Introduction to some Matrix Normal Forms

George Labahn and Arne Storjohann

Cheriton School of Computer Science
University of Waterloo

Recent Trends in Computer Algebra, Institut Henri Poincaré,

September 2023

Outline

Motivation

Popov Normal Form
  - Differential Popov Normal Form
  - Polynomial Popov Form

Linking Hermite and Popov

Motivation

Recall: Gcds and Bezout equation

Given $b, c \in \mathbb{Z}$

Finding $\mathbf{gcd}(b, c) = d$ typically involves:

(1) find v_1, v_2 such that $b = v_1 d$ and $c = v_2 d$

(2) find u_1, u_2 such that $u_1 b + u_2 c = d$

(3) d is 'normalized'

Recall: Gcds and Bezout equation

Given $b, c \in \mathbb{Z}$

Finding **gcd**$(b, c) = d$ typically involves:

(1) find v_1, v_2 such that $b = v_1 d$ and $c = v_2 d$

(2) find u_1, u_2 such that $u_1 b + u_2 c = d$

(3) d is 'normalized'

Note: (1) and (2) implies $u_1 v_1 + u_2 v_2 = 1$

Recall: Gcds and Bezout equation

Given $b, c \in \mathbb{Z}$

Finding **gcd**$(b, c) = d$ typically involves:

(1) find v_1, v_2 such that $b = v_1 d$ and $c = v_2 d$

(2) find u_1, u_2 such that $u_1 b + u_2 c = d$

(3) d is 'normalized'

Note: (1) and (2) implies $u_1 v_1 + u_2 v_2 = 1$

What about when b, c are both **matrices**?

Matrix gcds?

Given $b, c \in \mathbb{Z}^{n\times n}$. Find **rgcd**$(b, c) = d$

Matrix gcds?

Given $b, c \in \mathbb{Z}^{n\times n}$. Find **rgcd**$(b, c) = d$

$$\begin{bmatrix} b \\ c \end{bmatrix}$$

Matrix gcds?

Given $b, c \in \mathbb{Z}^{n\times n}$. Find **rgcd**$(b, c) = d$

Use row operations: $\begin{bmatrix} u_1 & u_2 \\ * & * \end{bmatrix} \begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} d \\ 0 \end{bmatrix}$

Matrix gcds?

Given $b, c \in \mathbb{Z}^{n\times n}$. Find **rgcd**$(b, c) = d$

Use row operations: $\begin{bmatrix} u_1 & u_2 \\ * & * \end{bmatrix} \begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} d \\ 0 \end{bmatrix}$

Row ops reversible: $\begin{bmatrix} v_1 & * \\ v_2 & * \end{bmatrix} \begin{bmatrix} u_1 & u_2 \\ * & * \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_m \end{bmatrix}$

Matrix gcds?

Given $b, c \in \mathbb{Z}^{n\times n}$: Find **rgcd**$(b, c) = d$.

$$\begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} v_1 & * \\ v_2 & * \end{bmatrix} \begin{bmatrix} d \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_1 & u_2 \\ * & * \end{bmatrix} \begin{bmatrix} v_1 & * \\ v_2 & * \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_m \end{bmatrix}$$

Matrix gcds?

Given $b, c \in \mathbb{Z}^{n\times n}$: Find **rgcd**$(b, c) = d$.

$$\begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} v_1 & * \\ v_2 & * \end{bmatrix} \begin{bmatrix} d \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_1 & u_2 \\ * & * \end{bmatrix} \begin{bmatrix} v_1 & * \\ v_2 & * \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_m \end{bmatrix}$$

Implies:

$$\begin{aligned} b &= v_1 d \\ c &= v_2 d \end{aligned}$$

Matrix gcds?

Given $b, c \in \mathbb{Z}^{n\times n}$: Find **rgcd**$(b, c) = d$.

$$\begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} v_1 & * \\ v_2 & * \end{bmatrix} \begin{bmatrix} d \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_1 & u_2 \\ * & * \end{bmatrix} \begin{bmatrix} v_1 & * \\ v_2 & * \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_m \end{bmatrix}$$

Implies:

$$\begin{aligned} b &= v_1 d \\ c &= v_2 d \\ u_1 v_1 + u_2 v_2 &= I_m \end{aligned}$$

Matrix gcds?

