom palindromic ϕ_i-Fibonacci words $w_n^{\phi_i}(A,C)$ for $\phi_i \in \{\theta_i,\mu_i\}$, $i \in \{1,2,4,5,10\}$ and $n \ge 3$.

Lemma 27. *Let Σ be an alphabet and $a,b \in \Sigma$ be letters. The following hold.*

(I) *For all $n \ge 1$, F_n mod $2 = 0$ iff n mod $3 = 0$.*

(II) *For $\phi \in \{\mu_2,\mu_8\}$ and for all $n \ge 2$, $w_{3n}^{\phi}(a,b) \ne x\phi(x)$ where $x \in \Sigma^+$.*

Proof. Statement (I) can be proved easily by induction and using properties of modulo operation. We now prove statement (II) for the case $\phi = \mu_2$ (the case when $\phi = \mu_8$ is similar). One can easily verify from Table 4 that the statement holds for $n = 2$. Assume now that $w_{3i} \ne x\mu_2(x)$, with $x \in \Sigma^+$, for all $3 \le i < k$. Then, by definition of w_n (Definition 4) and by the inductive hypothesis, we have $w_{3k} = \mu_2(w_{3k-1})\mu_2(w_{3k-2}) = w_{3k-2}w_{3k-3}\mu_2(w_{3k-2}) \ne w_{3k-2}x\mu_2(x)\mu_2(w_{3k-2})$. Hence, the result holds. □

Proposition 28. *Let $\phi = \mu_2$. For all $n \geq 1$ such that $n \bmod 3 = 0$, the atom palindromic ϕ-Fibonacci word $w_n^{\mu_2}(A,C)$ is primitive.*

Proof. By Table 4, w_n is primitive for $1 \leq n \leq 7$. Suppose w_n is not primitive for $n > 7$ and $n \bmod 3 = 0$, then we have $w_n = p^j$, where $j \geq 2$ and $p \in Q$. By definition of w_n (Definition 4), the word w_n can be decomposed as $w_n = \mu_2(w_{n-1})\mu_2(w_{n-2}) = w_{n-2}w_{n-3}w_{n-3}w_{n-4} = w_{n-4}w_{n-5}\mu_2(w_{n-4})w_{n-3}w_{n-3}w_{n-4}$. Let $y = w_{n-4}w_{n-5}\mu_2(w_{n-4})w_{n-3}w_{n-3}$ such that $w_n = yx$ and $x = w_{n-4}$. Note that $(n-4) \bmod 3 = 2$ and by Table 5, we have $|w_n|_A = |w_n|_C$, and $|x|_A + 1 = |x|_C$, so $|y|_A = |y|_C + 1$. Hence, by Lemma 3 and Lemma 12, both y and x are primitive. Since $x = w_{n-4}$, we have $|x| = F_{n-4} \geq \frac{7F_{n-4}-F_{n-8}}{7} - 1 = \frac{F_n}{7} - 1$ and for $j \geq 7$ we get, $|x| \geq \frac{F_n}{j} - 1 \geq \frac{F_n}{j} - \gcd(|p|,|y|) = |p| - \gcd(|p|,|y|)$, as $\gcd(|p|,|y|) \geq 1$. Then, by Proposition 7, we have, $p = y$, which is impossible. Hence, $w_n \neq p^j$ for $j \geq 7$ and $p \in Q$.

We now consider the cases when $2 \leq j \leq 6$. We split it into three cases when j is even, $j = 3$ and $j = 5$.

(I) If j is even, then w_n can be written as $w_n = pp$, where $|p| = \frac{F_n}{2} > 4$. Since by Lemma 27, $|w_{n-3}|$ is even, there exist $x, y \in \Delta^+$ such that $|x| = |y|$ and $w_{n-3} = xy$. Then, by definition $w_n = w_{n-2}w_{n-3}\mu_2(w_{n-2})$, we have $p = w_{n-2}x = y\mu_2(w_{n-2})$ for $w_{n-3} = xy$. Thus, $p = w_{n-2}x = \mu_2(w_{n-3})\mu_2(w_{n-4})x = \mu_2(x)\mu_2(y)\mu_2(w_{n-4})x = y\mu_2(w_{n-2})$ implies $y = \mu_2(x)$ and $w_{n-3} = x\mu_2(x)$ which is a contradiction to Lemma 27, and hence, j cannot be even.

(II) If $j = 3$, then by induction, we have $F_n \bmod 3 = 0$ iff $n \bmod 4 = 0$. Therefore, if $n \bmod 4 \neq 0$, then w_n cannot be written as $w_n = q^3$, where $q \in Q$, so we only need to consider the case where $n \bmod 12 = 0$. Note that the word w_n can be written as $w_n = \mu_2(w_{n-3})\mu_2(w_{n-4})w_{n-3}w_{n-3}w_{n-4} = w_{n-4}w_{n-5}\mu_2(w_{n-4})\mu_2(w_{n-4})\mu_2(w_{n-5})\mu_2(w_{n-4})\mu_2(w_{n-5})w_{n-4} = q^3$ and $|F_{n-4}|$ is divisible by 3. Then, there exist $x, y, r \in \Delta^+$ such that $|x| = |y| = |r|$ and $w_{n-4} = xyr$ and since $w_n = q^3$, we have, $q = xyrw_{n-5}\mu_2(xy) = \mu_2(r)\mu_2(xyr)\mu_2(w_{n-5})\mu_2(x) = \mu_2(yr)\mu_2(w_{n-5})xyr$. Thus, we have, $x = \mu_2(y) = \mu_2(r) = y$ which is a contradiction as μ_2 is not the identity mapping. Hence, $j \neq 3$.

