More on Hermite Normal Form

(focus on integer matrices)

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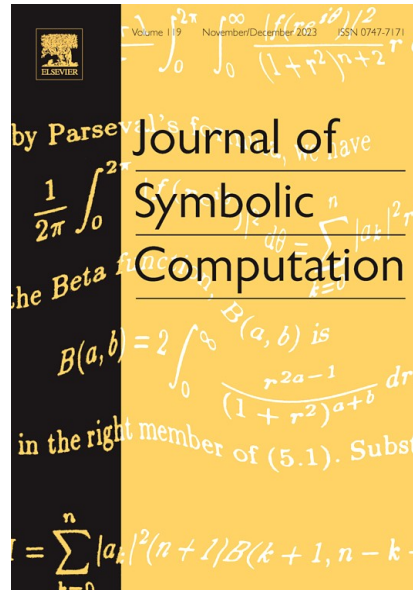


Outline

Hermite Normal Form

Recall: Hermite Normal Form

$A \in \mathbb{Z}^{m \times n}$, an integer matrix, full column rank. Hermite form of A:

$$UA = H = \begin{bmatrix} h_1 & h_{12} & \cdots & h_{1n} \\ & h_2 & \cdots & h_{2n} \\ & & \ddots & \vdots \\ & & & h_n \\ & & & \\ & & & \end{bmatrix}$$

- has all entries nonnegative
- in each column: $h_{ij} < h_j$
- A left equivalent to H, there exists U with $UA = H$
- $U \in \mathbb{Z}^{m \times m}$ unimodular, and represents the integer row operations.

Example 1:

$$\underset{U}{\begin{bmatrix} 235 & 454 & 256 & -84 & -269 & -577 \\ 194 & 374 & 209 & -70 & -221 & -473 \\ 704 & 1360 & 768 & -251 & -806 & -1730 \\ 461 & 890 & 501 & -165 & -527 & -1130 \\ 1241 & 2397 & 1352 & -443 & -1420 & -3047 \\ 1268 & 2450 & 1384 & -452 & -1452 & -3117 \end{bmatrix}} \underset{A}{\begin{bmatrix} -8 & -1 & 5 & 1 & 6 & 0 \\ 2 & -3 & -8 & -3 & 2 & -1 \\ -5 & -4 & -5 & 9 & -4 & 4 \\ 2 & -6 & -1 & -8 & 9 & -7 \\ -9 & 5 & -5 & -6 & 2 & -7 \\ 0 & -6 & -4 & 6 & 0 & -8 \end{bmatrix}} = \underset{H}{\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 7657 \\ & 1 & 0 & 1 & 4 & 6283 \\ & & 1 & 0 & 1 & 22951 \\ & & & 2 & 3 & 14998 \\ & & & & 5 & 40428 \\ & & & & & 41350 \end{bmatrix}}$$

where we can easily check that $UA = H$ and that det $U = -1$.

Example 2:

$$\underset{U}{\begin{bmatrix} -25 & -160 & 109 & 128 \\ -46 & -295 & 201 & 236 \\ -25 & -156 & 107 & 125 \\ -65 & -419 & 285 & 335 \end{bmatrix}} \underset{A}{\begin{bmatrix} -13 & 27 & 0 & -21 \\ 10 & 30 & 15 & 0 \\ -20 & 15 & 15 & -15 \\ 27 & 30 & 6 & 9 \end{bmatrix}} = \underset{H}{\begin{bmatrix} 1 & 0 & 3 & 42 \\ & 3 & 6 & 75 \\ & & 15 & 45 \\ & & & 105 \end{bmatrix}}$$

Integer matrix triangularization

- Over a field : work with bases of vector space of rows
- Over integers: work with bases of integer module of rows

  Triangular basis for the lattice generated by $\mathbb{Z}$-linear combinations of rows of an integer matrix A

Integer matrix triangularization

- Over a field : work with bases of vector space of rows
- Over integers: work with bases of integer module of rows

  Triangular basis for the lattice generated by $\mathbb{Z}$-linear combinations of rows of an integer matrix A
- Lattice is invariant under unimodular row operations
  1. Add an integer multiple of one row to a different row
  2. Multiply a row by -1
  3. Swap two rows
- Obtainable using "Gaussian elimination" with integer row operations

Integer matrix triangularization

Example:

- Triangularize A using integer row operations
- Record row operations in U, initially set to I_n

$$\underset{U}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} -4 & -4 & -1 \\ -6 & 3 & 3 \\ 2 & -7 & 6 \end{bmatrix}} = \underset{UA}{\begin{bmatrix} -4 & -4 & -1 \\ -6 & 3 & 3 \\ 2 & -7 & 6 \end{bmatrix}}$$

- Next row operation: Swap rows 1 and 3

Integer matrix triangularization

Example:

