Fast Hermite Form Computation for polynomial matrices

(also fast determinant)

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Recent Trends in Computer Algebra, Institut Henri Poincaré,

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Outline

Recall

Problem : Given nonsingular $\mathbf{A} \in \mathbb{K}[x]^{n \times n}$. Compute $\mathbf{U}$ and $\mathbf{H}$:

(i) $\mathbf{U}$ unimodular,

(ii) $\mathbf{H}$ in (column) Hermite form

(iii) $\mathbf{A} \cdot \mathbf{U} = \mathbf{H}$

Hermite Normal Form :

$$\mathbf{H} = \begin{bmatrix} h_{11} & 0 & \cdots & \cdots & 0 \\ h_{21} & h_{22} & 0 & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ h_{n1} & \cdots & & \cdots & h_{nn} \end{bmatrix} \qquad \deg h_{ij} < \deg h_{ii}.$$

Goal:

We describe:

- Fast, deterministic algorithms for $\mathbf{H}$
- Fast, deterministic algorithms for determinant of $(\mathbf{A})$
- Complexity : $O^{\sim}\left(n^{\omega}\lceil s\rceil\right)$ where s bounded by average
  : of row and column degrees of $\mathbf{A}$
  : here $O^{\sim}$ is big O without log factors.

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Details :

- G. Labahn, V. Neiger and W. Zhou,
  Fast, deterministic computation of determinants and Hermite normal forms of polynomial matrices, J. of Complexity 2018.

Recall: Previous work

- Polynomial-time over $\mathbb{Q}[x]$: Kannan 1985.
- $O^{\sim}(n^4 d)$: Hafner-McCurley 1991 deterministic
- $O^{\sim}(n^{\omega+1} d)$: Hafner-McCurley (1991), Villard (1996)
  Storjohann and Labahn (1996)
  deterministic
- $O^{\sim}(n^3 d^2)$: Mulders and Storjohann (2003) deterministic
- $O^{\sim}(n^3 d)$: Gupta and Storjohann (2012) probabilistic
- $O^{\sim}(n^{\omega} d)$: Gupta and Storjohann (2012) probabilistic
- $O^{\sim}(n^{\omega} s)$: Labahn-Neiger-Zhou (2018) deterministic

Approach used today for polynomial matrices

- Triangularize $\mathbf{A}$
  - Gives diagonal entries of $\mathbf{H}$ which can be large
  - Best a-priori bound $\deg h_{ii} \leq id$ with $\sum_i id \in O(n^2 d)$.

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- Reduce remaining off-diagonal entries
  - Need to avoid actually computing unimodular multiplier $\mathbf{U}$

    (since $\mathbf{U}$ can be too large)

Matrix Polynomial Algebra

Leading coefficients

Shifted Degrees

Reduced forms

Leading coefficients and degrees and shifts

$$A = 9z^8 + z^7 - 3z^3 + 2z = 9z^8 + \cdots$$

$$\mathrm{lcoeff}_z(A) = 9, \qquad \mathrm{degree}_z(A) = 8$$

Leading coefficients and degrees and shifts

$$A = \begin{bmatrix} 1 & z^2+3 & 7z^8+z \\ 1 & 2z^7+1 & 3 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} z^8 + \cdots$$

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Note in last two cases the leading coefficient was nonsingular.

A is column reduced and shifted column reduced (with shift $(0, 1, 8)$), respectively.

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- The column degree of a column vector $\mathbf{p}$ is

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- The $\vec{s}$-column degree of $\mathbf{p}$ is

$$\operatorname{cdeg}_{\vec{s}} \mathbf{p} \quad = \quad \max_{1 \le i \le n} \left[\deg p^{(i)} + s_i \right] \quad = \quad \operatorname{cdeg} x^{\vec{s}} \cdot \mathbf{p}.$$

Examples of Shifted degree

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- $\operatorname{cdeg}_{(0,1,8)}(A) = (8, 8, 8)$
- $\operatorname{cdeg}_{(1,0,0)}(A) = (1, 7, 9)$

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- $\operatorname{cdeg}_{(0,1,8)}(A) = (8, 8, 8)$
- $\operatorname{cdeg}_{(1,0,0)}(A) = (1, 7, 9)$
- For any matrix $\mathbf{A}$: $\operatorname{cdeg}_{-\vec{s}} \mathbf{A} \leq 0$ same as $\operatorname{rdeg} \mathbf{A} \leq \vec{s}$

Why are shifts useful?