(III) If $j = 5$, by induction, we have $F_n \bmod 5 = 0$ iff $n \bmod 5 = 0$. Therefore, if $n \bmod 5 \neq 0$, w_n cannot be written as $w_n = q^5$, where $q \in Q$, so we only need to consider the case where $n \bmod 15 = 0$. Note that the word w_n can be written as $w_n = w_{n-4}w_{n-5}\mu_2(w_{n-4})\mu_2(w_{n-4})\mu_2(w_{n-5})\mu_2(w_{n-4})\mu_2(w_{n-5})w_{n-4} = q^5$ and $|F_{n-5}|$ is divisible by 5. Then, there exist $x, y, r, s, t \in \Delta^+$ such that $|x| = |y| = |r| = |s| = |t|$ and $w_{n-5} = xyrst$ and since $w_n = q^5$, we have, $q = w_{n-4}xyr = st\mu_2(w_{n-4})s' = t'\mu_2(xyrs) = \mu_2(t)\mu_2(w_{n-4})\mu_2(xy) = \mu_2(rst)w_{n-4}$ where $\mu_2(w_{n-4}) = s't'$ for some $s', t' \in \Delta^+$. Then, we have, $r = \mu_2(r)$ which is a contradiction as μ_2 is not the identity mapping. Hence, $j \neq 5$.

□

Theorem 29. *Let $\phi \in \{\theta_1, \theta_2, \theta_4, \theta_5, \theta_{10}, \mu_2, \mu_4, \mu_5, \mu_{10}\}$. The primitivity properties of the atom palindromic ϕ-Fibonacci words $w_n^{\phi}(A,C)$, for $n \geq 3$, are as follows:*

(I) *For $\phi = \theta_2$, the atom ϕ-Fibonacci word $w_n^{\theta_2}(A,C)$ is primitive iff $n \bmod 3 \neq 0$.*

(II) *If $\phi \in \{\theta_1, \theta_4, \theta_5, \theta_{10}, \mu_2, \mu_4, \mu_5, \mu_{10}\}$, the atom ϕ-Fibonacci word $w_n^{\phi}(A,C)$ is primitive.*

Proof.

(I) Given that $\phi = \theta_2$. From Table 4, we have $w_3^{\theta_2} = AC$, which is primitive. By Proposition 8, the words $w_n^{\theta_2}$ are not primitive for $n > 3$ and $n \bmod 3 = 0$. Conversely, if $n \bmod 3 \neq 0$, then by Table 5, $\gcd(|w_n^{\theta_2}|_A, |w_n^{\theta_2}|_C) = 1$, for all $n \geq 3$. Therefore, by Lemma 12, the words $w_n^{\theta_2}$ are primitive for $n \geq 3$ and $n \bmod 3 \neq 0$.

(II) For $\phi = \mu_2$, by Proposition 28, the words $w_n^{\mu_2}$ are primitive for $n \bmod 3 = 0$. By Table 5, for the converse and for all other cases of ϕ, there exist two letters $a, b \in \Delta$, where $\gcd(|w_n^{\phi}|_a, |w_n^{\phi}|_b) = 1$, for all $n \geq 3$. Therefore, by Lemma 3 and Lemma 12, the words w_n^{ϕ} are primitive for all $n \geq 3$.

□

3.3. *Atom hairpin ϕ-Fibonacci words*

In Table 6, we begin by giving the first few values of the sequences $\{z_n^{\phi}(A,C)\}_{n\geq 1}$ for (anti)morphic involutions $\phi \in \{\mu_1, \mu_2, \mu_4, \mu_5, \mu_{10}, \theta_1, \theta_2, \theta_4, \theta_5, \theta_{10}\}$.

ϕ	$z_3^{\phi}(A,C)$	$z_4^{\phi}(A,C)$	$z_5^{\phi}(A,C)$	$z_6^{\phi}(A,C)$	$z_7^{\phi}(A,C)$
μ_1	CA	CAC	$CACCA$	$CACCACAC$	$CACCACACCACCA$
μ_2	CC	CCA	$CCAAA$	$CCAAAAAC$	$CCAAAAACAACCC$
μ_4	CT	CTC	$CTCCA$	$CTCCACAC$	$CTCCACACCACCT$
μ_5	CA	CAG	$CAGGA$	$CAGGAGAC$	$CAGGAGACGACCA$
μ_{10}	CT	CTG	$CTGGA$	$CTGGAGAC$	$CTGGAGACGACCT$
θ_1	CA	CAC	$CACAC$	$CACACCAC$	$CACACCACCACAC$
θ_2	CC	CCA	$CCAAA$	$CCAAACAA$	$CCAAACAACCCAA$
θ_4	CT	CTC	$CTCAC$	$CTCACCAC$	$CTCACCACCTCAC$
θ_5	CA	CAG	$CAGAG$	$CAGAGCAG$	$CAGAGCAGCACAG$
θ_{10}	CT	CTG	$CTGAG$	$CTGAGCAG$	$CTGAGCAGCTCAG$

Table 6: List of words $z_n^{\phi}(A,C)$, where $3 \leq n \leq 7$ and $\phi \in \{\mu_1, \mu_2, \mu_4, \mu_5, \mu_{10}, \theta_1, \theta_2, \theta_4, \theta_5, \theta_{10}\}$.

Similar to that of Table 5, we calculate the number of occurrences of letters in the words $z_n^{\phi}(A,C)$, for all $n \geq 3$ and $\phi \in \{\phi_1, \phi_4, \phi_5, \phi_{10}\}$, as summarized in Table 7.

Lemma 30. *Let $\phi \in \{\theta_2, \mu_2\}$. For all $n > 3$, the atom hairpin ϕ-Fibonacci word $z_n^{\phi}(A,C)$ cannot be a square.*

<table>
<tr><th></th><th>$i=1$</th><th colspan="4">$i=2$</th><th colspan="3">$i=5$</th></tr>
<tr><td>n mod 6</td><td>≥ 0</td><td>0</td><td>1,5</td><td>2,4</td><td>3</td><td>0,5</td><td>1,4</td><td>2,3</td></tr>
<tr><td>$|z_n^{\phi_i}(A,C)|_A$</td><td>F_{n-2}</td><td>$\frac{F_n}{2}+1$</td><td>$\frac{F_n+1}{2}$</td><td>$\frac{F_n-1}{2}$</td><td>$\frac{F_n}{2}-1$</td><td colspan="3">F_{n-2}</td></tr>
<tr><td>$|z_n^{\phi_i}(A,C)|_C$</td><td>F_{n-1}</td><td>$\frac{F_n}{2}-1$</td><td>$\frac{F_n-1}{2}$</td><td>$\frac{F_n+1}{2}$</td><td>$\frac{F_n}{2}+1$</td><td>$\frac{F_{n-1}-1}{2}$</td><td>$\frac{F_{n-1}}{2}$</td><td>$\frac{F_{n-1}+1}{2}$</td></tr>
<tr><td>$|z_n^{\phi_i}(A,C)|_G$</td><td>–</td><td colspan="4">–</td><td>$\frac{F_{n-1}+1}{2}$</td><td>$\frac{F_{n-1}}{2}$</td><td>$\frac{F_{n-1}-1}{2}$</td></tr>
</table>