- Triangularize A using integer row operations
- Record row operations in U, initially set to I_n

$$\underset{U}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} -4 & -4 & -1 \\ -6 & 3 & 3 \\ 2 & -7 & 6 \end{bmatrix}} = \underset{UA}{\begin{bmatrix} 2 & -7 & 6 \\ -6 & 3 & 3 \\ -4 & -4 & -1 \end{bmatrix}}$$

- Next row operation: Add 3 times row 1 to row 2

Integer matrix triangularization

Example:

- Triangularize A using integer row operations
- Record row operations in U, initially set to I_n

$$\underset{U}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 0 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} -4 & -4 & -1 \\ -6 & 3 & 3 \\ 2 & -7 & 6 \end{bmatrix}} = \underset{UA}{\begin{bmatrix} 2 & -7 & 6 \\ & -18 & 21 \\ -4 & -4 & -1 \end{bmatrix}}$$

- Next row operation: Add 2 times row 1 to row 3

Integer matrix triangularization

Example:

- Triangularize A using integer row operations
- Record row operations in U, initially set to I_n

$$\underset{U}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 0 & 2 \end{bmatrix}} \underset{A}{\begin{bmatrix} -4 & -4 & -1 \\ -6 & 3 & 3 \\ 2 & -7 & 6 \end{bmatrix}} = \underset{UA}{\begin{bmatrix} 2 & -7 & 6 \\ & -18 & 21 \\ & -18 & 11 \end{bmatrix}}$$

- Next row operation: Subtract row 2 from row 3

Integer matrix triangularization

Example:

- Triangularize A using integer row operations
- Record row operations in U, initially set to I_n

$$\underset{U}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & -1 & -1 \end{bmatrix}} \underset{A}{\begin{bmatrix} -4 & -4 & -1 \\ -6 & 3 & 3 \\ 2 & -7 & 6 \end{bmatrix}} = \underset{UA}{\begin{bmatrix} 2 & -7 & 6 \\ & -18 & 21 \\ & & -10 \end{bmatrix}}$$

- Next row operation: Multiply row 2 by -1

Integer matrix triangularization

Example:

- Triangularize A using integer row operations
- Record row operations in U, initially set to I_n

$$\underset{U}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -3 \\ 1 & -1 & -1 \end{bmatrix}} \underset{A}{\begin{bmatrix} -4 & -4 & -1 \\ -6 & 3 & 3 \\ 2 & -7 & 6 \end{bmatrix}} = \underset{UA}{\begin{bmatrix} 2 & -7 & 6 \\ & 18 & -21 \\ & & -10 \end{bmatrix}}$$

- Next row operation: Add row 2 to row 1

Integer matrix triangularization

Example:

- Triangularize A using integer row operations
- Record row operations in U, initially set to I_n

$$\underset{U}{\begin{bmatrix} 0 & -1 & -2 \\ 0 & -1 & -3 \\ 1 & -1 & -1 \end{bmatrix}} \underset{A}{\begin{bmatrix} -4 & -4 & -1 \\ -6 & 3 & 3 \\ 2 & -7 & 6 \end{bmatrix}} = \underset{UA}{\begin{bmatrix} 2 & 11 & -15 \\ & 18 & -21 \\ & & -10 \end{bmatrix}}$$

- Next row operations: Put last column in correct form

Integer matrix triangularization

Example:

- Triangularize A using integer row operations
- Record row operations in U, initially set to I_n

$$\underset{U}{\begin{bmatrix} -2 & 1 & 0 \\ -3 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} -4 & -4 & -1 \\ -6 & 3 & 3 \\ 2 & -7 & 6 \end{bmatrix}} = \underset{UA}{\begin{bmatrix} 2 & 11 & 5 \\ & 18 & 9 \\ & & 10 \end{bmatrix}}$$

- U is unimodular: $\det U = -1$
- $H := UA$ is the *Hermite form* for A

Computing a Hermite Form: A Bit of History

Elimination based methods

- Extended gcd: given $a, b \in \mathbb{Z}$ compute $s, t \in \mathbb{Z}$ such that

$$sa + tb = g = \gcd(a, b) \iff \begin{bmatrix} s & t \\ -b/g & a/g \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} g \\ \end{bmatrix}$$

- "Gaussian elimination" via unimodular transformations

$$\begin{bmatrix} s & & & t & \\ & 1 & & & \\ & & 1 & & \\ -b/g & & & a/g & \\ & & & & 1 \end{bmatrix} \begin{bmatrix} a & * \\ & * \\ & * \\ b & * \\ * & * \end{bmatrix} = \begin{bmatrix} g & * \\ & * \\ & * \\ & * \\ * & * \end{bmatrix}$$