Example:

Given $\mathbf{A}$ and an algorithm for column reduced kernel.

Find unimodular $\mathbf{U}$ such that $\mathbf{AU}$ is column reduced.

$$\mathbf{A} \cdot \mathbf{U} = \mathbf{B} \quad \text{same as} \quad [\mathbf{A}, \quad -\mathbf{I}_n] \cdot \begin{bmatrix} \mathbf{U} \\ \mathbf{B} \end{bmatrix} = 0$$

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But we can use column reduced kernel algorithm if k big enough!

Shifts allows us to focus on specific parts of a polynomial matrix.

Matrix Polynomial Domains

Kernel Bases

Order Bases

Column Bases

Minimal Kernel Bases

Given $\mathbf{A} \in \mathbb{K}[z]^{m \times n}, \quad m \leq n$:

A (right) *Kernel Basis* for $\mathbf{A}$ is a $\mathbb{K}[z]$-module basis for

$$\{ \mathbf{p} \in \mathbb{K}[z]^n \mid \mathbf{A} \cdot \mathbf{p} = 0 \}$$

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Minimal Kernel Basis if matrix $\mathbf{M}$ is column reduced.

- i.e. leading coeff matrix has full column rank

Shifted $\vec{s}$-Minimal Kernel Basis if $z^{\vec{s}} \cdot \mathbf{M}$ is column reduced.

Order Bases

An *Order Basis* for $\mathbf{A}$ and order $\vec{\sigma}$ is a $\mathbb{K}[z]$-module basis for

$$\{\ \mathbf{p} \in \mathbb{K}[z]^n \ \mid\ \mathbf{A} \cdot \mathbf{p} = O(z^{\vec{\sigma}})\ \}$$

Can represent basis as matrix $\mathbf{M} \in \mathbb{K}[z]^{n \times n}$.

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Again

(i) Represent column basis as full rank matrix $\mathbf{T} \in \mathbb{K}[z]^{m \times r}$.

(ii) Can find unimodular matrix $\mathbf{U}$ with $\mathbf{A} \cdot \mathbf{U} = [\mathbf{0}, \mathbf{T}]$.

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  - let σ be 3 times the average of the highest m entries of $\vec{s}$.

