

# A note on degree-constrained subgraphs

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## Abstract

Elementary proofs are presented for two graph theoretic results, originally proved by H. Shirazi and J. Verstraëte using the combinatorial Nullstellensatz.

In an undirected graph  $G = (V, E)$  we denote by  $d_G(v)$  the degree of  $v \in V$ . If  $F(v) \subseteq \mathbb{N}$  is a set of forbidden degrees for every  $v \in V$ , then a subgraph  $G' = (V, E')$  of  $G$  is called  **$F$ -avoiding** if  $d_{G'}(v) \notin F(v)$  for all  $v \in V$ .

**Theorem 1** (Shirazi, Verstraëte [5]). *If  $G = (V, E)$  is an undirected graph and*

$$|F(v)| \leq d_G(v)/2 \text{ for every node } v, \quad (1)$$

*then  $G$  has an  $F$ -avoiding subgraph.*

Theorem 1 appeared first under the name Louigi's conjecture in [1]. A version with  $d_G(v)/2$  replaced by  $d_G(v)/12$  was given in [1], while  $d_G(v)/8$  was proved in [2]. Louigi's conjecture was first settled in the affirmative by H. Shirazi and J. Verstraëte [5]. Their proof is based on the combinatorial Nullstellensatz of N. Alon [3]. We give an elementary proof, which uses Theorem 2 below. In a directed graph  $D = (V, \vec{E})$  we denote by  $\varrho_D(v)$  the in-degree of  $v \in V$ .

**Theorem 2.** *If  $G = (V, E)$  is an undirected graph and it has an orientation  $D$  for which  $\varrho_D(v) \geq |F(v)|$  for every node  $v$ , then  $G$  has an  $F$ -avoiding subgraph.*

*Proof.* For an undirected edge  $e$ , let  $\vec{e}$  denote the corresponding directed edge of  $D$ . We use induction on the number of edges. If 0 is not a forbidden degree at any node, then the empty subgraph  $(V, \emptyset)$  is  $F$ -avoiding. Suppose that  $0 \in F(t)$  for a node  $t$ . Then  $\varrho_D(t) \geq |F(t)| \geq 1$  and hence there is an edge  $e = st$  of  $G$  for which  $\vec{e}$  is directed toward  $t$ . Let  $G^- = G - e$  and  $D^- = D - \vec{e}$ . Define

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$F^-$  as follows. Let  $F^-(t) = \{i - 1 : i \in F(t) \setminus \{0\}\}$ ,  $F^-(s) = \{i - 1 : i \in F(s) \setminus \{0\}\}$ , and for  $z \in V - \{s, t\}$  let  $F^-(z) = F(z)$ . Since  $|F^-(t)| = |F(t)| - 1$ ,  $\varrho_D^-(v) \geq |F^-(v)|$  holds for every node  $v$ . By induction, there is an  $F^-$ -avoiding subgraph  $G''$  of  $G^-$ . By the construction of  $F^-$ , the subgraph  $G' := G'' + e$  of  $G$  is  $F$ -avoiding.  $\square$

*Proof. (of Theorem 1.)* It is well-known that every undirected graph  $G$  has an orientation  $D$  in which

$$\varrho_D(v) \geq \lfloor d_G(v)/2 \rfloor \text{ for every node } v. \quad (2)$$

Indeed, by adding a new node  $z$  to  $G$  and joining  $z$  to every node of  $G$  with odd degree, we obtain a graph  $G^+$  in which every degree is even. Hence  $G^+$  decomposes into edge-disjoint circuits and therefore it has an orientation in which the in-degree of every node equals its out-degree. The restriction of this orientation to  $G$  satisfies (2). (An orientation with property (2) is also used in [5].) Therefore Theorem 2 implies Theorem 1.  $\square$

S. L. Hakimi [4] proved that, given a function  $f : V \rightarrow \mathbb{Z}_+$ , an undirected graph  $G$  has an orientation for which  $\varrho(v) \geq f(v)$  for every node  $v$  if and only if  $e_G(X) \geq \sum[f(v) : v \in X]$  holds for every subset  $X \subseteq V$ , where  $e_G(X)$  denotes the number of edges with at least one end-node in  $X$ . By combining this with Theorem 2, one obtains the following.

**Corollary 3.** *If  $G = (V, E)$  is an undirected graph and  $e_G(X) \geq \sum[|F(v)| : v \in X]$  holds for every subset  $X \subseteq V$ , then  $G$  has an  $F$ -avoiding subgraph.*

Along with Theorem 1, the following result was also proved in [5] via the Combinatorial Nullstellensatz. A graph is called **empty** if it has no edges.

**Theorem 4** (Shirazi, Verstraëte [5]). *If  $G = (V, E)$  is an undirected graph,  $0 \notin F(v)$  for all  $v \in V$  and  $\sum_{v \in V} |F(v)| < |E|$ , then  $G$  has a nonempty  $F$ -avoiding subgraph  $G'$ .*

*Proof.* Again, we use induction on the number of edges. If  $d_G(v) \notin F(v)$  for all  $v \in V$ , then the nonempty  $G' = G$  will do. Otherwise there exists a node  $t \in V$  where  $d_G(t) \in F(t)$ . As  $0 \notin F(v)$ , there is an edge  $e$  of  $G$  incident to  $t$ . Let  $G^- = G - e$ , let  $F^-(t) = F(t) \setminus \{d_G(t)\}$  and for  $z \in V - \{t\}$  let  $F^-(z) = F(z)$ . By induction, there is a nonempty  $F^-$ -avoiding subgraph  $G'$  of  $G^-$ . As  $d_{G'}(t) < d_G(t)$ , this  $G'$  is also  $F$ -avoiding.  $\square$

We remark that Theorems 2 and 4 clearly hold for hypergraphs, as well, with the same proofs. Combining this with the hypergraph variant of Hakimi's theorem, one concludes that also Corollary 3 applies to hypergraphs. However, in Theorem 1 one should replace the denominator 2 by the rank of the hypergraph (that is, the maximum size of a hyperedge). This is already observed by Shirazi and Verstraëte [5]. Note also that both proofs give rise to polynomial algorithms: such algorithms were not known before.

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**Note added in proof.** After submitting the paper, the authors learned that Adrian Bondy also formulated and proved Theorem 2. His proof goes along the same line as ours.

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