Given $b, c \in \mathbb{Z}^{n \times n}$: Find **rgcd**$(b, c) = d$.

$$\begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} v_1 & * \\ v_2 & * \end{bmatrix} \begin{bmatrix} d \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_1 & u_2 \\ * & * \end{bmatrix} \begin{bmatrix} v_1 & * \\ v_2 & * \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_m \end{bmatrix}$$

Implies:

$$\begin{aligned} b &= v_1 d \\ c &= v_2 d \\ u_1 v_1 + u_2 v_2 &= I_m \\ d & \quad \text{'normalized'} \end{aligned}$$

Matrix gcds?

Given $b, c \in \mathbb{Z}^{n\times n}$: Find **rgcd**$(b, c) = d$.

$$\begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} v_1 & * \\ v_2 & * \end{bmatrix} \begin{bmatrix} d \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_1 & u_2 \\ * & * \end{bmatrix} \begin{bmatrix} v_1 & * \\ v_2 & * \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_m \end{bmatrix}$$

Implies:

$$\begin{aligned} b &= v_1 d \\ c &= v_2 d \\ u_1 v_1 + u_2 v_2 &= I_m \\ d & \quad \text{'normalized'} \end{aligned}$$

Similarly for $b, c \in \mathbb{K}[z]^{m\times m}$.

Hermite Normal Form

Integer Hermite Form

$A \in \mathbb{Z}^{n \times n}$, a non-singular matrix. Hermite form of A:

$$H = \begin{bmatrix} h_1 & h_{12} & \cdots & h_{1n} \\ & h_2 & \cdots & h_{2n} \\ & & \ddots & \vdots \\ & & & h_n \end{bmatrix}$$

- has all entries nonnegative
- in each column: $h_{ij} < h_j$
- A left equivalent to H

Integer Hermite Form

$A \in \mathbb{Z}^{n\times n}$, a non-singular matrix. Hermite form of A:

$$UA = H = \begin{bmatrix} h_1 & h_{12} & \cdots & h_{1n} \\ & h_2 & \cdots & h_{2n} \\ & & \ddots & \vdots \\ & & & h_n \end{bmatrix}$$

- has all entries nonnegative
- in each column: $h_{ij} < h_j$
- A left equivalent to H

  $U \in \mathbb{Z}^{n\times n}$ unimodular, i.e. det $U = \pm 1$

  - U represents the row operations.

Polynomial Hermite Form

$A \in \mathbb{K}[x]^{n\times n}$, a non-singular polynomial matrix. Hermite form of A:

$$UA = H = \begin{bmatrix} h_1 & h_{12} & \cdots & h_{1n} \\ & h_2 & \cdots & h_{2n} \\ & & \ddots & \vdots \\ & & & h_n \end{bmatrix}$$

- in each column: $\deg h_{ij} < \deg h_j$
- A left equivalent to H
- $U \in \mathbb{K}[x]^{n\times n}$ unimodular, i.e. $\det U = \text{constant}$
    - U represents the row operations.

Example 1:

$$\begin{bmatrix} -8 & -1 & 5 & 1 & 6 & 0 \\ 2 & -3 & -8 & -3 & 2 & -1 \\ -5 & -4 & -5 & 9 & -4 & 4 \\ 2 & -6 & -1 & -8 & 9 & -7 \\ -9 & 5 & -5 & -6 & 2 & -7 \\ 0 & -6 & -4 & 6 & 0 & -8 \end{bmatrix}$$

Example 1:

$$\begin{bmatrix} -8 & -1 & 5 & 1 & 6 & 0 \\ 2 & -3 & -8 & -3 & 2 & -1 \\ -5 & -4 & -5 & 9 & -4 & 4 \\ 2 & -6 & -1 & -8 & 9 & -7 \\ -9 & 5 & -5 & -6 & 2 & -7 \\ 0 & -6 & -4 & 6 & 0 & -8 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 7657 \\ & 1 & 0 & 1 & 4 & 6283 \\ & & 1 & 0 & 1 & 22951 \\ & & & 2 & 3 & 14998 \\ & & & & 5 & 40428 \\ & & & & & 41350 \end{bmatrix}$$