Example

$$A$$
$$\begin{bmatrix} -13 & 27 & 0 & -21 \\ 10 & 30 & 15 & 0 \\ -20 & 15 & 15 & -15 \\ 27 & 30 & 6 & 9 \end{bmatrix}$$

Example

$$A = \begin{bmatrix} -13 & 27 & 0 & -21 \\ 10 & 30 & 15 & 0 \\ -20 & 15 & 15 & -15 \\ 27 & 30 & 6 & 9 \end{bmatrix}$$

- Extended Euclidean Algorithm gives:

$$\begin{aligned} 3(-13) + 4(10) &= 1 \\ 10(-13) + 13(10) &= 0 \end{aligned}$$

Example

$$\underset{U_1}{\begin{bmatrix} 3 & 4 & 0 & 0 \\ 10 & 13 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} -13 & 27 & 0 & -21 \\ 10 & 30 & 15 & 0 \\ -20 & 15 & 15 & -15 \\ 27 & 30 & 6 & 9 \end{bmatrix}} = \underset{A_1}{\begin{bmatrix} 1 & 201 & 60 & -63 \\ & 660 & 195 & -210 \\ -20 & 15 & 15 & -15 \\ 27 & 30 & 6 & 9 \end{bmatrix}}$$

- Extended Euclidean Algorithm gives:

$$\begin{aligned} 3(-13) + 4(10) &= 1 \\ 10(-13) + 13(10) &= 0 \end{aligned}$$

Example

$$\underset{U_4}{\begin{bmatrix} 3 & 4 & 0 & 0 \\ 10 & 13 & 0 & 0 \\ 60 & 80 & 1 & 0 \\ -81 & -108 & 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} -13 & 27 & 0 & -21 \\ 10 & 30 & 15 & 0 \\ -20 & 15 & 15 & -15 \\ 27 & 30 & 6 & 9 \end{bmatrix}} = \underset{A_4}{\begin{bmatrix} 1 & 201 & 60 & -63 \\ & 660 & 195 & -210 \\ & 4035 & 1215 & -1275 \\ & -5397 & -1614 & 1710 \end{bmatrix}}$$

- Extended Euclidean Algorithm used for elimination:

$$\begin{aligned} 3(-13) + 4(10) &= 1 \\ 10(-13) + 13(10) &= 0 \end{aligned}$$

Example

$$\underset{U}{\begin{bmatrix} -25 & -160 & 109 & 128 \\ -46 & -295 & 201 & 236 \\ -25 & -156 & 107 & 125 \\ -65 & -419 & 285 & 335 \end{bmatrix}} \quad \underset{A}{\begin{bmatrix} -13 & 27 & 0 & -21 \\ 10 & 30 & 15 & 0 \\ -20 & 15 & 15 & -15 \\ 27 & 30 & 6 & 9 \end{bmatrix}} = \underset{H}{\begin{bmatrix} 1 & 0 & 3 & 42 \\ & 3 & 6 & 75 \\ & & 15 & 45 \\ & & & 105 \end{bmatrix}}$$

- Extended Euclidean Algorithm used for elimination:

$$\begin{aligned} 3(-13) + 4(10) &= 1 \\ 10(-13) + 13(10) &= 0 \end{aligned}$$

- Issue: Exponential growth of intermediate integers

A Bit of History

Hermite form of full column rank $A \in \mathbb{Z}^{n \times m}$.

- Counting bit operations, assuming standard matrix multiplication.

- Kannan & Bachem (79)
  - polynomial time
- Chou and Collins (82)
  - $O\tilde{}(n^6 \log ||A||)$
- Domich *et al. (87), Iliopoulos (89), Hafner & McCurley (91)*
  - $O\tilde{}(nm^3 \log ||A||)$
- *Storjohann & Labahn (96), Storjohann (00)*
  - $O\tilde{}(nm^3 \log ||A||)$ *with transformation* U
- *Sims (84), Havas, Majewski & Matthews (98)*
  - *application of lattice reduction (LLL)*

Elimination based methods with potential expression swell

1. Triangularize column by column

$$\left[\begin{array}{ccccc} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{array}\right] \xrightarrow{1,2,3} \left[\begin{array}{ccc|cc} h_1 & * & * & * & * \\ & h_2 & * & * & * \\ & & h_3 & * & * \\ \hline & & & * & * \\ & & & * & * \end{array}\right] \xrightarrow{4,5} \left[\begin{array}{ccc|cc} h_1 & * & * & * & * \\ & h_2 & * & * & * \\ & & h_3 & * & * \\ \hline & & & h_4 & * \\ & & & & h_5 \end{array}\right]$$