<table>
<tr><th></th><th colspan="3">$i=4$</th><th colspan="6">$i=10$</th></tr>
<tr><td>n mod 6</td><td>0,1</td><td>2,5</td><td>3,4</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td></tr>
<tr><td>$|z_n^{\phi_i}(A,C)|_A$</td><td>$\frac{F_{n-2}+1}{2}$</td><td>$\frac{F_{n-2}}{2}$</td><td>$\frac{F_{n-2}-1}{2}$</td><td colspan="2">$\frac{F_{n-2}+1}{2}$</td><td>$\frac{F_{n-2}}{2}$</td><td colspan="2">$\frac{F_{n-2}-1}{2}$</td><td>$\frac{F_{n-2}}{2}$</td></tr>
<tr><td>$|z_n^{\phi_i}(A,C)|_C$</td><td colspan="3">F_{n-1}</td><td>$\frac{F_{n-1}-1}{2}$</td><td rowspan="2">$\frac{F_{n-1}}{2}$</td><td colspan="2">$\frac{F_{n-1}+1}{2}$</td><td rowspan="2">$\frac{F_{n-1}}{2}$</td><td>$\frac{F_{n-1}-1}{2}$</td></tr>
<tr><td>$|z_n^{\phi_i}(A,C)|_G$</td><td colspan="3">–</td><td>$\frac{F_{n-1}+1}{2}$</td><td colspan="2">$\frac{F_{n-1}-1}{2}$</td><td>$\frac{F_{n-1}+1}{2}$</td></tr>
<tr><td>$|z_n^{\phi_i}(A,C)|_T$</td><td>$\frac{F_{n-2}-1}{2}$</td><td>$\frac{F_{n-2}}{2}$</td><td>$\frac{F_{n-2}+1}{2}$</td><td colspan="2">$\frac{F_{n-2}-1}{2}$</td><td>$\frac{F_{n-2}}{2}$</td><td colspan="2">$\frac{F_{n-2}+1}{2}$</td><td>$\frac{F_{n-2}}{2}$</td></tr>
</table>

Table 7: The numbers of occurrences of letters in the atom hairpin ϕ_i-Fibonacci words $z_n^{\phi_i}(A,C)$ for $\phi_i \in \{\theta_i, \mu_i\}$, $i \in \{1,2,4,5,10\}$ and $n \geq 3$.

Proof. First, we consider the case where $\phi = \theta_2$. By Table 6, the statement is true for $4 \leq n \leq 7$. Assuming that the statement holds for $z_i^{\phi}(A,C)$, where $7 \leq i < k$, we now prove it for $z_k^{\phi}(A,C)$. We only need to consider the condition where k mod $3 = 0$, since by induction, we have F_k mod $2 = 0$ iff k mod $3 = 0$. By definition of z_n (Definition 4), we have $z_k = z_{k-2}\theta_2(z_{k-3})\theta_2(z_{k-2}) = pp$. As, $(k-3)$ mod $3 = F_{k-3}$ mod $2 = 0$ and hence, there exist $x, y \in \Delta^+$ such that $z_{k-3} = xy$ and $|x| = |y|$. We have, $p = z_{k-2}\theta_2(y)$, and $p = \theta_2(x)\theta_2(z_{k-2}) = \theta_2(x)z_{k-4}\theta_2(z_{k-3}) = \theta_2(x)z_{k-4}\theta_2(y)\theta_2(x)$. Therefore, $\theta_2(x) = \theta_2(y)$, so $z_{k-3} = x^2$, which contradicts the inductive hypothesis.

Next, we consider the case where $\phi = \mu_2$. By Table 6, the statement is true for $4 \leq n \leq 7$. Assuming the statement holds for $z_i^{\phi}(A,C)$, where $7 \leq i < k$, we now prove it for $z_k^{\phi}(A,C)$. We only need to consider the condition where $k \bmod 3 = 0$, since by induction, we have F_k mod $2 = 0$ iff k mod $3 = 0$. By definition of z_n (Definition 4), we have $z_k = z_{k-2}\mu_2(z_{k-3})\mu_2(z_{k-2}) = pp$. As, $(k-3)$ mod $3 = F_{k-3}$ mod $2 = 0$ and hence, there exist $x, y \in \Delta^+$ such that $z_{k-3} = xy$ and $|x| = |y|$. We now have $p = z_{k-2}\mu_2(x) = z_{k-3}\mu_2(z_{k-4})\mu_2(x) = xy\mu_2(z_{k-4})\mu_2(x)$, and $p = \mu_2(y)\mu_2(z_{k-2}) = \mu_2(y)\mu_2(z_{k-3})z_{k-4} = \mu_2(yxy)z_{k-4}$. This implies, $x = \mu_2(y)$ and $\mu_2(z_{k-4})\mu_2(x) = xz_{k-4}$. Note that, $z_{k-3} = z_{k-4}\mu_2(z_{k-5}) = xy$, and hence, $z_{k-4} = x\mu_2(y_1)$ for some $y = y_1y_2$. Thus, $xz_{k-4} = xx\mu_2(y_1) = \mu_2(z_{k-4})\mu_2(x) = \mu_2(xy_1)\mu_2(x)$, which implies $x = \mu_2(x)$, and hence, $z_{k-3} = x^2$, a contradiction. □

Theorem 31. *Let $\phi \in \{\theta_1, \theta_2, \theta_4, \theta_5, \theta_{10}, \mu_2, \mu_4, \mu_5, \mu_{10}\}$. For all $n \geq 1$, we have:*

(I) *If $\phi \in \{\theta_2, \mu_2\}$ and $n \neq 3$, then the atom hairpin ϕ-Fibonacci word $z_n^{\phi}(A,C)$ is primitive.*

(II) *If $\phi \in \{\theta_1, \theta_4, \theta_5, \theta_{10}, \mu_4, \mu_5, \mu_{10}\}$, then the atom hairpin ϕ-Fibonacci word $z_n^{\phi}(A,C)$ is primitive.*

Proof.

