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Complexity of Finding Graph Roots with Girth Conditions

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Abstract Graph G is the square of graph H if two vertices x, y have an edge in G if and only if x, y are of distance at most two in H. Given H it is easy to compute its square H^2, however Motwani and Sudan proved that it is NP-complete to determine if a given graph G is the square of some graph H (of girth 3). In this paper we consider the characterization and recognition problems of graphs that are squares of graphs of small girth, i.e. to determine if $G = H^2$ for some graph H of small girth. The main results are the following.

- There is a graph theoretical characterization for graphs that are squares of some graph of girth at least 7. A corollary is that if a graph G has a square root H of girth at least 7 then H is unique up to isomorphism.
- There is a polynomial time algorithm to recognize if $G = H^2$ for some graph H of girth at least 6.
- It is NP-complete to recognize if $G = H^2$ for some graph H of girth 4.

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These results almost provide a dichotomy theorem for the complexity of the recognition problem in terms of girth of the square roots. The algorithmic and graph theoretical results generalize previous results on tree square roots, and provide polynomial time algorithms to compute a graph square root of small girth if it exists. Some open questions and conjectures will also be discussed.

Keywords Graph roots · Graph powers · Recognition algorithms · NP-completeness

1 Introduction

Root and *root finding* are concepts familiar to most branches of mathematics. In graph theory, H is a *square root* of G and G is the *square* of H if two vertices x, y have an edge in G if and only if x, y are of distance at most two in H. Graph square is a basic operation with a number of results about its properties in the literature. In this paper we are interested in the characterization and recognition problems of graph squares. Ross and Harary [21] characterized squares of trees and showed that tree square roots, when they exist, are unique up to isomorphism. Mukhopadhyay [19] provided a characterization of graphs which have a square root, but this is not a good characterization in the sense that it does not give a short certificate when a graph does not have a square root. In fact, such a good characterization may not exist as Motwani and Sudan proved that it is NP-complete to determine if a given graph has a square root [18]. On the other hand, there are polynomial time algorithms to compute the tree square root [3, 4, 13, 14, 16], a bipartite graph square root [14], and a proper interval graph square root [15].

The algorithms for computing tree square roots and bipartite graph square roots are based on the fact that the square roots have no cycles and no odd cycles respectively. Since computing the graph square uses only local information from the first and the second neighborhood, it is plausible that there are polynomial time algorithms to compute square roots that have no short cycles (locally tree-like), and more generally to compute square roots that have no short odd cycles (locally bipartite). The *girth* of a graph is the length of a shortest cycle. In this paper we consider the characterization and recognition problems of graphs that are squares of graphs of small girth, i.e. to determine if $G = H^2$ for some graph H of small girth.

The main results of this paper are the following. In Sect. 2 we will provide a good characterization for graphs that are squares of some graph of girth at least 7. This characterization not only leads to a simple algorithm to compute a square root of girth at least 7 but also shows such a square root, if it exists, is unique up to isomorphism. Then, in Sect. 3, we will present a polynomial time algorithm to compute a square root of girth at least 6, or report that none exists. In Sect. 4 we will show that it is NP-complete to determine if a graph G has a square root of girth 4. Finally, we discuss some open questions and conjectures.

These results almost provide a dichotomy theorem for the complexity of the recognition problem in terms of girth of the square roots. The algorithmic and graph theoretical results considerably generalize previous results on tree square roots. We believe that our algorithms can be extended to compute square roots with no short odd cycles (locally bipartite), and in fact one part of the algorithm for computing square roots of girth at least 6 uses only the assumption that the square roots have no 3-cycles or 5-cycles. Coloring properties of squares in terms of girth of the roots have been considered in the literature [2, 6, 10]; our algorithms would allow those results to apply even though a square root was not known a priori.

Definitions and notation All graphs considered are finite, undirected and simple. Let $G = (V_G, E_G)$ be a graph. We often write $xy \in E_G$ for $\{x, y\} \in E_G$. Following [15, 18], we sometimes also write $x \leftrightarrow y$ for the adjacency of x and y in the graph in question; this is particularly the case when we describe reductions in NP-completeness proofs.

The *neighborhood* $N_G(v)$ in G of a vertex v is the set of all vertices in G being adjacent to v and the *closed neighborhood* of v in G is $N_G[v] = N_G(v) \cup \{v\}$. Set $\deg_G(v) = |N_G(v)|$, the *degree* of v in G. We call vertices of degree one in G *end-vertices* of G. A *center vertex* of G is one that is adjacent to all other vertices.

Let $d_G(x, y)$ be the length, i.e., number of edges, of a shortest path in G between x and y. Let $G^k = (V_G, E^k)$ with $xy \in E^k$ if and only if $1 \le d_G(x, y) \le k$ denote the *k-th power of* G. If $G = H^k$ then G is the k-th power of the graph H and H is a *k-th root* of G. Since the power of a graph H is the union of the powers of the connected components of H, we may assume that all graphs considered are connected.

A set of vertices $Q \subseteq V_G$ is called a *clique* in G if every two distinct vertices in Q are adjacent; a *maximal clique* is a clique that is not properly contained in another clique. A *stable set* is a set of pairwise non-adjacent vertices. Given a set of vertices $X \subseteq V_G$, the subgraph induced by X is written $G[X]$ and $G - X$ stands for $G[V \setminus X]$. If $X = \{a, b, c, \ldots\}$, we write $G[a, b, c, \ldots]$ for $G[X]$. Also, we often identify a subset of vertices with the subgraph induced by that subset, and vice versa.

The *girth* of G, $\text{girth}(G)$, is the smallest length of a cycle in G; in case G has no cycles, we set $\text{girth}(G) = \infty$. In other words, G has girth k if and only if G contains a cycle of length k but does not contain any (induced) cycle of length $\ell = 3, \ldots, k - 1$. Note that the girth of a graph can be computed in $O(nm)$ time, where n and m are the number of vertices, respectively, edges of the input graph [12].