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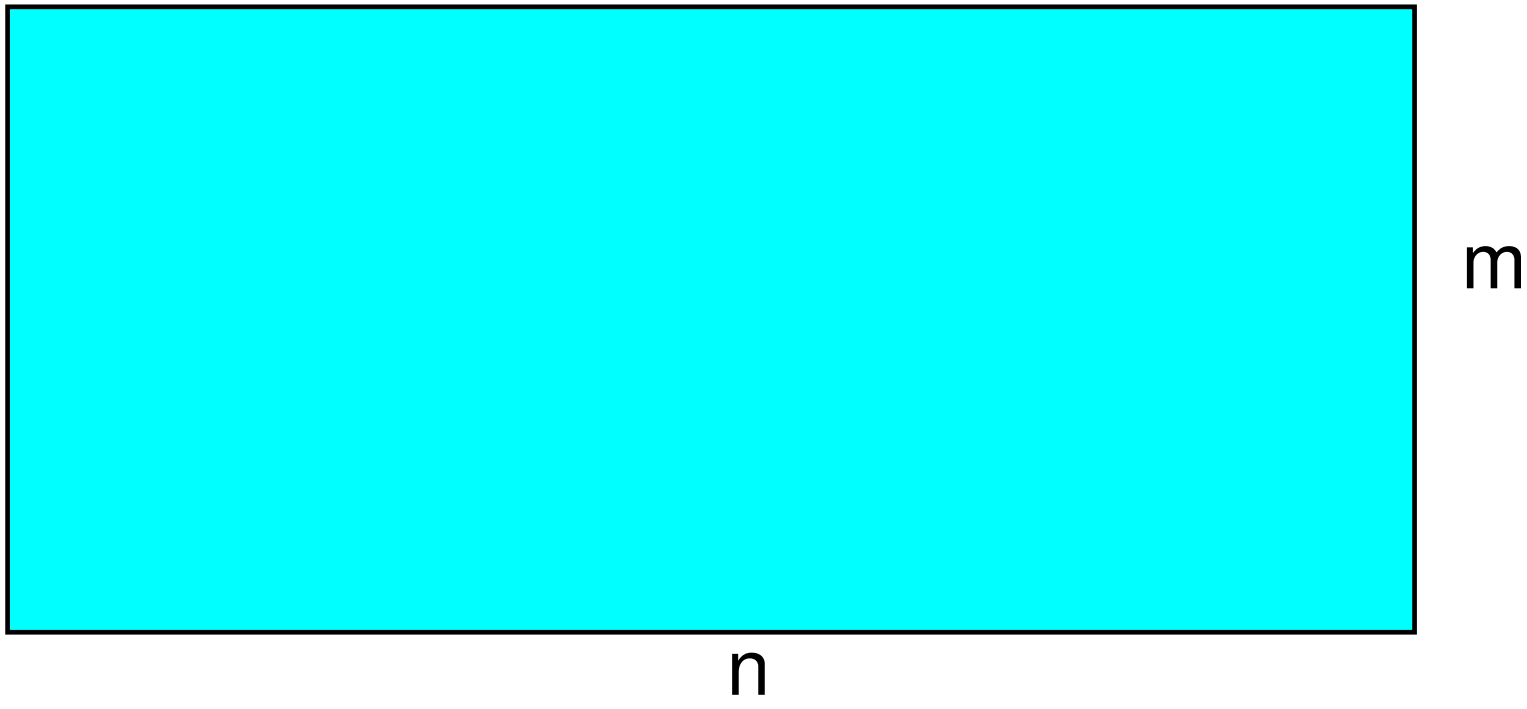
  - implies only $3m/2$ kernel basis columns remaining