(I) Let $\phi \in \{\theta_2, \mu_2\}$. Note that $z_3 = CC$ is not primitive. By Table 7, for $n \bmod 6 \in \{1,2,4,5\}$, we have $\gcd(|z_n^\phi|_A, |z_n^\phi|_C) = 1$, so by Lemma 12, the word z_n^ϕ is primitive in these cases. For $n \bmod 6 \in \{0,3\}$, the number of occurrences of A and C are consecutive positive even or odd integers. If they are consecutive positive odd integers, by Lemma 12, the word z_n^ϕ is primitive in this case. If they are consecutive positive even integers, the only non-trivial common divisor is 2, and we can prove that z_n^ϕ is primitive by contradiction. Assume z_n^ϕ is not primitive in this case, which means that $z_n^\phi = p^i$ (refer to proof of Lemma 12 to show that $i = 2$) where $p \in Q$, and this contradicts Lemma 30.

(II) By Table 7, for all other cases of ϕ, there exist two letters $a, b \in \Delta$, where $\gcd(|z_n^\phi|_a, |z_n^\phi|_b) = 1$, for all $n \geq 3$. Therefore, by Lemma 12, the word z_n^ϕ is primitive for all $n \geq 3$.

□

Let $\alpha_n \in \{g_n, w_n, z_n\}$ for all $n \geq 1$ and let $\phi \in \{\theta_1, \theta_2, \theta_4, \theta_5, \theta_{10}, \mu_2, \mu_4, \mu_5, \mu_{10}\}$. Based on Corollary 21, Theorems 24, 25, 26, 29 and 31, we conclude that the words $g_n^{\theta_2}$ and $g_n^{\mu_2}$ are not primitive for all $n \geq 3$, the word $w_n^{\mu_2}$ is primitive if $n \bmod 3 \neq 0$ and $n \geq 3$, and for all other cases, the word α_n^ϕ is primitive. These are summarized in Table 8.

	$i \in \{1,7\}$	$i \in \{2,8\}$	$i \in \{3,4\}$	$i \in \{5,6\}$	$i \in \{9,10\}$
$g_n^{\theta_i}(A,C)$	✓	✗(except $n = 1,2$)	✓	✓	✓
$g_n^{\mu_i}(A,C)$	✓	✗(except $n = 1,2$)	✓	✓	✓
$w_n^{\theta_i}(A,C)$	✓	✓(except $n \bmod 3 = 0, n > 3$)	✓	✓	✓
$w_n^{\mu_i}(A,C)$	✓	✓	✓	✓	✓
$z_n^{\theta_i}(A,C)$	✓	✓(except $n = 3$)	✓	✓	✓
$z_n^{\mu_i}(A,C)$	✓	✓(except $n = 3$)	✓	✓	✓

Table 8: Primitivity of atom ϕ-Fibonacci words $\alpha_n^\phi(A,C)$ for all $n \geq 1$, with different initial letters $A, C \in \Delta$, where $\alpha_n \in \{g_n, w_n, z_n\}$ for all $n \geq 1$, and $\phi \in \{\theta_i, \mu_i \mid 1 \leq i \leq 10\}$

(here, ✓ means that the words are primitive, and ✗ means that they are not primitive).

4. Primitivity of atom ϕ-Fibonacci words with identical initial letters

In this section, we discuss the primitivity of atom ϕ-Fibonacci words $\alpha_n^\phi(a,a)$ with identical initial letters, for all $n \geq 1$, where $a \in \Sigma_4$, $\phi \in \{\theta_i \mid 1 \leq i \leq 10\} \cup \{\mu_i \mid 1 \leq i \leq 10\}$, and $\alpha_n \in \{g_n, w_n, z_n\}$ for all $n \geq 1$. The primitivity results of this section hold for any choice of (x_1, x_2, x_3, x_4) and with x_1 and x_1 as the initial letters of the ϕ-Fibonacci sequence. Therefore, we only prove primitivity results for one of the cases, the sequence (A, C, G, T) over the DNA alphabet Δ, with A and A as the first two initial letters.

We can classify ϕ into two categories, where $\phi(a) = a$ and $\phi(a) \neq a$. If $\phi(a) = a$, then for all $n \geq 1$ we have that $\alpha_n^\phi(a,a) = a^{F_n}$, where $\alpha_n \in \{g_n, w_n, z_n\}$ for all $n \geq 1$. Hence,

$\alpha_n^{\phi}(a,a)$ is not primitive for all $n \geq 3$. Therefore, we only need to consider the case where $\phi(a) \neq a$. The set of all (anti)morphic involutions ϕ for the sequence (x_1,x_2,x_3,x_4) with $x_1 = a$ and $\phi_i(a) \neq a$ is $\{\phi_i \mid i = 2,3,4,8,9,10\}$. It is enough if we discuss the primitivity for one of such ϕ_i, say ϕ_2. We first show that it is sufficient to study the primitivity only for $\alpha_n^{\phi_2}(a,a)$, where $n \geq 1$.

We use the following lemma.

Lemma 32. *Let $\phi_i \in \{\mu_i,\theta_i\}$, $i \in \{2,3,4,8,9,10\}$, be an (anti)morphic involution on Σ_4^* and $\alpha_n \in \{g_n,w_n,z_n\}$ for all $n \geq 1$. For all $n \geq 1$, the following equalities hold regarding ϕ_i-Fibonacci words $\alpha_n^{\phi_i}(x_1,x_1)$:*

(I) $\phi_2(\alpha_n^{\phi_2}(x_1,x_1)) = \phi_8(\alpha_n^{\phi_8}(x_1,x_1))$ *iff* $\alpha_n^{\phi_2}(x_1,x_1) = \alpha_n^{\phi_8}(x_1,x_1)$.