A complete graph is one in which every two distinct vertices are adjacent; a complete graph on k vertices is also denoted by K_k. A *star* is a graph with *at least two* vertices that has a center vertex and the other vertices are pairwise non-adjacent. Note that a star contains at least one edge and at least one center vertex; the center vertex is unique whenever the star has more than two vertices.

2 Squares of Graphs with Girth at Least Seven

In this section, we give a good characterization of graphs that are squares of a graph of girth at least seven. The key idea is that every maximal clique in the square of a graph G of girth at least seven corresponds to the neighbourhood of a vertex in G. Our characterization leads to a simple polynomial-time recognition for such graphs.

Proposition 2.1 *Let G be a connected, non-complete graph such that $G = H^2$ for some graph H.*

(i) *If* $girth(H) \geq 6$ *and* v *is a vertex with* $\deg_H(v) \geq 2$ *then* $N_H[v]$ *is a maximal clique in* G;
(ii) *If* $girth(H) \geq 7$ *and* Q *is a maximal clique in* G *then* $Q = N_H[v]$ *for some vertex* v *where* $\deg_H(v) \geq 2$.

Proof (i) Let v be a vertex with $\deg_H(v) \geq 2$. Clearly, $Q = N_H[v]$ is a clique in G. Consider an arbitrary vertex w outside Q; in particular, w is non-adjacent in H to v. If w is non-adjacent in H to all vertices in Q, then $d_H(w, v) > 2$. If w is adjacent in H to a vertex $x \in Q - v$, let $y \in Q \setminus \{v, x\}$. Then $N_H[w] \cap N_H[y] = \emptyset$ (otherwise H would contain a cycle of length at most five), hence $d_H(w, y) > 2$. Thus, in any case, w cannot be adjacent, in G, to all vertices in Q, and so Q is a maximal clique in G.

(ii) Let Q be a maximal clique in G and $v \in Q$ be a vertex that maximizes $|Q \cap N_H[v]|$. We prove that $Q = N_H[v]$. It can be seen that by the maximality of Q, $\deg_H(v) \geq 2$. Now, we show that if $w \in Q \setminus N_H[v]$ and $x \in Q \cap N_H[v]$, then $wx \notin E_H$: As $w \notin N_H[v]$, this is clear in case $x = v$. So, let $x \neq v$ and assume to the contrary that $wx \in E_H$. Then, by the choice of v, there exists a vertex $w' \in Q \setminus N_H[x]$, $w' \in N_H[v]$. Note that $w'x, w'w \notin E_H$ because H has no C_3, C_4. As $ww' \in E_G \setminus E_H$, there exists a vertex $u \notin \{w, w', x, v\}$ with $uw, uw' \in E_H$. But then $H[w, w', x, v, u]$ contains a C_4 or C_5. Contradiction.

Finally, we show that $Q \subseteq N_H[v]$, and so, by the maximality of Q, $Q = N_H[v]$: Assume otherwise and let $w \in Q \setminus N_H[v]$. As $wv \in E_G \setminus E_H$, there exists a vertex x such that $xw, xv \in E_H$, and so, $x \in N_H[v] \setminus Q$. By the maximality of Q, x must be non-adjacent (in G) to a vertex $w' \in Q$. In fact, $w' \in Q \setminus N_H[v]$ as x is adjacent in G to every vertex in $N_H[v]$. Since $w'v \in E_G \setminus E_H$, there exists a vertex a such that $aw', av \in E_H$; note that $a \notin \{x, w\}$. Now, if $ww' \in E_H$ then $H[w, w', a, v, x]$ contains a cycle of length at most five. If $ww' \notin E_H$, let b be a vertex such that $bw, bw' \in E_H$; possibly $b = a$. Then $H[w, w', a, b, v, x]$ contains a cycle of length at most six. In any case we have a contradiction, hence $Q \setminus N_H[v] = \emptyset$. □

The 5-cycle C_5 and the 6-cycle C_6 show that (i), respectively, (ii) in Proposition 2.1 is best possible with respect to the girth condition of the root. More generally, the maximal cliques in the square of the subdivision of any complete graph on $n \geq 3$ vertices do not satisfy Condition (ii).

Let G be an arbitrary graph. An edge of G is called *forced* if it is contained in (at least) two distinct maximal cliques in G.

Proposition 2.2 *Let* G *be a connected, non-complete graph such that* $G = H^2$ *for some graph* H *with girth at least seven, and let* F *be the subgraph of* G *consisting of all forced edges of* G. *Then*

(i) F *is obtained from* H *by deleting all end-vertices in* H;
(ii) *for every maximal clique* Q *in* G, $F[Q \cap V_F]$ *is a star*; *and*
(iii) *every vertex in* $V_G - V_F$ *belongs to exactly one maximal clique in* G.

Proof We first make the following two observations:

(1) Consider a forced edge xy in G. Let $Q_1 \neq Q_2$ be two maximal cliques in G containing xy. By Proposition 2.1, there exist vertices v_i, $i = 1, 2$, with $\deg_H(v_i) \geq 2$ and $Q_i = N_H[v_i]$. As $Q_1 \neq Q_2$, $v_1 \neq v_2$. As $x, y \in N_H[v_1] \cap N_H[v_2]$ and H has no

C_3, C_4, $\{x, y\} = \{v_1, v_2\}$ and $xy = v_1v_2 \in E_H$. Thus, every forced edge xy in G is an edge in H with $\deg_H(x) \geq 2$ and $\deg_H(y) \geq 2$.

(2) Let xy be an edge in H. If x or y is an end-vertex in H, then clearly xy belongs to exactly one maximal clique in G, hence xy is not a forced edge in G. If $\deg_H(x) \geq 2$ and $\deg_H(y) \geq 2$, then by Proposition 2.1, $N_H[x]$ and $N_H[y]$ are two (distinct) maximal cliques in G containing xy, hence xy is a forced edge in G.