Example 1:

$$\overset{U}{\begin{bmatrix} 235 & 454 & 256 & -84 & -269 & -577 \\ 194 & 374 & 209 & -70 & -221 & -473 \\ 704 & 1360 & 768 & -251 & -806 & -1730 \\ 461 & 890 & 501 & -165 & -527 & -1130 \\ 1241 & 2397 & 1352 & -443 & -1420 & -3047 \\ 1268 & 2450 & 1384 & -452 & -1452 & -3117 \end{bmatrix}} \overset{A}{\begin{bmatrix} -8 & -1 & 5 & 1 & 6 & 0 \\ 2 & -3 & -8 & -3 & 2 & -1 \\ -5 & -4 & -5 & 9 & -4 & 4 \\ 2 & -6 & -1 & -8 & 9 & -7 \\ -9 & 5 & -5 & -6 & 2 & -7 \\ 0 & -6 & -4 & 6 & 0 & -8 \end{bmatrix}} = \overset{H}{\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 7657 \\ & 1 & 0 & 1 & 4 & 6283 \\ & & 1 & 0 & 1 & 22951 \\ & & & 2 & 3 & 14998 \\ & & & & 5 & 40428 \\ & & & & & 41350 \end{bmatrix}}$$

Can check that $UA = H$ and that det $U = -1$.

Example 1:

$$\overset{U}{\begin{bmatrix} 235 & 454 & 256 & -84 & -269 & -577 \\ 194 & 374 & 209 & -70 & -221 & -473 \\ 704 & 1360 & 768 & -251 & -806 & -1730 \\ 461 & 890 & 501 & -165 & -527 & -1130 \\ 1241 & 2397 & 1352 & -443 & -1420 & -3047 \\ 1268 & 2450 & 1384 & -452 & -1452 & -3117 \end{bmatrix}} \overset{A}{\begin{bmatrix} -8 & -1 & 5 & 1 & 6 & 0 \\ 2 & -3 & -8 & -3 & 2 & -1 \\ -5 & -4 & -5 & 9 & -4 & 4 \\ 2 & -6 & -1 & -8 & 9 & -7 \\ -9 & 5 & -5 & -6 & 2 & -7 \\ 0 & -6 & -4 & 6 & 0 & -8 \end{bmatrix}} = \overset{H}{\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 7657 \\ & 1 & 0 & 1 & 4 & 6283 \\ & & 1 & 0 & 1 & 22951 \\ & & & 2 & 3 & 14998 \\ & & & & 5 & 40428 \\ & & & & & 41350 \end{bmatrix}}$$

Can check that $UA = H$ and that det $U = -1$.

Example 2: (simpler) :

$$\overset{U}{\begin{bmatrix} -25 & -160 & 109 & 128 \\ -46 & -295 & 201 & 236 \\ -25 & -156 & 107 & 125 \\ -65 & -419 & 285 & 335 \end{bmatrix}} \overset{A}{\begin{bmatrix} -13 & 27 & 0 & -21 \\ 10 & 30 & 15 & 0 \\ -20 & 15 & 15 & -15 \\ 27 & 30 & 6 & 9 \end{bmatrix}} = \overset{H}{\begin{bmatrix} 1 & 0 & 3 & 42 \\ & 3 & 6 & 75 \\ & & 15 & 45 \\ & & & 105 \end{bmatrix}}$$

Interesting Points:

(1) HNF dates back to Hermite in 1851

(2) Can also define HNF for singular matrices

(3) A nonsingular $A \in \mathbb{Z}^{n\times n}$ implies HNF is unique:

(4) Also have column Hermite forms

(5) Lots of other variations

- lower triangular rather than upper triangular
- first rather than last zero rows
- etc

Applications

- Solving linear systems of equations
- Rational scaling invariants
    - will illustrate in a later lecture
- Rational invariants of Abelian group actions
- . . .

Popov Normal Form

Popov Form

- Hermite Normal Form does not have controlled degrees
  - e.g. degrees of HNF can be larger than input degree
- Popov's form (1969) :

  Purpose was to allow for simple conversion of state space to transfer functions in linear systems theory
- Popov form related to Gröbner bases
- Extends to noncommutative domains (e.g. Ore domains)

Outline

Motivation

Popov Normal Form
  - Differential Popov Normal Form
  - Polynomial Popov Form

Linking Hermite and Popov

Differential Popov Form

$$\mathbf{A}(\mathbf{D}) = \begin{bmatrix} a_1 & a_{1,2} & a_{1,3} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_2 & a_{3,4} & \cdots & a_{2,n-1} & a_{2,n} \\ a_{3,1} & a_{3,2} & a_3 & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & & & & & \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \cdots & \cdots & a_{n,n-1} & a_n \end{bmatrix}$$