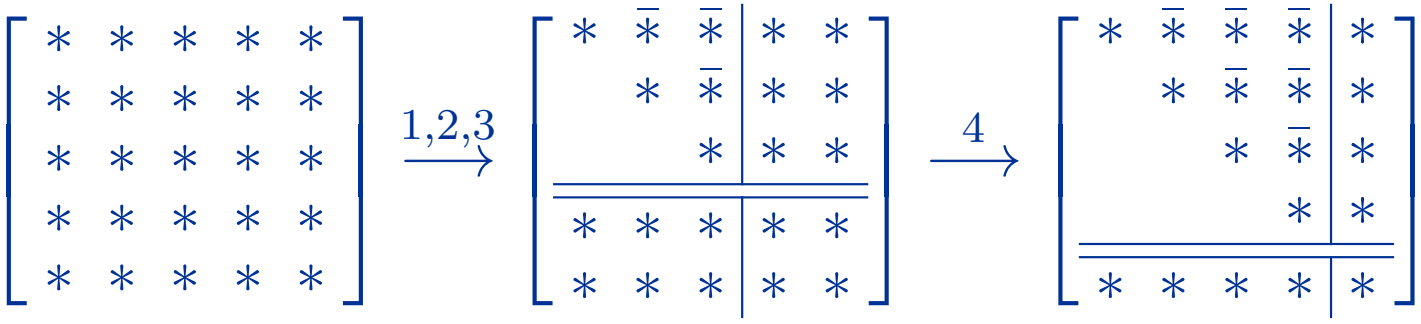
2. Reduce off diagonals column by column

$$\left[\begin{array}{ccccc} h_1 & * & * & * & * \\ & h_2 & * & * & * \\ & & h_3 & * & * \\ & & & h_4 & * \\ & & & & h_5 \end{array}\right] \xrightarrow{1,2,3} \left[\begin{array}{ccc|cc} h_1 & \bar{*} & \bar{*} & * & * \\ & h_2 & \bar{*} & * & * \\ & & h_3 & * & * \\ \hline & & & h_4 & * \\ & & & & h_5 \end{array}\right] \xrightarrow{4,5} \left[\begin{array}{ccc|cc} h_1 & \bar{*} & \bar{*} & \bar{*} & \bar{*} \\ & h_2 & \bar{*} & \bar{*} & \bar{*} \\ & & h_3 & \bar{*} & \bar{*} \\ \hline & & & h_4 & \bar{*} \\ & & & & h_5 \end{array}\right]$$

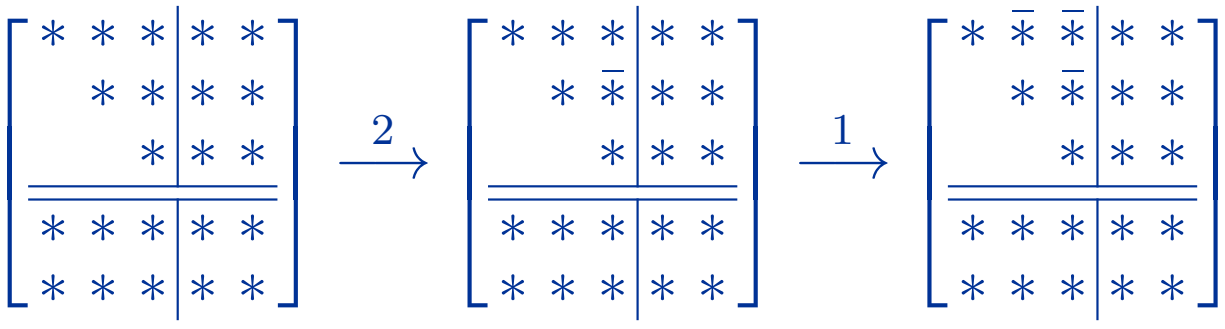
- Fang & Havas (ISSAC 97): Exponential growth in bit length

Elimination based methods with controlled growth

1. Triangularize row by row



2. During triangularization reduce off diagonals row by row



- Chou & Collins (86): $O\tilde{}(n^6 \log ||A||)$ bit operations

Modulo determinant methods

Let $A \in \mathbb{Z}^{n \times n}$ be nonsingular.

1. Compute $d = \det A$.
2. Triangularize working modulo d
3. "Fix up" diagonal entries in using extended gcd computations
4. Reduce off diagonal entries working modulo d

Cost is $O(n^3)$ operations modulo d: $O\tilde{}(n^4 \log ||A||)$ bit operations

why would this work?

Suppose $d = \det A$

- If $UA = H$ with HNF $A = H$ then

  Easy to see that $\text{HNF} \begin{bmatrix} A & 0 \\ dI_n & I_n \end{bmatrix} = \begin{bmatrix} H & 0 \\ 0 & I_n \end{bmatrix}$.