Now, (i) follows directly from the above observations. For (ii) and (iii), consider a maximal clique Q in G. By Proposition 2.1, $Q = N_H[v]$ for some vertex v with $\deg_H(v) \geq 2$. Let X be the set of all neighbors of v in H that are end-vertices in H and $Y = N_H(v) \setminus X$. Since G is not complete, $Y \neq \emptyset$. By (i), $X \cap V_F = \emptyset$, hence $F[Q \cap V_F] = F[\{v\} \cup Y]$ which implies (ii). For (iii), consider a vertex $u \in V_G - V_F$ and a maximal clique Q containing u. Then, u cannot belong to Y and therefore Q is the only maximal clique containing u. □

We now are able to characterize squares of graphs with girth at least seven as follows.

Theorem 2.3 *Let G be a connected, non-complete graph. Let F be the subgraph of G consisting of all forced edges in G. Then G is the square of a graph with girth at least seven if and only if the following conditions hold.*

(i) *Every vertex in $V_G - V_F$ belongs to exactly one maximal clique in G.*
(ii) *Every edge in F belongs to exactly two distinct maximal cliques in G.*
(iii) *Every two non-disjoint edges in F belong to a common maximal clique in G.*
(iv) *For each maximal clique Q of G, $F[Q \cap V_F]$ is a star.*
(v) *F is connected and has girth at least seven.*

Proof For the only if-part, (ii) and (iii) follow easily from Proposition 2.1, and (i), (iv) and (v) follow directly from Proposition 2.2.

For the if-part, let G be a connected graph satisfying (i)–(v). We will construct a spanning subgraph H of G with girth at least seven such that $G = H^2$ as follows. For each edge xy in F let, by (ii) and (iv), $Q \neq Q'$ be the two maximal cliques in G with $Q \cap Q' = \{x, y\}$. Let, without loss of generality, $|Q \cap V_F| \geq |Q' \cap V_F|$. Assuming x is a center vertex of the star $F[Q \cap V_F]$, then y is a center vertex of the star $F[Q' \cap V_F]$: Otherwise, by (iv), x is the center vertex of the star $F[Q' \cap V_F]$ and there exists some $y' \in Q' \cap V_F$ such that $yy' \notin F$; note that $xy' \in F$ (by (iv)). As $|Q \cap V_F| \geq |Q' \cap V_F|$, there is an edge $xz \in F - xy$ in $Q - Q'$. By (iii), $zy' \in E_G$. Now, as Q' is maximal, the maximal clique Q'' containing x, y, z, y' is different from Q'. But then $\{y, y'\} \subseteq Q' \cap Q''$, i.e., $yy' \in F$, hence F contains a triangle xyy', contradicting (v).

Thus, assuming x is a center vertex of the star $F[Q \cap V_F]$, y is a center vertex of the star $F[Q' \cap V_F]$. Then put the edges xq, $q \in Q - x$, and yq', $q' \in Q' - y$, into H.

By construction, $F \subseteq H \subseteq G$ and by (i),

$$\text{for all vertices } u \in V_H \setminus V_F, \quad \deg_H(u) = 1, \tag{1}$$

$$\forall v \in V_F, \forall a, b \in V_H \text{ with } va, vb \in E_H: \quad a \text{ and } b \text{ belong to the same clique in } G. \tag{2}$$

Furthermore, as every maximal clique in G contains a forced edge (by (iv)), H is a spanning subgraph of G. Moreover, F is an induced subgraph of H: Consider an edge $xy \in E_H$ with $x, y \in V_F$. By construction of H, x or y is a center vertex of the star $F[Q \cap V_F]$ for some maximal clique Q in G. Since $x, y \in V_F$, xy must be an edge of this star, i.e., $xy \in E_F$. Thus, F is an induced subgraph of H. In particular, by (1) and (v), H is connected and $\text{girth}(H) = \text{girth}(F) \geq 7$.

Now, we complete the proof of Theorem 2.3 by showing that $G = H^2$. Let $uv \in E_G \setminus E_H$ and let Q be a maximal clique in G containing uv. By (iv), Q contains a forced edge xy and x or y is a center vertex of the star $F[Q \cap V_F]$. By construction of H, xu and xv, or else yu and yv are edges of H, hence $uv \in E_{H^2}$. This proves $E_G \subseteq E_{H^2}$. Now, let $ab \in E_{H^2} \setminus E_H$. Then there exists a vertex x such that $xa, xb \in E_H$. By (1), $x \in V_F$, and by (2), $ab \in E_G$. This proves $E_{H^2} \subseteq E_G$. □

Corollary 2.4 *Given a graph $G = (V_G, E_G)$, it can be recognized in $O(|V_G|^2 \cdot |E_G|)$ time if G is the square of a graph H with girth at least seven. Moreover, such a square root, if any, can be computed in the same time.*

Proof Note that by Proposition 2.1, any square of an n-vertex graph with girth at least seven has at most n maximal cliques. Now, to avoid triviality, assume G is connected and non-complete. We first use the algorithm in [22] to list the maximal cliques in G in time $O(n^2 m)$, where $m = |E_G|$. If there are more than n maximal cliques, G is not the square of any graph with girth at least seven. Otherwise, compute the forced edges of G to form the subgraph F of G. This can be done in time $O(n^2)$ in an obvious way. Conditions (i)–(v) in Theorem 2.3 then can be tested within the same time bound, and the square root can be constructed, in case all conditions are satisfied, according to the proof of Theorem 2.3. □

Corollary 2.5 *The square roots with girth at least seven of squares of graphs with girth at least seven are unique, up to isomorphism.*

Proof Let G be the square of some graph H with girth ≥ 7. If G is complete, clearly, every square root with girth ≥ 6 of G must be isomorphic to the star $K_{1,n-1}$ where n is the number of vertices of G.