Differential Popov Form

$$\mathbf{A}(\mathbf{D}) = \begin{bmatrix} a_1 & a_{1,2} & a_{1,3} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_2 & a_{3,4} & \cdots & a_{2,n-1} & a_{2,n} \\ a_{3,1} & a_{3,2} & a_3 & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & & & & & \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \cdots & \cdots & a_{n,n-1} & a_n \end{bmatrix}$$

- Diagonal entries monic

Differential Popov Form

$$\mathbf{A}(\mathbf{D}) = \begin{bmatrix} a_1 & a_{1,2} & a_{1,3} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_2 & a_{3,4} & \cdots & a_{2,n-1} & a_{2,n} \\ a_{3,1} & a_{3,2} & a_3 & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & & & & & \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \cdots & \cdots & a_{n,n-1} & a_n \end{bmatrix}$$

- Diagonal entries monic and of row degree.
  - that is deg $a_{i,j} \leq$ deg a_i

Differential Popov Form

$$\mathbf{A}(\mathbf{D}) = \begin{bmatrix} a_1 & a_{1,2} & a_{1,3} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_2 & a_{3,4} & \cdots & a_{2,n-1} & a_{2,n} \\ a_{3,1} & a_{3,2} & a_3 & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & & & & & \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \cdots & \cdots & a_{n,n-1} & a_n \end{bmatrix}$$

- Diagonal entries monic and of row degree.
  - that is deg $a_{i,j} \leq$ deg a_i
- deg $a_{j,i} <$ deg a_i for $j \neq i$
  - that is for each column degrees above or below diagonal less

Example

Given a system of differential equations

$$\begin{array}{rcrcrcl} y_1''(t) - (t+2)y_1(t) & - & t^2 y_2''(t) - y_2(t) & - & y_3'(t) - y_3(t) & = & n_1(t) \\ y_1'(t) + 3y_1(t) & + & y_2'''(t) - 2y_2'(t) - y_2(t) & - & y_3'''(t) - 2t^2 y_3(t) & = & n_2(t) \\ y_1'(t) + y_1(t) & - & t y_2''(t) - 2t y_2'(t) - y_2(t) & + & y_3''''(t) & = & n_3(t). \end{array}$$

Algorithms for different systems typically want first order systems.

Example

The system of differential equations

$$\begin{array}{rcrcrcl} y_1''(t) - (t+2)y_1(t) & - & t^2 y_2''(t) - y_2(t) & - & y_3'(t) - y_3(t) & = & n_1(t) \\ y_1'(t) + 3y_1(t) & + & y_2'''(t) - 2y_2'(t) - y_2(t) & - & y_3'''(t) - 2t^2 y_3(t) & = & n_2(t) \\ y_1'(t) + y_1(t) & - & t y_2''(t) - 2t y_2'(t) - y_2(t) & + & y_3''''(t) & = & n_3(t). \end{array}$$

Notice we can represent this as an Ore matrix equation.

$$\left[\begin{array}{ccc} D^2 - (t+2) & -t^2 D^2 - 1 & D - 1 \\ D + 3 & D^3 - 2D - 1 & D^3 - 2t^2 \\ D + 1 & -tD^2 - 2tD - 1 & D^4 \end{array}\right] \cdot \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} n_1(t) \\ n_2(t) \\ n_3(t) \end{bmatrix}.$$

Example

The system of differential equations

$$\begin{array}{rcrcrcl} y_1''(t) - (t+2)y_1(t) & - & t^2 y_2''(t) - y_2(t) & - & y_3'(t) - y_3(t) & = & n_1(t) \\ y_1'(t) + 3y_1(t) & + & y_2'''(t) - 2y_2'(t) - y_2(t) & - & y_3'''(t) - 2t^2 y_3(t) & = & n_2(t) \\ y_1'(t) + y_1(t) & - & t y_2''(t) - 2t y_2'(t) - y_2(t) & + & y_3''''(t) & = & n_3(t). \end{array}$$

Notice we can represent this as an Ore matrix equation.

$$\begin{bmatrix} D^2 - (t+2) & -t^2 D^2 - 1 & D - 1 \\ D + 3 & D^3 - 2D - 1 & D^3 - 2t^2 \\ D + 1 & -tD^2 - 2tD - 1 & D^4 \end{bmatrix} \cdot \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} n_1(t) \\ n_2(t) \\ n_3(t) \end{bmatrix}.$$

Why interesting?