Process



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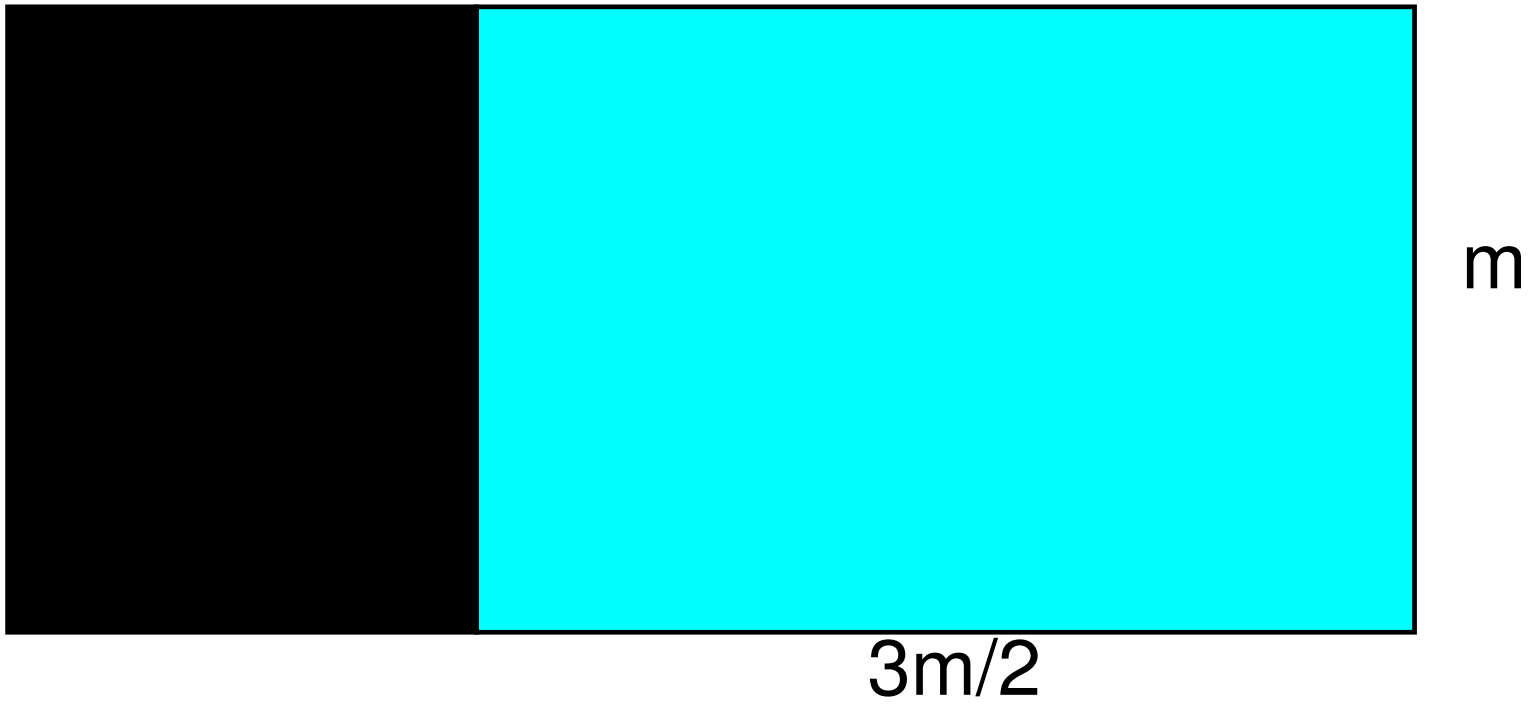

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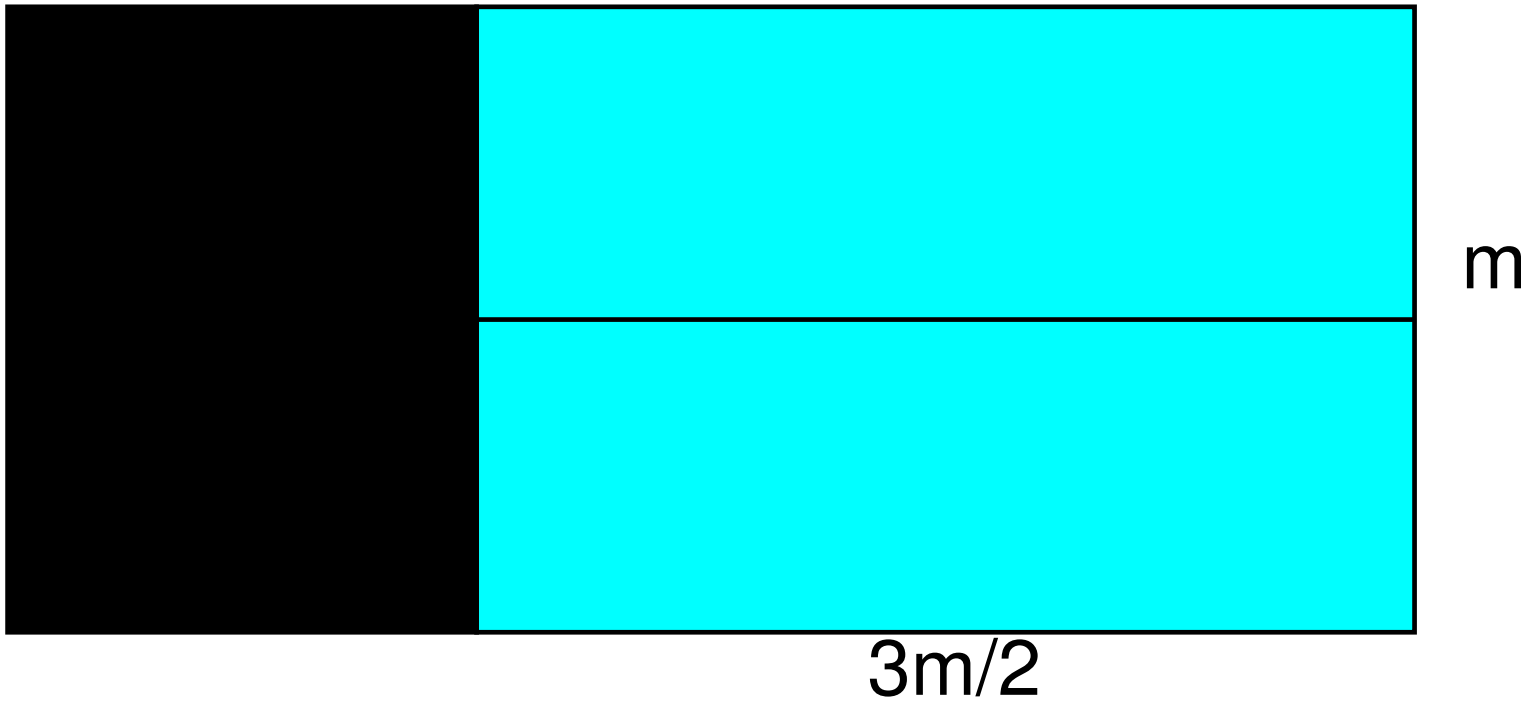