Thus, let G be non-complete, and let F be the subgraph of G formed by the forced edges. If F has only one edge, G clearly consists of exactly two maximal cliques, Q_1, Q_2, say, and $Q_1 \cap Q_2$ is the only forced edge of G. Then, it is easily seen that every square root with girth ≥ 6 of G must be isomorphic to the double star T having center edge $v_1 v_2$ and $\deg_T(v_i) = |Q_i|$.

So, assume F has at least two edges. Then for each two maximal cliques Q, Q' in G with $Q \cap Q' = \{x, y\}$, x or y is the unique center vertex of the star $F[V_F \cap Q]$ or $F[V_F \cap Q']$. Hence, for any end-vertex u of H, i.e., $u \in V_G - V_F$, the neighbor of u in F is unique. Since F is the graph resulting from H by deleting all end-vertices, H is therefore unique. □

2.1 Further Considerations

Squares of bipartite graphs can be recognized in $O(\Delta \cdot M(n))$ time in [14], where $\Delta = \Delta(G)$ is the maximum degree of the n-vertex input graph G and $M(n)$ is the time needed to perform the multiplication of two $n \times n$-matrices. However, no good characterization is known so far. As bipartite graphs with girth at least seven are exactly the (C_4, C_6)-free bipartite graphs, we immediately have:

Corollary 2.6 *Let G be a connected, non-complete graph. Let F be the subgraph of G consisting of all forced edges in G. Then G is the square of a (C_4, C_6)-free bipartite graph if and only if the following conditions hold.*

(i) *Every vertex in $V_G - V_F$ belongs to exactly one maximal clique in G.*
(ii) *Every edge in F belongs to exactly two distinct maximal cliques in G.*
(iii) *Every two non-disjoint edges in F belong to a common maximal clique in G.*
(iv) *For each maximal clique Q of G, $F[Q \cap V_F]$ is a star.*
(v) *F is a connected (C_4, C_6)-free bipartite graph.*

Moreover, squares of (C_4, C_6)-free bipartite graphs can be recognized in $O(n^2m)$ time, and the (C_4, C_6)-free square bipartite roots of such squares are unique, up to isomorphism.

Using the results in this section, we obtain a new characterization for tree squares that allows us to derive the known results on tree square roots easily.

It was shown in [16] that the problems CLIQUE and STABLE SET, i.e., finding a maximum stable set and finding a maximum stable set, respectively, remain NP-complete on squares of graphs (of girth three). Another consequence of our results is.

Corollary 2.7 *The weighted version of* CLIQUE *can be solved in $O(n^2m)$ time on squares of graphs with girth at least* 7, *where n and m are the number of vertices, respectively, edges of the input graph.*

Proof Let $G = (V_G, E_G)$ be the square of some graph with girth at least seven. By Proposition 2.1, G has $O(|V_G|)$ maximal cliques. By [22], all maximal cliques in G then can be listed in time $O(|V_G| \cdot |E_G| \cdot |V_G|)$. □

In [11], it was shown that STABLE SET is NP-complete on squares of the subdivision of some graph (i.e. the squares of the total graph of some graph). As the subdivision of a graph has girth at least six, STABLE SET therefore is NP-complete on squares of graphs with girth at least six.

3 Squares of Graphs with Girth at Least Six

In this section we will show that squares of graphs with girth at least six can be recognized efficiently. Formally, we will show that the following problem

SQUARE OF GRAPH WITH GIRTH AT LEAST SIX

Instance: A graph G.
Question: Does there exist a graph H with girth at least 6 such that $G = H^2$?

is polynomially solvable (Theorem 3.5).

Similar to the algorithm in [14], our recognition algorithm consists of two steps. The first step (Sect. 3.1) is to show that if we fix a vertex $v \in V_G$ and a subset $U \subseteq N_G(v)$, then there is at most one $\{C_3, C_5\}$-free (locally bipartite) square root graph H of G with $N_H(v) = U$. Then, in the second step (Sect. 3.2), we show that if we fix an edge $e = uv \in E_G$, then there are at most two possibilities of $N_H(v)$ for a square root H with girth at least 6. Furthermore, both steps can be implemented efficiently, and thus it will imply that SQUARE OF GRAPH WITH GIRTH AT LEAST SIX is polynomially solvable.

3.1 Square Root with a Specified Neighborhood

This subsection deals with the first auxiliary problem.

$\{C_3, C_5\}$-FREE SQUARE ROOT WITH A SPECIFIED NEIGHBORHOOD

Instance: A graph G, $v \in V_G$ and $U \subseteq N_G(v)$.
Question: Does there exist a $\{C_3, C_5\}$-free graph H such that $H^2 = G$ and $N_H(v) = U$?

An efficient recognition algorithm for $\{C_3, C_5\}$-FREE SQUARE ROOT WITH A SPECIFIED NEIGHBORHOOD relies on the following fact.

Lemma 3.1 *Let $G = H^2$ for some $\{C_3, C_5\}$-free graph H. Then, for all vertices $x \in V_G$ and all vertices $y \in N_H(x)$, $N_H(y) = N_G(y) \cap (N_G[x] \setminus N_H(x))$.*

Proof First, consider an arbitrary vertex $w \in N_H(y) - x$. Clearly, $w \in N_G(y)$, as well $w \in N_G(x)$. Also, since H is C_3-free, $wx \notin E_H$. Thus $w \in N_G(y) \cap (N_G(x) \setminus N_H(x))$.

Conversely, let w be an arbitrary vertex in $N_G(y) \cap (N_G[x] \setminus N_H(x))$. Assuming $wy \notin E_H$, then $w \neq x$ and there exist vertices z and z' such that $zx, zw \in E_H$ and $z'y, z'w \in E_H$. As H is C_3-free, $zy \notin E_H$, $z'x \notin E_H$, and $zz' \notin E_H$. But then x, y, w, z and z' induce a C_5 in H, a contradiction. Thus $w \in N_H(y)$. □

Recall that $M(n)$ stands for the time needed to perform a matrix multiplication of two $n \times n$ matrices; currently, $M(n) = O(n^{2.376})$ [5].