(II) $\phi_3(\alpha_n^{\phi_3}(x_1,x_1)) = \phi_9(\alpha_n^{\phi_9}(x_1,x_1))$ *iff* $\alpha_n^{\phi_3}(x_1,x_1) = \alpha_n^{\phi_9}(x_1,x_1)$.

(III) $\phi_4(\alpha_n^{\phi_4}(x_1,x_1)) = \phi_{10}(\alpha_n^{\phi_{10}}(x_1,x_1))$ *iff* $\alpha_n^{\phi_4}(x_1,x_1) = \alpha_n^{\phi_{10}}(x_1,x_1)$.

(IV) $[\phi_2(\alpha_n^{\phi_2}(x_1,x_2))]_{x_2\to x_3} = \phi_3(\alpha_n^{\phi_3}(x_1,x_2))$ *iff* $[\alpha_n^{\phi_2}(x_1,x_2)]_{x_2\to x_3} = \alpha_n^{\phi_3}(x_1,x_2)$.

(V) $[\phi_2(\alpha_n^{\phi_2}(x_1,x_2))]_{x_2\to x_4} = \phi_4(\alpha_n^{\phi_4}(x_1,x_2))$ *iff* $[\alpha_n^{\phi_2}(x_1,x_2)]_{x_2\to x_4} = \alpha_n^{\phi_4}(x_1,x_2)$.

Proof. The proof, by induction on n, is similar to that of Lemma 17. □

Based on Lemma 32, we have the following result.

Lemma 33. *Let $\phi_i \in \{\theta_i,\mu_i\}$, $i \in \{2,3,4,8,9,10\}$ be an (anti)morphic involution on Σ_4^*, and let $\alpha_n \in \{g_n,w_n,z_n\}$ for all $n \geq 1$. For all $n \geq 1$, the following equalities regarding ϕ_i-Fibonacci words $\alpha_n^{\phi_i}(x_1,x_1)$ hold:*

(I) $\alpha_n^{\phi_2}(x_1,x_1) = \alpha_n^{\phi_8}(x_1,x_1)$.

(II) $[\alpha_n^{\phi_2}(x_1,x_1)]_{x_2\to x_3} = \alpha_n^{\phi_3}(x_1,x_1) = \alpha_n^{\phi_9}(x_1,x_1)$.

(III) $[\alpha_n^{\phi_2}(x_1,x_1)]_{x_2\to x_4} = \alpha_n^{\phi_4}(x_1,x_1) = \alpha_n^{\phi_{10}}(x_1,x_1)$.

Proof. The proof, by induction on n and using Lemma 32, is similar to that of Lemma 18. □

One can easily observe that for $\alpha_n^{\phi}(a,a)$, we have $|\mathrm{Alph}(\alpha_n^{\phi}(a,a))| \leq 2$, for all $n \geq 3$, ϕ an (anti)morphic involution, and $\alpha_n \in \{g_n,w_n,z_n\}$ for all $n \geq 3$. Thus, by Lemma 33, it is enough to discuss the primitivity for $\alpha_n^{\phi_2}(x_1,x_1)$, $n \geq 3$. As the choice of initial letter does not matter, we choose for convenience, A and A to be the two initial letters. We now study the primitivity of the words $\alpha_n^{\phi_2}(A,A)$ for all $n \geq 3$, where $\alpha_n \in \{g_n,w_n,z_n\}$ for all $n \geq 3$.

4.1. Atom alternating ϕ-Fibonacci words

The first few values of the sequence $\{g_n^{\phi}(A,A)\}_{n\geq 1}$ for $\phi \in \{\mu_2,\theta_2\}$ are given in Table 9.

We first show that for a morphic involution μ_2, the μ_2-Fibonacci words g_n with identical initial letters are primitive. The proof is similar to that of Theorem 25.

Theorem 34. *Let $\phi = \mu_2$. For all $n \geq 1$, the atom alternating ϕ-Fibonacci word $g_n^{\mu_2}(A,A)$ is primitive.*

ϕ	$g_3^{\phi}(A,A)$	$g_4^{\phi}(A,A)$	$g_5^{\phi}(A,A)$	$g_6^{\phi}(A,A)$	$g_7^{\phi}(A,A)$
μ_2	CA	ACA	$CACCA$	$ACAACACA$	$CACCACACCACCA$
θ_2	CA	CAA	$CCACA$	$CACAACAA$	$CCACCACACCACA$

Table 9: List of words $g_n^{\phi}(A,A)$, where $3 \le n \le 7$ and $\phi \in \{\mu_2, \theta_2\}$.

Proof. By Theorem 9, we have $g_n^{\phi}(A,A) = f_n(A,\phi(A))$ if n is odd, and $g_n^{\phi}(A,A) = f_n(\phi(A),A)$, otherwise. Note that, $\phi(A) \neq A$ for $\phi = \mu_2$. Then, by Theorem 5, the word $g_n^{\phi}(A,A)$ is primitive for $\phi = \mu_2$ and $n \ge 1$. □

We now discuss the primitivity of θ_2-Fibonacci words $g_n^{\theta_2}(A,A)$, $n \ge 3$. The following result can be proved by induction and we omit the proof.

Theorem 35. *Let $\phi \in \{\theta_1, \theta_2\}$. For all $n \ge 1$, the atom alternating ϕ-Fibonacci word $g_n^{\theta_2}(A,A)$ can be represented by atom alternating ϕ-Fibonacci words $g_n^{\theta_1}(A,C)$ and $g_n^{\theta_1}(C,A)$ as follows:*

$$g_n^{\theta_2}(A,A) = \begin{cases} g_n^{\theta_1}(C,A) \;:\; n \bmod 2 = 0, \\ g_n^{\theta_1}(A,C) \;:\; n \bmod 2 = 1. \end{cases}$$

Proof. The proof, by induction on n and using Lemma 22, is similar to that of Lemma 23. □

Using Theorem 35 and Corollary 21, we have the following corollary.