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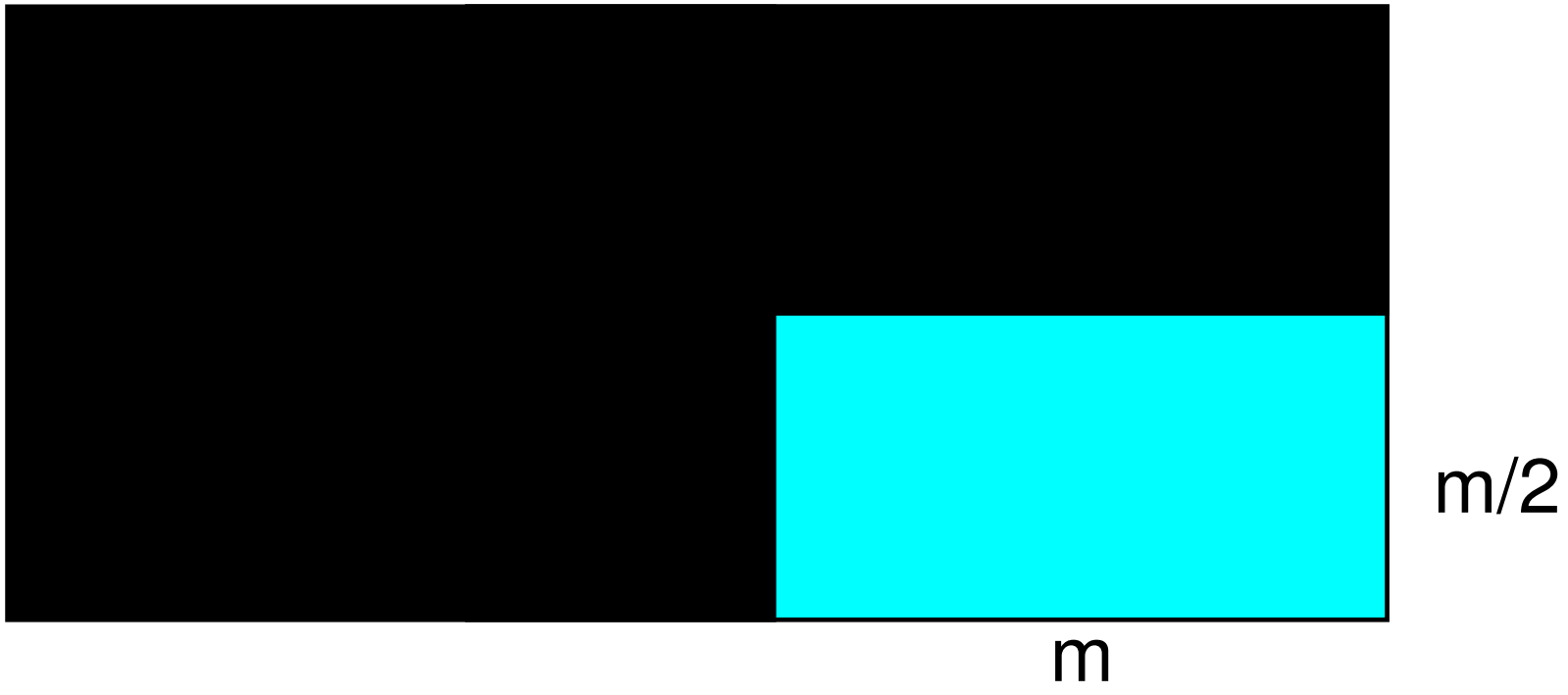