Let D be the differentiation operator on t. If the system of equations is

$$\begin{bmatrix} D^2-(t+2) & -t^2D^2-1 & D-1 \\ D+3 & D^3-2D-1 & D^3-2t^2 \\ D+1 & -tD^2-2tD-1 & D^4 \end{bmatrix} \cdot \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} n_1(t) \\ n_2(t) \\ n_3(t) \end{bmatrix},$$

then we can rewrite

$$\begin{aligned} y_1''(t) &= (t+2)y_1(t) + t^2y_2''(t) + y_2(t) + y_3'(t) + y_3(t) + n_1(t) \\ y_2'''(t) &= -y_1'(t) - 3y_1(t) + 2y_2'(t) + y_2(t) + y_3'''(t) + 2t^2y_3(t) + n_2(t) \\ y_3''''(t) &= -y_1'(t) - y_1(t) + ty_2''(t) + 2ty_2'(t) + y_2(t) + n_3(t) \end{aligned}$$

Canonical form : $Y'(t) = A(t)Y(t) + B(t)$

$$\left[\begin{array}{c} y_1(t) \\ y_1'(t) \\ \hline y_2(t) \\ y_2'(t) \\ y_2''(t) \\ \hline y_3(t) \\ y_3'(t) \\ y_3''(t) \\ y_3'''(t) \end{array}\right]' = \left[\begin{array}{ccccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (t+2) & 0 & -1 & 0 & t^2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -3 & -1 & 1 & 2 & 0 & 2t^2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & 1 & 2t & t & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} y_1(t) \\ y_1'(t) \\ \hline y_2(t) \\ y_2'(t) \\ y_2''(t) \\ \hline y_3(t) \\ y_3'(t) \\ y_3''(t) \\ y_3'''(t) \end{array}\right] + \left[\begin{array}{c} 0 \\ n_1(t) \\ \hline 0 \\ 0 \\ n_2(t) \\ \hline 0 \\ 0 \\ 0 \\ n_3(t) \end{array}\right]$$

Canonical form : $Y'(t) = A(t)Y(t) + B(t)$

$$\left[\begin{array}{c} y_1(t) \\ y_1'(t) \\ \hline y_2(t) \\ y_2'(t) \\ y_2''(t) \\ \hline y_3(t) \\ y_3'(t) \\ y_3''(t) \\ y_3'''(t) \end{array}\right]' = \left[\begin{array}{ccccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (t+2) & 0 & -1 & 0 & t^2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -3 & -1 & 1 & 2 & 0 & 2t^2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & 1 & 2t & t & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} y_1(t) \\ y_1'(t) \\ \hline y_2(t) \\ y_2'(t) \\ y_2''(t) \\ \hline y_3(t) \\ y_3'(t) \\ y_3''(t) \\ y_3'''(t) \end{array}\right] + \left[\begin{array}{c} 0 \\ n_1(t) \\ \hline 0 \\ 0 \\ n_2(t) \\ \hline 0 \\ 0 \\ 0 \\ n_3(t) \end{array}\right]$$

Implies one can now use tools for first order systems

Canonical form : $Y'(t) = A(t)Y(t) + B(t)$

$$\left[\begin{array}{c} y_1(t) \\ y_1'(t) \\ \hline y_2(t) \\ y_2'(t) \\ y_2''(t) \\ \hline y_3(t) \\ y_3'(t) \\ y_3''(t) \\ y_3'''(t) \end{array}\right]' = \left[\begin{array}{ccccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (t+2) & 0 & -1 & 0 & t^2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -3 & -1 & 1 & 2 & 0 & 2t^2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & 1 & 2t & t & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} y_1(t) \\ y_1'(t) \\ \hline y_2(t) \\ y_2'(t) \\ y_2''(t) \\ \hline y_3(t) \\ y_3'(t) \\ y_3''(t) \\ y_3'''(t) \end{array}\right] + \left[\begin{array}{c} 0 \\ n_1(t) \\ \hline 0 \\ 0 \\ n_2(t) \\ \hline 0 \\ 0 \\ 0 \\ n_3(t) \end{array}\right]$$

Implies one can now use tools for first order systems

Even smarter. In some cases one can do:

(a) Use Popov to convert to first order system.

