

Kate Larson

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# Introduction to Social Choice

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University of Waterloo

# Outline

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# What Is Social Choice Theory

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Summary

- Study of decision problems in which a group has to make the decision
- The decision affects all members of the group
  - Their opinions should count!
- Applications
  - Political elections
  - Other elections
  - Allocations problems (e.g. allocation of money among agents, allocation of goods, tasks, resources....)
  - ...

# Formal Model

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Summary

- Set of agents  $N = \{1, 2, \dots, n\}$
- Set of outcomes  $O$
- Set of strict total orders on  $O$ ,  $L$
- Social choice function:  $f : L^n \rightarrow O$
- Social welfare function:  $f : L^n \rightarrow L^-$  where  $L^-$  is the set of weak total orders on  $O$

# Assumptions

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- Agents have preferences over alternatives
  - Agents can rank order outcomes
- Voters are sincere
  - They truthfully tell the center their preferences
- Outcome is enforced on all agents

Assume that there are only two alternatives,  $x$  and  $y$ . We can represent the family of preferences by

$$(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$$

where  $\alpha_i$  is 1, 0, or -1 according to whether agent  $i$  prefers  $x$  to  $y$ , is ambivalent between them, or prefers  $y$  to  $x$ .

### Definition (Paretian)

*A social choice function is **paretian** if it respects unanimity of strict preferences on the part of the agents.*

# Majority Voting

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$$f(\alpha_1, \dots, \alpha_n) = \text{sign} \sum_j \alpha_j$$

$f(\alpha) = 1$  if and only if more agents prefer  $x$  to  $y$  and  $-1$  if and only if more agents prefer  $y$  to  $x$ . Clearly majority voting is paretian.

# Additional Properties

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Summary

- Symmetric among agents
- Neutral between alternatives
- Positively responsive

## Theorem (May's Theorem)

*A social choice function  $f$  is a majority voting rule if and only if it is symmetric among agents, neutral between alternatives, and positively responsive.*



# Plurality Voting

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The rules of plurality voting are probably familiar to you (e.g. the Canadian election system)

- One name is ticked on a ballot
- One round of voting
- One candidate is chosen
  - Candidate with the most votes

Is this a “good” voting system?

# Plurality Example

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- 3 candidates
  - Lib, NDP, C
- 21 voters with the following preferences
  - 10 C>NDP>Lib
  - 6 NDP>Lib>C
  - 5 Lib>NDP>C
- Result: C 10, NDP 6, Lib 5

The Conservative candidate wins, but a majority of voters (11) prefer all other parties more than the Conservatives.

# What Can We Do?

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Majority system works well when there are two alternatives, but has problems when there are more alternatives.

Proposal: Organize a series of votes between 2 alternatives at a time

# Agendas

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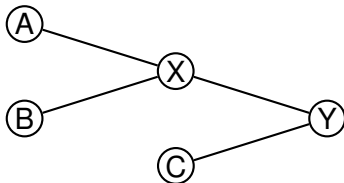
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Summary

- 3 alternatives  $\{A, B, C\}$
- Agenda:  $\langle A, B, C \rangle$



where  $X$  is the outcome of majority vote between  $A$  and  $B$ , and  $Y$  is the outcome of majority vote between  $X$  and  $C$ .

# Agenda Paradox: Power of the Agenda Setter

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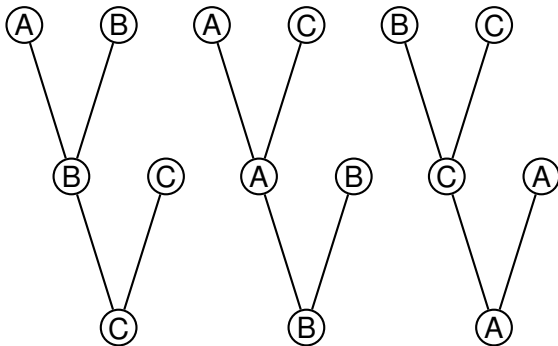
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Summary

3 types of agents:  $A > C > B$  (35%),  $B > A > C$  (33%),  
 $C > B > A$  (32%).

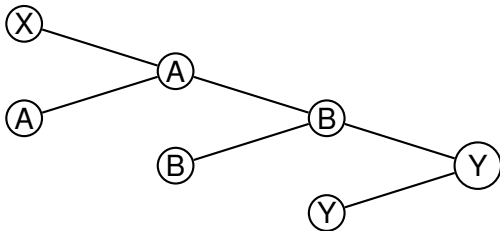
3 different agendas:



# Pareto Dominated Winner Paradox

## 4 alternatives and 3 agents

- $X > Y > B > A$
- $A > X > Y > B$
- $B > A > X > Y$



BUT Everyone prefers X to Y

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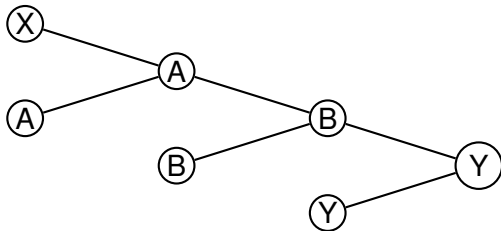
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4 alternatives and 3 agents

- $X > Y > B > A$
- $A > X > Y > B$
- $B > A > X > Y$



BUT Everyone prefers X to Y

# Maybe the problem is with the ballots

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Now have agents reveal their entire preference ordering.  
Condorcet proposed the following

- Compare each pair of alternatives
- Declare “A” is socially preferred to “B” if more voters strictly prefer A to B