Theorem 3.2 $\{C_3, C_5\}$-FREE SQUARE ROOT WITH A SPECIFIED NEIGHBORHOOD *has at most one solution. The unique solution, if any, can be constructed in time $O(M(n))$.*

Proof Given G, $v \in V_G$ and $U \subseteq N_G(v)$, assume H is a $\{C_3, C_5\}$-free square root of G such that $N_H(v) = U$. Then, by Lemma 3.1, the neighborhood in H of each vertex $u \in U$ is uniquely determined by $N_H(u) = N_G(u) \cap (N_G[v] \setminus U)$. By repeatedly applying Lemma 3.1 for each $v' \in U$ and $U' = N_H(v')$ and noting that all considered graphs are connected, we can conclude that H is unique.

ALGORITHM 1

Input: A graph G, a vertex $v \in V_G$ and a subset $U \subseteq N_G(v)$.
Output: A $\{C_3, C_5\}$-free graph H with $H^2 = G$ and $N_H(v) = U$, or else 'NO' if such a square root H of G does not exist.

```
1.  Add all edges vu, u ∈ U, to E_H
2.  Q ← ∅
3.  for each u ∈ U do
4.      enqueue(Q, u)
5.      parent(u) ← v
6.  while Q ≠ ∅ do
7.      u ← dequeue(Q)
8.      set W := N_G(u) ∩ (N_G(parent(u)) \ N_H(parent(u)))
9.      for each w ∈ W do
10.         add uw to E_H
11.         if parent(w) = ∅
12.         then parent(w) ← u
13.              enqueue(Q, w)
14. if G = H^2 then return H
15.   else return 'NO'
```

Lemma 3.1 also suggests the following BFS-like procedure, Algorithm 1 below, for constructing the $\{C_3, C_5\}$-free square root H of G with $U = N_H(v)$, if any.

It can be seen, by construction, that H is $\{C_3, C_5\}$-free, and thus the correctness of Algorithm 1 follows from Lemma 3.1. Moreover, since every vertex is enqueued at most once, lines 1–13 take $O(m)$ steps, $m = |E_G|$. Checking if $G = H^2$ (line 14) takes $O(M(n))$ steps, $n = |V_G|$. □

3.2 Square Root with a Specified Edge

This subsection discusses the second auxiliary problem.

GIRTH ≥ 6 ROOT GRAPH WITH ONE SPECIFIED EDGE

Instance: A graph G and an edge $xy \in E_G$.
Question: Does there exist a graph H with girth at least six such that $H^2 = G$ and $xy \in E_H$?

The question is easy if $|G| \leq 2$. So, for the rest of this section, assume that $|G| > 2$. Then, we will reduce this problem to $\{C_3, C_5\}$-FREE SQUARE ROOT WITH A SPECIFIED NEIGHBORHOOD. Given a graph G and an edge xy of G, write $C_{xy} = N_G(x) \cap N_G(y)$, i.e., C_{xy} is the set of common neighbors of x and y in G.

Lemma 3.3 *Suppose H is of girth at least 6, $xy \in E_H$ and $H^2 = G$. Then $G[C_{xy}]$ has at most two connected components. Moreover, if A and B are the connected components of $G[C_{xy}]$ (one of them may be empty) then* (i) *$A = N_H(x) - y$ and $B = N_H(y) - x$, or* (ii) *$B = N_H(x) - y$ and $A = N_H(y) - x$.*

Proof Set $X = N_H(x) - y$ and $Y = N_H(y) - x$. Notice that X or Y (but not both) may be empty. First we show that $X \cup Y = C_{xy}$. Consider an arbitrary vertex $v \in C_{xy}$;

we claim that v is either in X or Y. Otherwise, there is a length 2 path from v to x and a length 2 path from v to y, which implies that there is either a 3-cycle or a 5-cycle, a contradiction. So we have $C_{xy} \subseteq X \cup Y$.

On the other hand, consider an arbitrary vertex $u \in X$. It is obvious that $u \in N_{H^2}(x)$. Also, since $xy \in E_H$, $u \in N_{H^2}(y)$. A similar argument applies if $u \in Y$. Therefore, $u \in N_{H^2}(x) \cap N_{H^2}(y)$. Since $H^2 = G$, $u \in C_{xy}$. Hence $X \cup Y = C_{xy}$.

Next, observe that X and Y induce cliques in H^2 and thus in G. Moreover, $X \cap Y = \emptyset$ (as H has no 3-cycle) and no vertex in X is adjacent in H to a vertex in Y (as H has no 4-cycle). Now, no vertex $u \in X$ is adjacent in G to a vertex $w \in Y$: Otherwise, there is a vertex $v \notin X \cup Y$ adjacent in H to u and to w, implying that x, y, u, w, v induce a 5-cycle in H, a contradiction.

Thus, the cliques $G[X]$ and $G[Y]$ are exactly the connected components of $G[C_{xy}]$ and the lemma follows. □

By Lemma 3.3, we can solve GIRTH ≥ 6 ROOT GRAPH WITH ONE SPECIFIED EDGE as follows: Compute C_{xy}. If $G[C_{xy}]$ has more than two connected components, there is no solution. If $G[C_{xy}]$ is connected, solve $\{C_3, C_5\}$-FREE SQUARE ROOT WITH A SPECIFIED NEIGHBORHOOD for inputs $I_1 = (G, v = x, U = C_{xy} + y)$ and $I_2 = (G, v = y, U = C_{xy} + x)$. If, for I_1 or I_2, Algorithm 1 outputs H and if H is C_4-free, then H is a solution. In other cases there is no solution. If $G[C_{xy}]$ has two connected components, A and B, solve $\{C_3, C_5\}$-FREE SQUARE ROOT WITH A SPECIFIED NEIGHBORHOOD for inputs $I_1 = (G, v = x, U = A + y)$, $I_2 = (G, v = x, U = B + y)$, $I_3 = (G, v = y, U = A + x)$, $I_4 = (G, v = y, U = B + x)$, and make a decision similarly. In this way, checking if a graph is C_4-free is the most expensive step, and we obtain

Theorem 3.4 GIRTH ≥ 6 ROOT GRAPH WITH ONE SPECIFIED EDGE *can be solved in time* $O(n^4)$.