(b) Apply 1st order algorithm of some kind

(c) Interpret back in higher order.

Outline

Motivation

Popov Normal Form
    Differential Popov Normal Form
    Polynomial Popov Form

Linking Hermite and Popov

Motivation: Transfer Functions

If discrete inputs are : $v_0, v_1, \dots$ and outputs are $u_0, u_1, \dots$ then

$$\begin{aligned}
u_0 &= f_0 v_0 \\
u_1 &= f_0 v_1 + f_1 v_0 \\
u_2 &= f_0 v_2 + f_1 v_1 + f_2 v_0 \\
&\vdots
\end{aligned}$$

Model as $u(z) = f(z)v(z)$ or equivalently as.$u(z^{-1}) = f(z^{-1})v(z^{-1})$

Motivation: Transfer Functions

If discrete inputs are : $v_0, v_1, \ldots$ and outputs are $u_0, u_1, \ldots$ then:

Use $u(z^{-1}) = f(z^{-1})v(z^{-1})$ and form $f(z^{-1}) = \frac{p(z)}{q(z)}$, deg $c(z) = n$

Then $q(z)u(z^{-1}) = p(z)v(z^{-1})$ so

Motivation: Transfer Functions

If discrete inputs are : $v_0, v_1, \ldots$ and outputs are $u_0, u_1, \ldots$ then:

Use $u(z^{-1}) = f(z^{-1})v(z^{-1})$ and form $f(z^{-1}) = \frac{p(z)}{q(z)}$, deg $c(z) = n$

Then $q(z)u(z^{-1}) = p(z)v(z^{-1})$ so

$$q_n u_{m+n} + \cdots + q_0 u_m \quad = \quad p_n v_{m+n} + \cdots + p_0 v_m$$

Motivation: Transfer Functions

If discrete inputs are : $v_0, v_1, \ldots$ and outputs are $u_0, u_1, \ldots$ then:

Use $u(z^{-1}) = f(z^{-1})v(z^{-1})$ and form $f(z^{-1}) = \frac{p(z)}{q(z)}$, deg $c(z) = n$

Then $q(z)u(z^{-1}) = p(z)v(z^{-1})$ so

$$\begin{aligned} q_n u_{m+n} + \cdots + q_0 u_m &= p_n v_{m+n} + \cdots + p_0 v_m \\ u_{m+n} &= (p_n v_{m+n} + \cdots + p_0 v_m \\ &\qquad - q_{n-1} u_{m+n-1} - \cdots - q_0 u_m)/q_n \end{aligned}$$

Modeled as feedback loop: Output at each step depends on previous $n+1$ inputs and n outputs

Transfer Functions (cont.)

$$\vec{v} \longrightarrow \boxed{} \longrightarrow F\vec{v}$$

- $F(z^{-1}) \approx Q(z)^{-1}P(z)$ to get input-output model.
- Multi-dimensions: $P(z)$ and $Q(z)$ are matrix polynomials.
- Goal is to view process in state-space form

$$\begin{aligned} x'(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

  with minimum number of states $u(t)$.
- Implies $P(z)$ and $Q(z)$ should have no common left factor
- $Q(z)$ in Popov form

A second way to view Popov form

An polynomial matrix $\mathbf{A}(z)$ is in Popov Form if:

1. it has rank $\mathbf{A}(z)$ non-zero rows;
2. the leading row coefficient is triangular, with monic leading entries;
3. the leading entry of each row has the highest degree in its columns.

A second way to view Popov form

An polynomial matrix $\mathbf{A}(z)$ is in Popov Form if:

1. it has rank $\mathbf{A}(z)$ non-zero rows;
2. the leading row coefficient is triangular, with monic leading entries;
3. the leading entry of each row has the highest degree in its columns.

Also called a Polynomial Echelon Form (Kailath book [1980]).

A second way to view Popov form

An polynomial matrix $\mathbf{A}(z)$ is in Popov Form if:

1. it has rank $\mathbf{A}(z)$ non-zero rows;
2. the leading row coefficient is triangular, with monic leading entries;
3. the leading entry of each row has the highest degree in its columns.

Also called a Polynomial Echelon Form (Kailath book [1980]).

Any $\mathbf{A}(z)$ can be transformed into a unique Popov form by row operations.