Corollary 36. *Let $\phi = \theta_2$. For all $n \ge 1$, the atom alternating ϕ-Fibonacci word $g_n^{\theta_2}(A,A)$ is primitive.*

4.2. *Atom palindromic ϕ-Fibonacci words*

The first few values of the sequences $\{w_n^{\phi}(A,A)\}_{n\ge1}$ for $\phi \in \{\mu_2, \theta_2\}$ are given in Table 10.

ϕ	$w_3^{\phi}(A,A)$	$w_4^{\phi}(A,A)$	$w_5^{\phi}(A,A)$	$w_6^{\phi}(A,A)$	$w_7^{\phi}(A,A)$
μ_2	CC	AAC	$CCAAA$	$AACCCCCA$	$CCAAAAACAACCC$
θ_2	CC	AAC	$ACCAA$	$CCAACACC$	$AACACCAACCAAC$

Table 10: List of words $w_n^{\phi}(A,A)$, where $3 \le n \le 7$ and $\phi \in \{\mu_2, \theta_2\}$.

Therefore, we have the following theorem.

Theorem 37. *Let $\phi = \mu_2$. For all $n \ge 1$, the atom palindromic ϕ-Fibonacci word $w_n^{\mu_2}(A,A)$ is primitive.*

Proof. We have by Theorem 9, $w_n^{\mu_2}(A,A) = z_n^{\mu_2}(A,C)$ if n is odd, and $w_n^{\mu_2}(A,A) = z_n^{\mu_2}(C,A)$, otherwise, and hence, by Theorem 31, $w_n^{\mu_2}(A,A)$ is primitive for $n \ge 1$. □

We now count the number of occurrences of the letters A and C in the atom palindromic ϕ_2-Fibonacci words.

Lemma 38. *Let $\phi \in \{\mu_2, \theta_2\}$. For all $n \geq 1$, the numbers of occurrences of letters in the atom palindromic ϕ-Fibonacci word $w_n^{\phi}(A,A)$ satisfy:*

$$|w_n^{\phi}|_A = \begin{cases} \frac{F_n}{2} - 1 & : n \bmod 3 = 0, \\ \frac{F_n+1}{2} & : n \bmod 3 = 1,2, \end{cases} \qquad |w_n^{\phi}|_C = \begin{cases} \frac{F_n}{2} + 1 & : n \bmod 3 = 0, \\ \frac{F_n-1}{2} & : n \bmod 3 = 1,2. \end{cases}$$

Proof. The proof uses the fact that $|w_{n+2}^{\phi}|_A = |w_{n+1}^{\phi}|_C + |w_n^{\phi}|_C$, $|w_{n+2}^{\phi}|_C = |w_{n+1}^{\phi}|_A + |w_n^{\phi}|_A$ and is by induction on n. $\square$

By induction on n, we have the following lemma.

Lemma 39. *Let $\phi = \theta_2$. For all $n > 3$, the atom palindromic ϕ-Fibonacci word $w_n^{\theta_2}(A,A)$ cannot be a square.*

Proof. The proof is by induction on n. One can easily check that the statement is true for $4 \leq n \leq 10$. We now assume that $w_i^{\theta_2}(A,A)$ is not a square for all $10 \leq i < k$. We prove that this is true for $w_k^{\theta_2}(A,A)$. We only need to consider the condition where $k \bmod 3 = 0$, since by induction, we have $F_k \bmod 2 = 0$ iff $k \bmod 3 = 0$. By definition of w_n (Definition 4), we have $w_k = \theta_2(w_{k-1})\theta_2(w_{k-2}) = w_{k-3}w_{k-2}w_{k-4}w_{k-3} = w_{k-3}\theta_2(w_{k-3})\theta_2(w_{k-4})w_{k-4}w_{k-3} = w_{k-3}w_{k-5}w_{k-4}w_{k-6}w_{k-5}w_{k-4}w_{k-3} = pp$. As, $(k-6) \bmod 3 = F_{k-6} \bmod 2 = 0$, and hence, there exist $x, y \in \Delta^+$ such that $w_{k-6} = xy$ and $|x| = |y|$. We now have $p = w_{k-3}w_{k-5}w_{k-4}x = \theta_2(w_{k-4})\theta_2(w_{k-5})w_{k-5}w_{k-4}x = w_{k-6}w_{k-5}\theta_2(w_{k-5})w_{k-5}w_{k-4}x = xyw_{k-5}\theta_2(w_{k-5})w_{k-5}w_{k-4}x$, and $p = yw_{k-5}w_{k-4}w_{k-3}$. Therefore, we have $x = y$, so $w_{k-6} = x^2$, which contradicts the inductive hypothesis. $\square$

Theorem 40. *Let $\phi = \theta_2$. For $n \geq 1$, the atom palindromic ϕ-Fibonacci word $w_n^{\theta_2}(A,A)$ is primitive iff $n \neq 3$.*

Proof. The proof is similar to part (I) of the proof of Theorem 31, and uses Lemma 39. $\square$

4.3. Atom hairpin ϕ-Fibonacci words

In Table 11, we give the first few values of the sequences $\{z_n^{\phi}(A,A)\}_{n\geq 1}$ for $\phi \in \{\mu_2, \theta_2\}$.

ϕ	$z_3^{\phi}(A,A)$	$z_4^{\phi}(A,A)$	$z_5^{\phi}(A,A)$	$z_6^{\phi}(A,A)$	$z_7^{\phi}(A,A)$
μ_2	AC	ACC	$ACCCA$	$ACCCACAA$	$ACCCACAACAAAC$
θ_2	AC	ACC	$ACCAC$	$ACCACAAC$	$ACCACAACACAAC$

Table 11: List of words $z_n^{\phi}(A,A)$, where $3 \leq n \leq 7$ and $\phi \in \{\mu_2, \theta_2\}$.

Theorem 41. *Let $\phi = \mu_2$. For all $n \geq 1$, the atom hairpin ϕ-Fibonacci word $z_n^{\mu_2}(A,A)$ is primitive.*

Proof. We have by Theorem 9, $z_n^{\mu_2}(A,A) = w_n^{\mu_2}(A,C)$ if n is odd, and $z_n^{\mu_2}(A,A) = w_n^{\mu_2}(C,A)$, otherwise, and hence, by Theorem 29, $z_n^{\mu_2}(A,A)$ is primitive for all $n \geq 1$. $\square$

We now count the number of occurrences of the letters A and C in the atom hairpin ϕ_2-Fibonacci words. The proof is by induction, and we omit it.