Let $\delta = \delta(G)$ denote the minimum vertex degree in G. Now we can state the main result of this section as follows.

Theorem 3.5 SQUARE OF GRAPH WITH GIRTH AT LEAST SIX *can be solved in time* $O(\delta \cdot n^4)$, *where n is the number of vertices.*

Proof Given graph G, let $x \in V_G$ be a vertex of minimum degree. For each vertex $y \in N_G(x)$ check if the instance $(G, xy \in E_G)$ for GIRTH ≥ 6 ROOT GRAPH WITH ONE SPECIFIED EDGE has a solution. □

Notice that $O(n^4)$ comes from the time needed for testing C_4-freeness.

4 Squares of Graphs with Girth Four

Note that the reductions for proving the NP-completeness results by Motwani and Sudan [18] show that recognizing squares of graphs with girth three is NP-complete. In this section we show that the following problem is NP-complete.

SQUARE OF GRAPH WITH GIRTH FOUR

Instance: A graph G.
Question: Does there exist a graph H with girth 4 such that $G = H^2$?

Observe that SQUARE OF GRAPH WITH GIRTH FOUR is in NP. We will reduce the NP-complete problem SET SPLITTING [8, Problem SP4], also known as HYPERGRAPH 2-COLORABILITY, to it.

SET SPLITTING

Instance: Collection D of subsets of a finite set S.
Question: Is there a partition of S into two subsets S_1 and S_2 such that each subset in D intersects both S_1 and S_2?

Our reduction is a modification of the reductions for proving the NP-completeness of SQUARE OF CHORDAL GRAPH [15, Theorem 3.5] and for CUBE OF BIPARTITE GRAPH [14, Theorem 7.6]. We also apply the tail structure of a vertex v, first described in [18], to ensure that v has the same neighbors in any square root H of G.

Lemma 4.1 [18] *Let a, b, c be vertices of a graph G such that* (i) *the only neighbors of a are b and c*, (ii) *the only neighbors of b are a, c, and d, and* (iii) *c and d are adjacent. Then the neighbors, in $V_G - \{a, b, c\}$, of d in any square root of G are the same as the neighbors, in $V_G - \{a, b, d\}$, of c in G; see Fig. 1.*

We now are going to describe the reduction. Let $S = \{u_1, \ldots, u_n\}$, $D = \{d_1, \ldots, d_m\}$ where $d_j \subseteq S$, $1 \le j \le m$, be an instance of SET SPLITTING. We construct an instance $G = G(D, S)$ for SQUARE OF GRAPH WITH GIRTH FOUR as follows.

The vertex set of graph G consists of:

(I) U_i, $1 \le i \le n$. Each 'element vertex' U_i corresponds to the element u_i in S.
(II) D_j, $1 \le j \le m$. Each 'subset vertex' D_j corresponds to the subset d_j in D.
(III) D_j^1, D_j^2, D_j^3, $1 \le j \le m$. Each three 'tail vertices' D_j^1, D_j^2, D_j^3 of the subset vertex D_j correspond to the subset d_j in D.
(IV) S_1, S_1', S_2, S_2', four 'partition vertices'.
(V) X, a 'connection vertex'.

The edge set of graph G consists of:

(I) Edges of tail vertices of subset vertices:
For all $1 \le j \le m$: $D_j^3 \leftrightarrow D_j^2$, $D_j^3 \leftrightarrow D_j^1$, $D_j^2 \leftrightarrow D_j^1$, $D_j^2 \leftrightarrow D_j$, $D_j^1 \leftrightarrow D_j$, and for all i, $D_j \leftrightarrow U_i$ whenever $u_i \in d_j$.

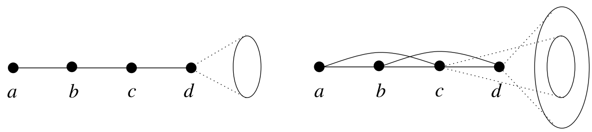

Fig. 1 Tail in H (*left*) and in $G = H^2$ (*right*)

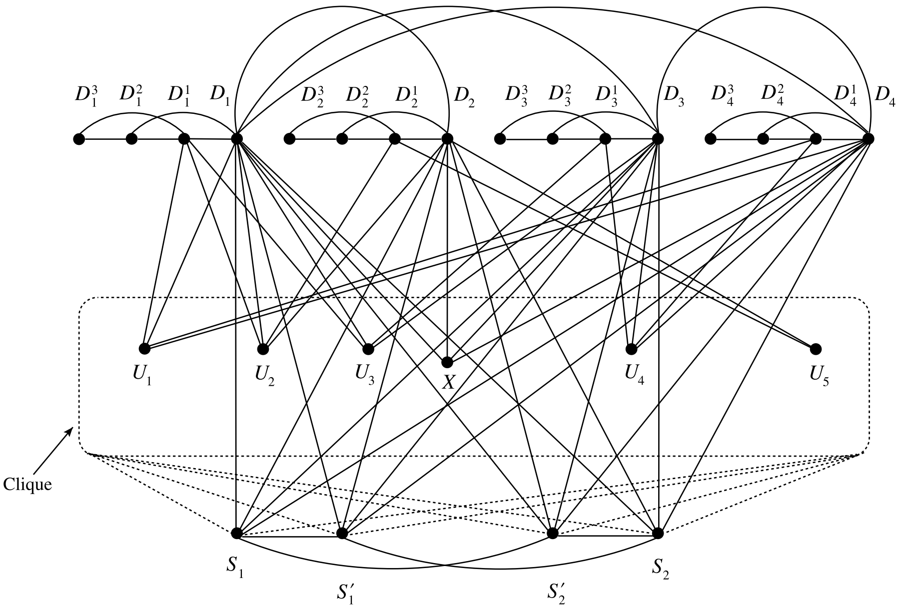

Fig. 2 An example of G

(II) Edges of subset vertices:
For all $1 \le j \le m$: $D_j \leftrightarrow S_1$, $D_j \leftrightarrow S_1'$, $D_j \leftrightarrow S_2$, $D_j \leftrightarrow S_2'$, $D_j \leftrightarrow X$, $D_j \leftrightarrow U_i$ for all i, and $D_j \leftrightarrow D_k$ for all k with $d_j \cap d_k \neq \emptyset$.