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  since $\begin{bmatrix} U & 0 \\ -\mathrm{adj}(A) & I_n \end{bmatrix} \begin{bmatrix} A & 0 \\ dI_n & I_n \end{bmatrix} = \begin{bmatrix} H & 0 \\ 0 & I_n \end{bmatrix}$

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  since $\begin{bmatrix} U & 0 \\ -\mathrm{adj}(A) & I_n \end{bmatrix} \begin{bmatrix} A & 0 \\ dI_n & I_n \end{bmatrix} = \begin{bmatrix} H & 0 \\ 0 & I_n \end{bmatrix}$

  and so $\begin{bmatrix} U & 0 \\ -\mathrm{adj}(A) & I_n \end{bmatrix} \begin{bmatrix} A \\ dI_n \end{bmatrix} = \begin{bmatrix} H \\ 0 \end{bmatrix}$

- Implies we can compute HNF of $\begin{bmatrix} A \\ dI_n \end{bmatrix}$ working modulo d

Example: Domich et al algorithm

$$
\begin{bmatrix}
a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\
\vdots & & & & \vdots \\
a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \\
d & & & & \\
 & d & & & \\
 & & \ddots & & \\
 & & & \ddots & \\
 & & & & d
\end{bmatrix}
$$

Example: Domich et al algorithm

$$
\begin{bmatrix}
h_{11} & b_{12} & \cdots & \cdots & b_{1n} \\
 & b_{22} & \cdots & \cdots & b_{2n} \\
 & \vdots & & & \vdots \\
 & b_{n2} & \cdots & \cdots & b_{nn} \\
d & * & * & * & * \\
 & d & & & \\
 & & \ddots & & \\
 & & & \ddots & \\
 & & & & d
\end{bmatrix}
$$

Example: Domich et al algorithm

$$
\begin{bmatrix}
h_{11} & b_{12} & \cdots & \cdots & b_{1n} \\
 & b_{22} & \cdots & \cdots & b_{2n} \\
 & \vdots & & & \vdots \\
 & b_{n2} & \cdots & \cdots & b_{nn} \\
 & * & * & * & * \\
 & d & & & \\
 & & \ddots & & \\
 & & & \ddots & \\
 & & & & d
\end{bmatrix}
$$

Example: Domich et al algorithm

$$
\begin{bmatrix}
h_{11} & h_{12} & c_{13} & \cdots & c_{1n} \\
 & h_{22} & c_{23} & \cdots & c_{2n} \\
 & & \vdots & & \vdots \\
 & & c_{n3} & \cdots & c_{nn} \\
 & & * & * & * \\
 & d & * & * & * \\
 & & d & & \\
 & & & \ddots & \\
 & & & & d
\end{bmatrix}
$$

Example: Domich et al algorithm

$$\begin{bmatrix} h_{11} & h_{12} & c_{13} & \cdots & c_{1n} \\ & h_{22} & c_{23} & \cdots & c_{2n} \\ & & \vdots & & \vdots \\ & & c_{n3} & \cdots & c_{nn} \\ & & * & * & * \\ & & * & * & * \\ & & d & & \\ & & & \ddots & \\ & & & & d \end{bmatrix}$$

Example: Domich et al algorithm

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} & \cdots & h_{1n} \\ & h_{22} & h_{23} & \cdots & h_{2n} \\ & & & \ddots & \vdots \\ & & & & h_{nn} \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix}$$

Quick aside: Smith normal form

Given $A \in \mathbb{Z}^{n \times n}$ a nonsingular integer matrix.

The Smith normal form

- $S = \operatorname{diag}(s_1, s_2, \ldots, s_n) \in \mathbb{Z}^{n \times n}$.
- $s_1 \,|s_2\, | \ldots |\, s_n$. (invariant factors)

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$$\underset{A}{\begin{bmatrix} -8 & -1 & 5 & 1 & 6 & 0 \\ 2 & -3 & -8 & -3 & 2 & -1 \\ -5 & -4 & -5 & 9 & -4 & 4 \\ 2 & -6 & -1 & -8 & 9 & -7 \\ -9 & 5 & -5 & -6 & 2 & -7 \\ 0 & -6 & -4 & 6 & 0 & -8 \end{bmatrix}} \rightarrow \underset{S}{\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 2 & \\ & & & & & 206750 \end{bmatrix}}$$

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The Smith normal form

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$$\underset{A}{\begin{bmatrix} -8 & -1 & 5 & 1 & 6 & 0 \\ 2 & -3 & -8 & -3 & 2 & -1 \\ -5 & -4 & -5 & 9 & -4 & 4 \\ 2 & -6 & -1 & -8 & 9 & -7 \\ -9 & 5 & -5 & -6 & 2 & -7 \\ 0 & -6 & -4 & 6 & 0 & -8 \end{bmatrix}} \rightarrow \underset{S}{\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 2 & \\ & & & & & 206750 \end{bmatrix}}$$

- S obtained using unimodular row and column operations.
- typically $UAV = S$ or $AV = WS$

Some Oddities

Oddity 1: Faster algorithms for polynomial HNF:

Labahn-Neiger-Zhou (2017)	$O^{\sim}(n^{\omega}s)$	Deterministic

- s minimum of the average of column/row degrees

Oddity 2: Faster algorithms for SNF (with multipliers)

Birmpilis, Labahn, Storjohann (2023)	$O^{\sim}(n^{\omega}\log\|\|A\|\|)$	LV

- Complexity is given without the extra $\log n$ and $\log\log||A||$ factors.