Lemma 42. *Let $\phi \in \{\mu_2, \theta_2\}$. For all $n \geq 1$, the numbers of occurrences of letters in the atom hairpin ϕ-Fibonacci word $z_n^{\phi}(A,A)$ satisfy:*

$$|z_n^{\phi}|_A = \begin{cases} \frac{F_n}{2} & : n \bmod 6 = 0,3, \\ \frac{F_n+1}{2} & : n \bmod 6 = 1,2, \\ \frac{F_n-1}{2} & : n \bmod 6 = 4,5, \end{cases} \quad |z_n^{\phi}|_C = \begin{cases} \frac{F_n}{2} & : n \bmod 6 = 0,3, \\ \frac{F_n-1}{2} & : n \bmod 6 = 1,2, \\ \frac{F_n+1}{2} & : n \bmod 6 = 4,5. \end{cases}$$

Proof. The proof is by induction on n and uses the fact that for all $n \geq 1$, $|z_{n+2}^{\phi}|_A = |z_{n+1}^{\phi}|_A + |z_n^{\phi}|_C$, $|z_{n+2}^{\phi}|_C = |z_{n+1}^{\phi}|_C + |z_n^{\phi}|_A$, where $|z_1^{\phi}|_A = |z_2^{\phi}|_A = 1$ and $|z_1^{\phi}|_C = |z_2^{\phi}|_C = 0$. Hence, using inductive hypothesis, one can obtain the result for $|z_k^{\phi}|_A$ and $|z_k^{\phi}|_C$. $\square$

We next show that the atom hairpin μ_2-Fibonacci words $z_n^{\mu_2}(A,A)$ are primitive for $n \geq 3$. We need the following lemma which can be proved by induction on n.

Lemma 43. *Let $\phi = \theta_2$. The following hold for all $n \geq 1$:*

(I) *If $n \bmod 3 = 0$, then $z_n^{\theta_2}(A,A) \neq q^2$ for any θ_2-palindrome q.*
(II) *If $n \bmod 4 = 0$, then $z_n^{\theta_2}(A,A) \neq q^3$ for any θ_2-palindrome q.*
(III) *If $n \bmod 5 = 0$, then $z_n^{\theta_2}(A,A) \neq q^5$ for any θ_2-palindrome q.*

Proof. We only prove statement (I). Given that $n \bmod 3 = 0$. Then, $F_n \bmod 2 = 0$, and there exist $x, y \in \Delta^+$ such that $|z_n| = xy$ and $|x| = |y|$. One can easily verify the statement for $n = 3$ and $n = 6$. Assume the statement to be true for $z_i^{\theta_2}(A,A)$, where $i \bmod 3 = 0$ and $3 \leq i < k$. Let k be a number such that $k \bmod 3 = 0$ and $k > n$. Suppose, $z_k = z_{k-1}\theta_2(z_{k-2}) = p^2$ where p is a θ_2-palindrome, then, $z_k = z_{k-1}\theta_2(z_{k-2}) = z_{k-2}\theta_2(z_{k-3})\theta_2(z_{k-2})$ and $p = z_{k-2}x = y\theta_2(z_{k-2})$ for $\theta_2(z_{k-3}) = xy$. Since, p is a θ_2-palindrome, $x = \theta_2(y)$ and $z_{k-2} = z_{k-3}\theta_2(z_{k-4}) = x\theta_2(x)\theta_2(z_{k-4})$. This implies that $x = y = \theta_2(x)$ and hence $z_{k-3} = x^2$, a contradiction to our induction hypothesis. Hence, the result. $\square$

The following result uses Lemma 43 and has a proof similar to that of Proposition 28.

Theorem 44. *Let $\phi = \theta_2$. For all $n \geq 1$, the atom hairpin ϕ-Fibonacci word $z_n^{\theta_2}(A,A)$ is primitive.*

Based on Corollary 36 and Theorems 34, 37, 40, 41 and 44, the results can be generalized to non-trivial ϕ-Fibonacci words with the same initial letters. The primitivity properties of the n-th atom ϕ-Fibonacci word with the same two initial letters, for all $n \geq 1$, are summarized in Table 12.

	$i \in \{2,3,4,8,9,10\}$	$i \in \{1,5,6,7\}$
$g_n^{\phi_i}(A,A)$	✓	✗
$w_n^{\phi_i}(A,A)$	✓(except $n=3$)	✗
$z_n^{\phi_i}(A,A)$	✓	✗

Table 12: Primitivity of atom ϕ-Fibonacci words $\alpha_n^{\phi}(A,A)$ for all $n \geq 1$, with identical initial letters $A \in \Delta$, where $\alpha_n \in \{g_n, w_n, z_n\}$ for all $n \geq 1$, and $\phi \in \{\theta_i, \mu_i \mid 1 \leq i \leq 10\}$

(here, ✓ means that the words are primitive, and ✗ means that they are not primitive).

5. Conclusions and future work

This paper analyzed the primitivity properties of atom involutive Fibonacci words over a four-letter alphabet and concluded that, for some (anti)morphic involutions, some initial letters, and some indices n, we have that the n-th ϕ-Fibonacci word is primitive, while for some others, it is not. In the particular case of the Watson-Crick complementarity involution θ_{DNA} over the DNA alphabet $\Delta = \{A, C, G, T\}$, our results imply that regardless of the initial two letters in the Fibonacci recursion (different, or the same), the n-th atom Watson-Crick Fibonacci word is primitive for all $n > 3$.

Future topics of research include studying the ϕ-primitivity of ϕ-Fibonacci words, as well combinatorial properties of ϕ-Fibonacci words (counting their distinct factors, squares, ϕ-squares, cubes, ϕ-cubes, palindromes, ϕ-palindromes, etc.).