(III) Edges of element vertices:
For all $1 \le i \le n$: $U_i \leftrightarrow X$, $U_i \leftrightarrow S_1$, $U_i \leftrightarrow S_2$, $U_i \leftrightarrow S_1'$, $U_i \leftrightarrow S_2'$, and $U_i \leftrightarrow U_{i'}$ for all $i' \neq i$.

(IV) Edges of partition vertices:
$S_1 \leftrightarrow X$, $S_1 \leftrightarrow S_1'$, $S_1 \leftrightarrow S_2'$, $S_2 \leftrightarrow X$, $S_2 \leftrightarrow S_1'$, $S_2 \leftrightarrow S_2'$, $S_1' \leftrightarrow X$, $S_2' \leftrightarrow X$.

Clearly, G can be constructed from D, S in polynomial time. For an illustration, given $S = \{u_1, u_2, u_3, u_4, u_5\}$ and $D = \{d_1, d_2, d_3, d_4\}$ with $d_1 = \{u_1, u_2, u_3\}$, $d_2 = \{u_2, u_5\}$, $d_3 = \{u_3, u_4\}$, and $d_4 = \{u_1, u_4\}$, the graph G is depicted in Fig. 2. In the figure, the two dotted lines from a vertex to the clique $\{U_1, U_2, U_3, U_4, U_5, X\}$ mean that the vertex is adjacent to all vertices in that clique.

Note that, apart from the three vertices X, S_1', and S_2' (or, symmetrically, X, S_1, and S_2), our construction is the same as those in [15, §3.1.1]. While S_1 and S_2 will represent a partition of the ground set S (Lemma 4.3), the vertices X, S_1', and S_2' allow us to make a square root of G being C_3-free (Lemma 4.2).

Lemma 4.2 *If there exists a partition of S into two disjoint subsets S_1 and S_2 such that each subset in D intersects both S_1 and S_2, then there exists a graph H with girth four such that $G = H^2$.*

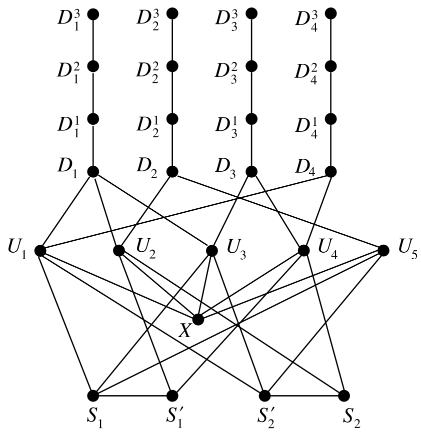

Fig. 3 An example of root H with girth 4

Proof Let H have the same vertex set as G. The edges of H are as follows.

- Edges of subset vertices and their tail vertices:
  For all $1 \le j \le m$: $D_j^3 \leftrightarrow D_j^2$, $D_j^2 \leftrightarrow D_j^1$, $D_j^1 \leftrightarrow D_j$, and for all i, $D_j \leftrightarrow U_i$ whenever $u_i \in d_j$.
- Edges of partition vertices:
  $S_1 \leftrightarrow S_1'$, $S_2 \leftrightarrow S_2'$, and for all i, $S_1 \leftrightarrow U_i$ and $S_2' \leftrightarrow U_i$ whenever $u_i \in S_1$, and $S_2 \leftrightarrow U_i$ and $S_1' \leftrightarrow U_i$ whenever $u_i \in S_2$.
- Edges of the connection vertex:
  $X \leftrightarrow U_i$ for all $1 \le i \le n$.

It is straightforward to check that $G = H^2$; see also Fig. 3 for an example.

By construction, the neighborhood in H of any vertex is a stable set, hence H has no C_3. Observe that H has girth four as it contains a C_4 consisting of X, D_i, an element vertex that corresponds to an element in $d_i \cap S_1$, and another element vertex that corresponds to an element in $d_i \cap S_2$. □

In the above example, $S_1 = \{u_1, u_3, u_5\}$ and $S_2 = \{u_2, u_4\}$ is a possible legal partition of S. The corresponding graph H constructed in the proof of Lemma 4.2 is depicted in Fig. 3.

Lemma 4.3 *Let H be the graph constructed in the proof of Lemma 4.2. If H is a square root of G, then there exists a partition of S into two disjoint subsets S_1 and S_2 such that each subset in D intersects both S_1 and S_2.*

Proof First, observe that for each j, D_j^3, D_j^2, D_j^1, D_j satisfy the properties of Lemma 4.1. Hence, in H, D_j is adjacent to exactly D_j^1 and U_i for which $u_i \in d_j$. This and the fact that, in G, the partition vertices S_1, S_1', S_2, S_2' are non-adjacent to the tail vertices, show that $N_H(S_1) - \{S_1', S_2', X\}$ and $N_H(S_2) - \{S_1', S_2', X\}$ consist of element vertices only.