Hermite form with arbitrary shape and rank

The Hermite form: arbitrary shape and rank input

- Example: 6×9 input matrix with rank 3

$$A \xrightarrow{HNF} \begin{bmatrix} h_1 & * & * & \bar{*} & \bar{*} & * & * & * & * \\ & & & h_2 & \bar{*} & * & * & * & * \\ & & & & h_3 & * & * & * & * \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \end{bmatrix}$$

- Column rank profile of A is $[1, 4, 5]$
- Define Hermite basis $\bar{H} = \begin{bmatrix} h_1 & \bar{*} & \bar{*} \\ & h_2 & \bar{*} \\ & & h_3 \end{bmatrix}$

The Hermite form: Full row rank

- Example: 3×9 input matrix with rank 3

$$A \xrightarrow{HNF} \begin{bmatrix} h_1 & * & * & \bar{*} & \bar{*} & * & * & * & * \\ & & & h_2 & \bar{*} & * & * & * & * \\ & & & & h_3 & * & * & * & * \end{bmatrix}$$

- Let $\bar{A} = \left[\begin{array}{c|c} A_1 & A_2 \end{array}\right] = AC$, perm. C based on rank profile

$$\bar{A} \xrightarrow{HNF} \left[\begin{array}{ccc||cccccc} h_1 & \bar{*} & \bar{*} & * & * & * & * & * & * \\ & h_2 & \bar{*} & * & * & * & * & * & * \\ & & h_3 & * & * & * & * & * & * \end{array}\right]$$

The Hermite form: Full row rank

- Example: 3×9 input matrix with rank 3

$$A \longrightarrow \begin{bmatrix} h_1 & * & * & \bar{*} & \bar{*} & * & * & * & * \\ & & & h_2 & \bar{*} & * & * & * & * \\ & & & & h_3 & * & * & * & * \end{bmatrix}$$

- Let $\bar{A} = \left[\begin{array}{c|c} A_1 & A_2 \end{array}\right] = AC$, perm. C based on rank profile

$$\bar{A} \quad \stackrel{HNF}{\longrightarrow} \quad \left[\begin{array}{c|c} \bar{H} & \bar{H}A_1^{-1}A_2 \end{array}\right]$$

The Hermite form: Arbitrary rank

- Input $A \in \mathbb{Z}^{n \times m}$ of arbitrary rank r

Let

$$\left[\begin{array}{c||c} A_{11} & A_{12} \\ \hline\hline A_{21} & A_{22} \end{array}\right] = RAC$$

where

- C a permutation based on column rank profile
- R *any* row permutation such that $A_{11} \in \mathbb{Z}^{r \times r}$ is nonsingular

Then

$$\left[\begin{array}{c||c} A_{11} & A_{12} \\ \hline\hline A_{21} & A_{22} \end{array}\right] \quad \xrightarrow{HNF} \quad \left[\begin{array}{c||c} \bar{H} & \bar{H}A_{11}^{-1}A_{12} \\ \hline\hline & \end{array}\right]$$

Computing the transforming matrix

What about the transforming matrix? $UA = H$

Case $A \in \mathbb{Z}^{n \times n}$ nonsingular

- U is unique and can be found using linear solving: $U = HA^{-1}$
- also note that $\left[\begin{array}{c|c} A & I_n \end{array}\right] \overset{HNF}{\longrightarrow} \left[\begin{array}{c|c} H & U \end{array}\right]$

Case $A \in \mathbb{Z}^{n \times m}$ of full column rank

- use rank computation to reduce to nonsingular case

$$A \xrightarrow{(1)} \left[\begin{array}{c} A_1 \\ \hline A_2 \end{array}\right] \xrightarrow{(2)} \left[\begin{array}{c|c} A_1 & \\ \hline A_2 & I_{n-m} \end{array}\right] \overset{HNF}{\longrightarrow} \left[\begin{array}{c|c} \bar{H} & U_1 \\ \hline & U_2 \end{array}\right]$$

Then

$$U = \left[\begin{array}{c|c} \bar{H} - U_1 A_2 & U_1 \\ \hline -U_2 A_2 & U_2 \end{array}\right] \left[\begin{array}{c|c} A_1^{-1} & \\ \hline & I_{n-m} \end{array}\right]$$