References

[1] J. BERSTEL, Mots de Fibonacci. In: *Séminaire d'Informatique Théorique*. Rapport L.I.T.P., 1980, 57–78.

[2] J. BERSTEL, Fibonacci words - A survey. In: *The Book of L*. Springer, 1986, 13–27.

[3] K. BŘINDA, E. PELANTOVÁ, O. TUREK, Balances of m-bonacci words. *Fundamenta Informaticae* **132** (2014) 1, 33–61.

[4] J. A. BRZOZOWSKI, Derivatives of regular expressions. *Journal of the ACM* **11** (1964) 4, 481–494.

[5] W.-F. CHUAN, Fibonacci words. *Fibonacci Quarterly* **30** (1992) 1, 68–76.

[6] W.-F. CHUAN, Generating Fibonacci words. *Fibonacci Quarterly* **33** (1995) 2, 104–112.

[7] E. CZEIZLER, L. KARI, S. SEKI, On a special class of primitive words. *Theoretical Computer Science* **411** (2010) 3, 617–630.

[8] A. DE LUCA, A combinatorial property of the Fibonacci words. *Information Processing Letters* **12** (1981) 4, 193–195.

[9] C.-M. FAN, H. J. SHYR, Some properties of Fibonacci languages. *Tamkang Journal of Mathematics* **27** (1996) 2, 165–182.

[10] T. HEAD, Relativized code concepts and multi-tube DNA dictionaries. In: C. CALUDE, G. PĂUN (eds.), *Finite Versus Infinite: Contributions to an Eternal Dilemma*. Discrete Mathematics and Theoretical Computer Science, Springer, London, 2000, 175–186.

[11] H.-K. HSIAO, S.-S. YU, Mapped shuffled Fibonacci languages. *Fibonacci Quarterly* **41** (2003) 5, 421–430.

[12] N. JONOSKA, K. MAHALINGAM, Languages of DNA based code words. In: J. CHEN, J. REIF (eds.), *DNA Computing, (DNA 2003)*. LNCS 2943, Springer, Berlin, Heidelberg, 2004, 61–73.

[13] L. KARI, R. KITTO, G. THIERRIN, Codes, involutions, and DNA encodings. In: W. BRAUER, H. EHRIG, J. KARHUMÄKI, A. SALOMAA (eds.), *Formal and Natural Computing: Essays Dedicated to Grzegorz Rozenberg*. LNCS 2300, Springer, 2002, 376–393.

[14] L. KARI, S. KONSTANTINIDIS, P. SOSÍK, Bond-free languages: Formalizations, maximality and construction methods. *International Journal of Foundations of Computer Science* **16** (2005) 05, 1039–1070.

[15] L. KARI, M. S. KULKARNI, K. MAHALINGAM, Z. WANG, Involutive Fibonacci words. *Journal of Automata, Languages and Combinatorics* **26** (2021) 3–4, 255–280.

[16] D. E. KNUTH, *The Art of Computer Programming (Volume 1: Fundamental Algorithms)*. First edition, Addison-Wesley, 1968.

[17] D. E. KNUTH, J. H. MORRIS, JR., V. R. PRATT, Fast pattern matching in strings. *SIAM Journal on Computing* **6** (1977) 2, 323–350.

[18] R. C. LYNDON, M. P. SCHÜTZENBERGER, The equation $a^M = b^N c^P$ in a free group. *Michigan Mathematical Journal* **9** (1962), 289–298.

[19] A. MARATHE, A. E. CONDON, R. M. CORN, On combinatorial DNA word design. *Journal of Computational Biology* **8** (2001) 3, 201–219.

[20] G. MAURI, C. FERRETTI, Word design for molecular computing: A survey. In: J. CHEN, J. REIF (eds.), *DNA Computing, (DNA 2003)*. LNCS 2943, Springer, Berlin, Heidelberg, 2004, 37–47.

[21] H. MOUSAVI, L. SCHAEFFER, J. SHALLIT, Decision algorithms for Fibonacci-automatic words, I: Basic results. *RAIRO Theoretical Informatics and Applications* **50** (2016) 1, 39–66.

[22] G. PĂUN, G. ROZENBERG, A. SALOMAA, *DNA computing: New Computing Paradigms*. Texts in Theoretical Computer Science, Springer, Berlin, Heidelberg, 1998.

[23] G. PĂUN, G. ROZENBERG, Y. TAKASHI, Hairpin languages. *International Journal of Foundations of Computer Science* **12** (2001) 06, 837–847.

[24] J. L. RAMÍREZ, G. N. RUBIANO, On the k-Fibonacci words. *Acta Universitatis Sapientiae, Informatica* **5** (2013) 2, 212–226.

[25] J. L. RAMÍREZ, G. N. RUBIANO, Properties and generalizations of the Fibonacci word fractal, exploring fractal curves. *The Mathematica Journal* **16** (2014), 1–25.

[26] J. L. RAMÍREZ, G. N. RUBIANO, R. D. CASTRO, A generalization of the Fibonacci word fractal and the Fibonacci snowflake. *Theoretical Computer Science* **528** (2014), 40–56.

[27] P. SÉÉBOLD, Sequences generated by infinitely iterated morphisms. *Discrete Applied Mathematics* **11** (1985) 3, 255–264.

[28] J. SHALLIT, A generalization of automatic sequences. *Theoretical Computer Science* **61** (1988) 1, 1–16.

[29] K. B. STOLARSKY, Beatty sequences, continued fractions, and certain shift operators. *Canadian Mathematical Bulletin* **19** (1976) 4, 473–482.

[30] D. C. TULPAN, H. H. HOOS, A. E. CONDON, Stochastic local search algorithms for DNA word design. In: M. HAGIYA, A. OHUCHI (eds.), *DNA Computing, (DNA 2002)*. LNCS 2568, Springer, Berlin, Heidelberg, 2003, 229–241.

[31] H. L. WU, *On the Properties of Primitive Words*. Master's thesis, Institute of Applied Mathematics, Chung-Yuan Christian University, 1992.

[32] S.-S. YU, Y.-K. ZHAO, Properties of Fibonacci languages. *Discrete Mathematics* **224** (2000) 1, 215–223.