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5. Return $[\mathbf{P}_1, \mathbf{P}_2\mathbf{N}_1\mathbf{N}_2]$.

Cost of Tools

$\mathbf{A} \in \mathbb{K}[z]^{n \times n}$, $\vec{s} \in \mathbb{Z}^n$ bounds column degrees, $\sum \vec{s} \leq \xi$

Theorem (Zhou-L-Storjohann, ISSAC 2012)

$\vec{s}$-Minimal kernel basis computation costs $O^{\sim}(n^{\omega} s)$.

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Column basis computation costs $O^{\sim}(n^{\omega} s)$.

Triangularization

- Finding Diagonals
- Complexity
- Aside: Computing the determinant

Finding Diagonal Elements

Partition $\mathbf{A}$ and $\mathbf{U}$ and reduce via

$$\mathbf{A} \cdot \mathbf{U} = \begin{bmatrix} \mathbf{A}_u \\ \mathbf{A}_d \end{bmatrix} \begin{bmatrix} \mathbf{U}_\ell & \mathbf{U}_r \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 & 0 \\ * & \mathbf{B}_2 \end{bmatrix}.$$

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Here

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Important to control size (measured by column degrees).

Example

$$\mathbf{A} = \begin{bmatrix} z & -z^3 & -2z^4 & 2z & -z^2 \\ 1 & -1 & -2z & 2 & -z \\ -3 & 3z^2+z & 2z^2 & -z^4+1 & 3z \\ 0 & 1 & z^2+2z-2 & z^3+2z-2 & 0 \\ 1 & -z^2+2 & -2z^3-3z+3 & 2z+2 & 0 \end{bmatrix} \in \mathbb{Z}_7[z]^{5\times 5}.$$

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$$\begin{bmatrix} \mathbf{A}_u \\ \mathbf{A}_d \end{bmatrix} \cdot [\mathbf{U}_\ell^{(2)}, \mathbf{U}_r^{(2)}] = \begin{bmatrix} z & 0 & & & \\ 1 & z^2-1 & & & \\ * & * & z^3 & & \\ * & * & * & z^3-1 & 0 \\ * & * & * & -z & z \end{bmatrix}$$

Example

$$\mathbf{A} = \begin{bmatrix} z & -z^3 & -2z^4 & 2z & -z^2 \\ 1 & -1 & -2z & 2 & -z \\ -3 & 3z^2+z & 2z^2 & -z^4+1 & 3z \\ 0 & 1 & z^2+2z-2 & z^3+2z-2 & 0 \\ 1 & -z^2+2 & -2z^3-3z+3 & 2z+2 & 0 \end{bmatrix} \in \mathbb{Z}_7[z]^{5\times 5}.$$

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Costs?