Computing a transforming matrix

- recursive kernel basis computation of Hafner & McCurley (91)
- example of $8m \times m$ input matrix $(n = 8m)$
- kernel produced has at most $(n/m)\log_2(n/m)$ nonzero blocks

$$\begin{bmatrix} I & & & & & & & \\ & I & & & & & & \\ & & I & & & & & \\ & & & I & & & & \\ & & & & I & & & \\ & & & & & I & & \\ & & & & & & I & \\ & & & & & & & I \end{bmatrix} \begin{bmatrix} * \\ * \\ * \\ * \\ * \\ * \\ * \\ * \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix}$$

Computing a transforming matrix

- recursive kernel basis computation of Hafner & McCurley (91)
- example of $8m \times m$ input matrix $(n = 8m)$
- kernel produced has at most $(n/m)\log_2(n/m)$ nonzero blocks

$$\begin{bmatrix} * & * & & & & & & \\ * & * & & & & & & \\ & & * & * & & & & \\ & & * & * & & & & \\ & & & & * & * & & \\ & & & & * & * & & \\ & & & & & & * & * \\ & & & & & & * & * \end{bmatrix} \begin{bmatrix} * \\ * \\ * \\ * \\ * \\ * \\ * \\ * \end{bmatrix} = \begin{bmatrix} b_1 \\ \\ b_2 \\ \\ b_3 \\ \\ b_4 \\ \end{bmatrix}$$

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$$\begin{bmatrix} * & * & * & * & & & & \\ * & * & & & & & & \\ * & * & * & * & & & & \\ & & * & * & & & & \\ & & & & * & * & * & * \\ & & & & * & * & & \\ & & & & * & * & * & * \\ & & & & & & * & * \end{bmatrix} \begin{bmatrix} * \\ * \\ * \\ * \\ * \\ * \\ * \\ * \end{bmatrix} = \begin{bmatrix} c_1 \\ \\ \\ \\ c_2 \\ \\ \\ \end{bmatrix}$$

Computing a transforming matrix

- recursive kernel basis computation of Hafner & McCurley (91)
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$$\left[\begin{array}{cccccccc} * & * & * & * & * & * & * & * \\ \hline * & * & & & & & & \\ * & * & * & * & & & & \\ & & * & * & & & & \\ * & * & * & * & * & * & * & * \\ & & & & * & * & & \\ & & & & * & * & * & * \\ & & & & & & * & * \end{array}\right] \left[\begin{array}{c} * \\ * \\ * \\ * \\ * \\ * \\ * \\ * \end{array}\right] = \left[\begin{array}{c} d_1 \\ \hline \\ \\ \\ \\ \\ \\ \\ \end{array}\right]$$

The transforming matrix: a closer view

- Let $A \in \mathbb{Z}^{n \times m}$ have full column rank
- Let $U \in \mathbb{Z}^{n \times n}$ be unimodular such that $UA = H$

$$\left[\begin{array}{c} U \\ \hline E \\ \hline \\ K \\ \\ \end{array}\right] \left[\begin{array}{c} A \\ \hline * \\ \hline \\ * \\ \\ \end{array}\right] = \left[\begin{array}{c} \bar{H} \\ \hline \\ \\ \\ \end{array}\right]$$

with $E \in \mathbb{Z}^{m \times n}$ and $K \in \mathbb{Z}^{(n-m) \times n}$ satisfying

1. $EA = \bar{H}$
2. $KA = 0_{(n-m) \times m}$

Other choices

- Let $A \in \mathbb{Z}^{n \times m}$ have full column rank
- Let $U \in \mathbb{Z}^{n \times n}$ be unimodular such that $UA = H$

$$\begin{array}{c} U \\ \left[\begin{array}{c} E \\ \hline \\ K_1 \\ \\ \end{array}\right] \end{array} \begin{array}{c} A \\ \left[\begin{array}{c} * \\ \hline \\ * \end{array}\right] \end{array} = \begin{array}{c} \\ \left[\begin{array}{c} \bar{H} \\ \hline \\ \end{array}\right] \end{array}$$

with $E_1 \in \mathbb{Z}^{m \times n}$ and $K_1 \in \mathbb{Z}^{(n-m) \times n}$ satisfying

1. $K_1 = VK$ with V unimodular

Other choices

- Let $A \in \mathbb{Z}^{n \times m}$ have full column rank
- Let $U \in \mathbb{Z}^{n \times n}$ be unimodular such that $UA = H$

$$\overset{U}{\left[\begin{array}{c} E_1 \\ \hline \\ K_1 \\ \\ \end{array}\right]} \overset{A}{\left[\begin{array}{c} * \\ \hline \\ * \\ \\ \end{array}\right]} = \left[\begin{array}{c} \bar{H} \\ \hline \\ \\ \\ \end{array}\right]$$

with $E_1 \in \mathbb{Z}^{m \times n}$ and $K_1 \in \mathbb{Z}^{(n-m) \times n}$ satisfying

1. $K_1 = VK$ with V unimodular
2. $E_1 = E - WK$, with $W \in \mathbb{Z}^{(n-m) \times (n-m)}$

Summary: From Arne's Thesis

Input $A \in \mathbb{Z}^{n \times m}$ of rank r.