Now, since S_1 and S_2 are non-adjacent in G, they have no common neighbor in H. Therefore, $N_H(S_1) - \{S_1', S_2', X\}$ and $N_H(S_2) - \{S_1', S_2', X\}$ will define a partition of the element set. Since the partition vertices are adjacent to all subset vertices in G but not in H, each of S_1 and S_2 has, in H, a common neighbor with D_j in the element set for all j. Thus, $N_H(S_1) - \{S_1', S_2', X\}$ and $N_H(S_2) - \{S_1', S_2', X\}$ define a desired partition of S. □

Note that in Lemma 4.3 above we did not require that H has girth four. Thus, any square root of G-particularly, any square root with girth four-will tell us how to do set splitting. Together with Lemma 4.2 we conclude:

Theorem 4.4 SQUARE OF GRAPH WITH GIRTH FOUR *is NP-complete*.

5 Squares of Trees Revisited

Given the fact that the squares of trees have been widely discussed in the literature, we will derive from the results in Sect. 2 a new characterization for tree squares from which known results on tree squares in the literature, such as chordality and linear-time recognition, follow easily.

Observe that the proof of Theorem 2.3 shows that if F is a tree, then also the square root H is a tree. This fact and Propositions 2.1 and 2.2 immediately imply the following good characterization for squares of trees in terms of forced edges. Recall the definition of forced edges in a graph (Sect. 2).

Theorem 5.1 *Let G be a connected, non-complete graph. Let F be the subgraph of G consisting of all forced edges in G. Then G is the square of a tree if and only if the following conditions hold.*

(i) *Every vertex in $V_G - V_F$ belongs to exactly one maximal clique in G;*
(ii) *Every edge in F belongs to exactly two distinct maximal cliques in G;*
(iii) *Every two non-disjoint edges in F belong to a common maximal clique in G;*
(iv) *For each maximal clique Q of G, $F[Q \cap V_F]$ is a star;*
(v) *F is a tree.*

A graph is *chordal* if it does not contain an induced cycle C_ℓ of any length $\ell \geq 4$. A chordal graph is *strongly chordal* if it does not contain any k-sun as an induced subgraph; here a *k-sun*, $k \geq 3$, consist of a clique $\{u_1, u_2, \ldots, u_k\}$ and a stable set $\{v_1, v_2, \ldots, v_k\}$ such that for $i \in \{1, \ldots, k\}$, v_i is adjacent to exactly u_i and u_{i+1} (index arithmetic modulo k).

In [7, 17, 20] it was shown that the square of a tree is strongly chordal; later, [1, 16] proved that the square of a tree is chordal. Our characterization of tree squares, Theorem 5.1, gives a new and short proof for this fact:

Corollary 5.2 [7, 17, 20] *Squares of trees are strongly chordal.*

Proof Let G be a non-complete graph that is the square of a tree, and let F be the forced subgraph of G. Then F satisfies (i)–(v) in Theorem 5.1. In particular, G cannot contain an induced sun otherwise F would contain a cycle, contradicting (v). Now, assume $v_1v_2\dots v_\ell v_1$ is an induced cycle in G with $\ell \geq 4$. Consider the maximal cliques Q_i in G containing the edge v_iv_{i+1}, $1 \leq i \leq \ell$ (modulo ℓ). Note that the Q_is are pairwise distinct, hence by (i), $v_i \in V_F$. Thus, with (iv), $F[Q_i \cap V_F]$ is a star containing v_i and v_{i+1}, $1 \leq i \leq \ell$, implying F contains a cycle; a contradiction to (v). □

Corollary 5.3 [3, 4, 13, 14, 16] *Given a graph $G = (V_G, E_G)$, it can be recognized in $O(|V_G| + |E_G|)$ time if G is the square of a tree. Moreover, a tree root of a square of a tree can be computed in the same time.*

Proof In order to obtain linear time bound, we use Corollary 5.2 saying that squares of trees are chordal, and that all maximal cliques of a chordal graph can be computed in linear time (see, for example, [9]).

Thus, given $G = (V_G, E_G)$, we may assume that G is chordal and all maximal cliques of G are available. To detect all forced edges in G, create for each edge e of G a linked list $L(e)$ consisting of all maximal cliques in G that contain e: Scan each maximal clique Q_i and for each edge e_j in Q_i add Q_i to $L(e_j)$; this can be done in time $O(n + m)$. If $|L(e)| \geq 3$ for some edge e, then (i) fails, and G is not the square of a tree. So, let $|L(e)| \leq 2$ for all edges e, and F consists of all edges e with $|L(e)| = 2$. Clearly, F can be obtained in $O(m)$ time, and (ii)–(iv) can be tested in $O(n + m)$ time. □

Corollary 5.4 [3, 14, 21] *The tree roots of squares of trees are unique, up to isomorphism.*

Proof By Corollary 2.5. □

Finally, we note that the characterizations for tree squares given in [3] also easily follow from our Theorem 5.1.

6 Conclusion and Open Problems

We have shown that squares of graphs with girth at least six can be recognized in polynomial time. We have found a good characterization for squares of graphs with girth at least seven that gives a faster recognition algorithm in this case. For squares of graphs with girth at most four we have shown that recognizing the squares of such graphs is NP-complete.

The complexity status of computing square root with girth (exactly) five is not yet determined. However, we believe that this problem should be efficiently solvable. Also, we believe that the algorithm to compute a square root of girth 6 can be extended to compute a square root with no C_3 or C_5. More generally, let k be a positive integer and consider the following problem.

k-POWER OF GRAPH WITH GIRTH $\geq 3k-1$

Instance: A graph G.
Question: Does there exist a graph H with girth $\geq 3k-1$ such that $G = H^k$?

Conjecture 6.1 k-POWER OF GRAPH WITH GIRTH $\geq 3k-1$ *is polynomially solvable.*

The truth of the above conjecture together with the results in this paper would imply a complete dichotomy theorem: SQUARES OF GRAPHS OF GIRTH g is polynomial if $g \geq 5$ and NP-complete otherwise.

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