- Compute (shifted) Kernel Basis : $\mathbf{U}_r$
- Compute Column Basis : $\mathbf{B}_1$
- Multiply two polynomial matrices : $\mathbf{B}_2 = \mathbf{A}_d \cdot \mathbf{U}_r$

Important Properties (ZLS ISSAC 2012)

$\mathbf{A} \in \mathbb{K}[z]^{m \times n}$, $\vec{s} \in \mathbb{Z}^n$ bounds column degrees, $\sum \vec{s} \leq \xi$

Theorem

For $\mathbf{M}$ *a* $\vec{s}$*-minimal kernel basis of* $\mathbf{A}$*:* $\sum \operatorname{cdeg}_{\vec{s}} \mathbf{M} \leq \sum \vec{s}$

Theorem

(i) $\mathbf{P} \in \mathbb{K}[z]^{m \times n}$, $m \leq n$, $\vec{s} \in \mathbb{Z}^n$ *bounding column degrees of* $\mathbf{P}$

(ii) $\mathbf{Q} \in \mathbb{K}[z]^{n \times k}$ *with* $k \in O(m)$, $\sum \operatorname{cdeg}_{\vec{s}} \mathbf{Q} \leq \sum \vec{s} \in O(\xi)$

Multiply $\mathbf{P}$ *and* $\mathbf{Q}$ *:* $O^{\sim}(n^2 m^{\omega-2} s) \subset O^{\sim}(n^{\omega} s)$, $s = \xi/n$.

Complexity

Theorem

$\mathbf{A} \in \mathbb{K}[z]^{n \times n}$. *Diagonals costs* $O^{\sim}(n^{\omega} \lceil s \rceil)$ *where* $s = \frac{\sum \operatorname{cdeg} \mathbf{A}}{n}$.

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If cost : $g(n)$ then recurrence relation: (with $s = \frac{\xi}{n}$)

$$g(n) \quad \in \quad O^{\sim}(n^{\omega}\lceil s \rceil) + g(\lceil n/2 \rceil) + g(\lfloor n/2 \rfloor)$$

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□

Finding Rest of $\mathbf{H}$

Know : $\vec{\delta}$ diagonal degrees of $\mathbf{H}$. Set $\mu = \max(\vec{\delta})$

$$\mathbf{x}^{\vec{\mu}-\vec{\delta}}\,\mathbf{A} \xrightarrow{\text{reduce}} \mathbf{x}^{\vec{\mu}-\vec{\delta}}\,\mathbf{R} \xrightarrow{\text{normalize}} \mathbf{H} = \mathbf{R} \cdot \mathrm{lc}_{-\vec{\delta}}(\mathbf{R})^{-1}$$

where $\mathbf{R}$ is any $-\vec{\delta}$-column reduced form of $\mathbf{A}$.

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Problem : Shift $\vec{\mu} - \vec{\delta}$ might be too large

Answer : Partial linearization of Storjohann (2007): $\mathbf{A} \to \mathcal{L}(\mathbf{A})$

Smooths shifts, keeps properties of $\mathbf{A}$ while enlarging a bit.

Partial Linearization

Consider $\mathbf{H}$ with diagonal degrees $(2, 37, 7, 18)$.

$$\mathbf{H} = \begin{bmatrix} (2) & & & \\ [36] & (37) & & \\ [6] & [6] & (7) & \\ [17] & [17] & [17] & (18) \end{bmatrix},$$

$[d]$: degree at most d and (d) : monic , degree exactly d.

$\delta = 1 + \lfloor (2 + 37 + 7 + 18)/4 \rfloor = 17.$ Construct by "expanding rows":

$$\widetilde{\mathbf{H}} = \begin{bmatrix} (2) & & & \\ [16] & [16] & & \\ [16] & [16] & & \\ [2] & (3) & & \\ [6] & [6] & (7) & \\ [16] & [16] & [16] & [16] \\ [0] & [0] & [0] & (1) \end{bmatrix}.$$

$\mathbf{H}$ and $\widetilde{\mathbf{H}}$ are related by $\mathbf{H} = \mathcal{E}_{\vec{\delta}} \cdot \widetilde{\mathbf{H}}$ where

$$\mathcal{E}_{\vec{\delta}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & x^{17} & x^{34} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & x^{17} \end{bmatrix}.$$