Summary: From Arne's Thesis

Input $A \in \mathbb{Z}^{n \times m}$ of rank r. Let $\beta = (\sqrt{r}||A||)^r$

$$\begin{bmatrix} U \\ E \\ \hline \\ K \\ \\ \end{bmatrix} \begin{bmatrix} A \\ * \\ \hline \\ * \\ \\ \end{bmatrix} = \begin{bmatrix} H \\ * \\ \hline \\ \\ \\ \end{bmatrix}$$

We can compute U, H in $O\tilde{}(nmr^2 \log ||A||)$ bit operations with

- $||H|| \leq \beta$
- $||E|| \leq \beta$
- $||K|| \leq r\beta^2$

Example: Using lattice reduction

- Sims (84): Use LLL after the fact to reduce size of U
- Example for random 8×4 input

```
>RandomMatrix(8,4);
```

$$A = \begin{bmatrix} 50 & -38 & -93 & 8 \\ 10 & -18 & -76 & 69 \\ -16 & 87 & -72 & 99 \\ -9 & 33 & -2 & 29 \\ -50 & -98 & -32 & 44 \\ -22 & -77 & -74 & 92 \\ 45 & 57 & -4 & -31 \\ -81 & 27 & 27 & 67 \end{bmatrix}$$

Example: Using lattice reduction

- Sims (84): Use LLL after the fact to reduce size of U
- Example for random 8×4 input

```
>HermiteForm(A,output=['U']);
```

$$U = \left[\begin{array}{c} E \\ \hline K \end{array}\right] = \left[\begin{array}{rrrrr} 9086206 & -8629726 & -5060749 & 29822997 & 361156 \\ 8363956 & -7943761 & -4658477 & 27452408 & 332448 \\ 2467635 & -2343664 & -1374400 & 8099340 & 980829 \\ 3307464 & -3141301 & -1842160 & 10855850 & 131464 \\ \hline 9507196 & -9029566 & -5295228 & 31204782 & 377889 \\ 9199746 & -8737563 & -5123987 & 30195660 & 365669 \\ 6367234 & -6047352 & -3546362 & 20898713 & 253083 \\ 3264133 & -3100146 & -1818026 & 10713625 & 129741 \end{array}\right]$$

Example: Using lattice reduction

- Sims (84): Use LLL after the fact to reduce size of U
- Example for random 8×4 input

```
>HermiteForm(A,output=['U']);
```

$$U = \left[\frac{E}{\mathrm{LLL}(K)}\right] = \left[\begin{array}{rrrrr} 9086206 & -8629726 & -5060749 & 29822997 & 361156 \\ 8363956 & -7943761 & -4658477 & 27452408 & 332448 \\ 2467635 & -2343664 & -1374400 & 8099340 & 980829 \\ 3307464 & -3141301 & -1842160 & 10855850 & 131464 \\ \hline 11 & -80 & -2 & 50 & 18 \\ 2 & 26 & -26 & 69 & 43 \\ -41 & 36 & 16 & 22 & 41 \\ 12 & 1 & 15 & 47 & -20 \end{array}\right]$$

Example: Using lattice reduction

- Sims (84): Use LLL after the fact to reduce size of U
- Example for random 8×4 input

```
>HermiteForm(A,output=['U'],method='integer[reduced]')
```

$$U = \left[\frac{E + M\,\mathsf{LLL}(K)}{\mathsf{LLL}(K)}\right] = \left[\begin{array}{cccccccc} 0 & 4 & -3 & 59 & 21 & -18 & -8 & -18 \\ -18 & 24 & 12 & 1 & 8 & -25 & -31 & -26 \\ 3 & 5 & -3 & -16 & -7 & 1 & 2 & 10 \\ -21 & -57 & 24 & -18 & 9 & 48 & 33 & -23 \\ \hline 11 & -80 & -2 & 50 & 18 & 58 & 65 & 1 \\ 2 & 26 & -26 & 69 & 43 & -24 & 47 & 8 \\ -41 & 36 & 16 & 22 & 41 & -36 & -9 & -47 \\ 12 & 1 & 15 & 47 & -20 & -31 & -101 & -36 \end{array}\right]$$