**Condorcet Principle:** If one alternative is preferred to *all other* candidates, then it should be selected.

## Definition (Condorcet Winner)

*An outcome  $o \in O$  is a Condorcet Winner if  $\forall o' \in O$ ,  $\#(o > o') \geq \#(o' > o)$ .*



# Condorcet Example

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- 3 candidates
  - Lib, NDP, C
- 21 voters with the following preferences
  - 10 C>NDP>Lib
  - 6 NDP>Lib>C
  - 5 Lib>NDP>C

**Result:** NDP win since 11/21 prefer them to the Conservatives and 16/21 prefer them to the Liberals.

# There Are Other Problems With Condorcet Winners

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Summary

- 3 candidates: Liberal, NDP, Conservative
- 3 voters with preferences
  - Liberal > NDP > Conservative
  - NDP > Conservative > Liberal
  - Conservative > Liberal > NDP

**Result:** Condorcet winners do not always exist.

# Borda Count

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Summary

- Each ballot is a list of ordered alternatives
- On each ballot, compute the rank of each alternative
- Rank order alternatives based on decreasing sum of their ranks

$$\begin{array}{l} A > B > C \\ A > C > B \\ C > A > B \end{array} \Rightarrow \begin{array}{l} A : 4 \\ B : 8 \\ C : 6 \end{array}$$

# Borda Count

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- The Borda Count is simple
- There is always a Borda winner
- BUT the Borda winner is not always the Condorcet winner

3 voters: 2 with preferences  $B > A > C > D$  and one with  $A > C > D > B$

Borda scores:  $A:5, B:6, C:8, D:11$

Therefore A wins, but B is the Condorcet winner.

# Other Borda Count Issues: Inverted-Order Paradox

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Summary

## Agents

- $X > C > B > A$
- $A > X > C > B$
- $B > A > X > C$
- $X > C > B > A$
- $A > X > C > B$
- $B > A > X > C$
- $X > C > B > A$

## Borda Scores

- $X:13, A:18, B:19, C:20$

## Remove X

- $C:13, B:14, A:15$

# Other Borda Count Issues: Inverted-Order Paradox

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## Agents

- $X > C > B > A$
- $A > X > C > B$
- $B > A > X > C$
- $X > C > B > A$
- $A > X > C > B$
- $B > A > X > C$
- $X > C > B > A$

## Borda Scores

- $X:13, A:18, B:19, C:20$

## Remove X

- $C:13, B:14, A:15$

# Vulnerability to Irrelevant Alternatives

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## 3 types of agents

- $X > Z > Y$  (35%)
- $Y > X > Z$  (33%)
- $Z > Y > X$  (32%)

The Borda winner is X.

Remove alternative Z. Then the Borda winner is Y.

# Vulnerability to Irrelevant Alternatives

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3 types of agents

- $X > Z > Y$  (35%)
- $Y > X > Z$  (33%)
- $Z > Y > X$  (32%)

The Borda winner is X.

Remove alternative Z. Then the Borda winner is Y.



# Other Examples of Rules

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- Copeland
  - Do pairwise comparisons of outcomes.
  - Assign 1 point if an outcome wins, 0 if it loses,  $\frac{1}{2}$  if it ties
  - Winner is the outcome with the highest summed score
- Kemeny
  - Given outcomes  $a$  and  $b$ , ranking  $r$  and vote  $v$ , define  $\delta_{a,b}(r, v) = 1$  if  $r$  and  $v$  agree on relative ranking of  $a$  and  $b$
  - *Kemeny ranking*  $r'$  maximises  $\sum_v \sum_{a,b} \delta_{a,b}(r, v)$

# Properties for Voting Protocols

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## Property (Universality)

*A voting protocol should work with any set of preferences.*

## Property (Transitivity)

*A voting protocol should produce an ordered list of alternatives (social welfare function).*

## Property (Pareto efficiency)

*If all agents prefer X to Y, then in the outcome X should be preferred to Y. That is, SWF  $f$  is pareto efficient if for any  $o_1, o_2 \in O, \forall i \in N, o_1 >_i o_2$  then  $o_1 >_f o_2$ .*

# More Properties

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## Property (Independence of Irrelevant Alternatives (IIA))

*Comparison of two alternatives depends only on their standings among agents' preferences, and not on the ranking of other alternatives.*

## Property (No Dictators)

*A SWF  $f$  has no dictator if  $\neg \exists i \forall o_1, o_2 \in O,$   
 $o_1 >_i o_2 \Rightarrow o_1 >_f o_2$*

# Arrow's Theorem

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## Theorem (Arrow's Theorem)

*If there are 3 or more alternatives and a finite number of agents, then there is no SWF which satisfies the 5 desired properties.*

# Is There Anything That Can Be Done?

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## Can we relax the properties?

- No dictator?
  - Fundamental for a voting protocol
- Paretian?
  - Also pretty fundamental
- Transitivity?
  - Maybe you only need to know the top ranked alternative?
    - Stronger form of Arrow's theorem says that you are still in trouble
- IIA?
- Universality
  - Some hope here (1 dimensional preferences, spacial preferences...)

# Take-home Message

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- Despair?
  - No ideal voting method
  - That would be boring!
- A group of more complex than an individual
- Weigh the pro's and cons of each system and understand the setting they will be used in
- Do not believe anyone who says they have the best voting system out there!

# Voting as MLE

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Summary

There is an alternative view of voting.

- There is some sense that some outcomes are *better* than others.
  - This ranking is not merely based on idiosyncratic preferences of voters.
- Voters preferences are *noisy* estimates of the quality.
- Voting is used to infer outcomes true ranking based on the voters' noisy signals (votes)