Insert elementary columns in $\widetilde{\mathbf{H}}$ by

$$\mathcal{L}_{\vec{\delta}}(\mathbf{H}) = \begin{bmatrix} (2) & & & & & & \\ [16] & x^{17} & & [16] & & & \\ [16] & -1 & x^{17} & [16] & & & \\ [2] & & -1 & (3) & & & \\ [6] & & & [6] & (7) & & \\ [16] & & & [16] & [16] & x^{17} & [16] \\ [0] & & & [0] & [0] & -1 & (1) \end{bmatrix}$$

Row degrees $\vec{d} = (2, 17, 17, 3, 7, 17, 1)$ - maximum 17.

Main property kept : shifted column reduction.

$$\begin{array}{ccccc}
\mathbf{x}^{\vec{\mu}-\vec{\delta}}\,\mathbf{A} & \xrightarrow{\text{reduce}} & \mathbf{x}^{\vec{\mu}-\vec{\delta}}\,\mathbf{R} & \xrightarrow{\text{normalize}} & \mathbf{H} = \mathbf{R}\,\mathrm{lc}_{-\vec{\delta}}(\mathbf{R})^{-1} \\
\Big| & & & & \Big| \\
\text{partial linearization} & & & & \text{partial linearization} \\
\downarrow & & & & \downarrow \\
\mathbf{x}^{\vec{m}-\vec{d}}\,\mathcal{L}_{\vec{\delta}}(\mathbf{A}) & \xrightarrow{\text{reduce}} & \mathbf{x}^{\vec{m}-\vec{d}}\,\hat{\mathbf{R}} & \xrightarrow{\text{normalize}} & \mathcal{L}_{\vec{\delta}}(\mathbf{H}) = \hat{\mathbf{R}}\,\mathrm{lc}_{-\vec{d}}(\hat{\mathbf{R}})^{-1}
\end{array} .$$

Theorem

Let $\mathbf{A} \in \mathbb{K}[x]^{n\times n}$ nonsingular with $\vec{\delta}$ the degrees of the diagonal entries of the Hermite form.

Then the Hermite form is computed using $O^{\sim}(n^{\omega}d)$ field operations.

Improving the Complexity

Repeat : partial linearization (this time with columns) :

(i) Enlarge : $\mathbf{A} \to \mathcal{L}^c(\mathbf{A})$

- size of $\mathcal{L}^c(\mathbf{A})$ at most twice size of $\mathbf{A}$
- degree $\mathcal{L}^c(\mathbf{A})$ at most average of $\mathbf{A}$

(ii) Compute Hermite form of $\mathcal{L}^c(\mathbf{A})$

(iii) $\mathbf{H}$ is found in lower right corner of Hermite form of $\mathcal{L}^c(\mathbf{A})$

Theorem

$\mathbf{A} \in \mathbb{K}[x]^{n \times n}$ *nonsingular. Hermite form computed:* $O^{\sim}(n^{\omega}\lceil s \rceil)$.

Determinants

Diagonals not enough - need to worry about unimodular part.

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For $\det \mathbf{U} = \det [\, \mathbf{U}_\ell \; \mathbf{U}_r]$ we do:

1. $\det \mathbf{U} = \det \mathbf{U} \bmod z = \det U = \det [U_\ell, U_r]$
2. $\mathbf{V} = \mathbf{U}^{-1} = \begin{bmatrix} \mathbf{V}_u \\ \mathbf{V}_d \end{bmatrix}$
3. $\mathbf{U}_r$ and $\mathbf{V}_u$ determined in column bases computation
4. Find $U_\ell{}^*$ such that $U^* = [U_\ell{}^*, U_r]$ is unimodular
5. Let $V_u = \mathbf{V}_u \bmod z$. Then $\det \mathbf{U} = \frac{\det U^*}{\det V_u U_\ell{}^*}$

Some References

Other relevant papers:

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