Example

Let $A \in \mathbb{Q}[z]^{3\times 3}$ be given by:

$$A = \begin{bmatrix} z^2 + \frac{5}{2} & 4z^2 + 1 & 2z^2 + 1 \\ 3z + 3 & z^3 + 2z - 1 & 3z^3 - 8 \\ 4z + 1 & 5z^2 + \frac{2}{3}z + 1 & z^4 \end{bmatrix}$$

Example

Let $A \in \mathbb{Q}[z]^{3\times 3}$ be given by:

$$\begin{aligned} A &= \begin{bmatrix} z^2 + \frac{5}{2} & 4z^2 + 1 & 2z^2 + 1 \\ 3z + 3 & z^3 + 2z - 1 & 3z^3 - 8 \\ 4z + 1 & 5z^2 + \frac{2}{3}z + 1 & z^4 \end{bmatrix} \\ &= \operatorname{diag}(z^2, z^3, z^4) \begin{bmatrix} 1 & 4 & 2 \\ & 1 & 3 \\ & & 1 \end{bmatrix} + \begin{bmatrix} \frac{5}{2} & 1 & 1 \\ 3z + 3 & 2z - 1 & -8 \\ 4z + 1 & 5z^2 + \frac{2}{3}z + 1 & 0 \end{bmatrix} \end{aligned}$$

Example

Let $A \in \mathbb{Q}[z]^{3\times 3}$ be given by:

$$\begin{aligned}
A &= \begin{bmatrix} z^2 + \frac{5}{2} & 4z^2+1 & 2z^2+1 \\ 3z+3 & z^3+2z-1 & 3z^3-8 \\ 4z+1 & 5z^2+\frac{2}{3}z+1 & z^4 \end{bmatrix} \\
&= \operatorname{diag}(z^2, z^3, z^4) \begin{bmatrix} 1 & 4 & 2 \\ & 1 & 3 \\ & & 1 \end{bmatrix} + \begin{bmatrix} \frac{5}{2} & 1 & 1 \\ 3z+3 & 2z-1 & -8 \\ 4z+1 & 5z^2+\frac{2}{3}z+1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \operatorname{diag}(z^2, z^3, z^4) + \begin{bmatrix} \frac{5}{2} & 4z^2+1 & 2z^2+1 \\ 3z+3 & 2z-1 & 3z^3-8 \\ 4z+1 & 5z^2+\frac{2}{3}z+1 & 0 \end{bmatrix}
\end{aligned}$$

Popov form vs Hermite form : Gröbner Bases

Monomials on vectors $\mathbb{K}^{1\times n}[z]$:

$$z^\alpha e_j = [0, \ldots, 0, z^\alpha, 0, \ldots, 0]$$

Ordering on monomials of $\mathbb{K}^{1\times n}[z]$:

- Position over Term (POT):

$$z^\alpha e_i < z^\beta e_j \quad \Longleftrightarrow \quad i < j \quad \text{or } i = j \text{ and } \alpha < \beta$$

Popov form vs Hermite form : Gröbner Bases

Monomials on vectors $\mathbb{K}^{1\times n}[z]$:

$$z^\alpha e_j = [0, \dots, 0, z^\alpha, 0, \dots, 0]$$

Ordering on monomials of $\mathbb{K}^{1\times n}[z]$:

- Position over Term (POT):

$$z^\alpha e_i < z^\beta e_j \quad \Longleftrightarrow \quad i < j \quad \text{or } i = j \text{ and } \alpha < \beta$$

- Term over Position (TOP):

$$z^\alpha e_i < z^\beta e_j \quad \Longleftrightarrow \quad \alpha < \beta \quad \text{or } \alpha = \beta \text{ and } i < j.$$

Popov form vs Hermite form : Gröbner Bases

Monomials on vectors $\mathbb{K}^{1\times n}[z]$:

$$z^{\alpha}e_j = [0,\ldots,0,z^{\alpha},0,\ldots,0]$$

Ordering on monomials of $\mathbb{K}^{1\times n}[z]$:

- Position over Term (POT):

$$z^{\alpha}e_i < z^{\beta}e_j \quad \Longleftrightarrow \quad i < j \quad \text{or } i = j \text{ and } \alpha < \beta$$

- Term over Position (TOP):

$$z^{\alpha}e_i < z^{\beta}e_j \quad \Longleftrightarrow \quad \alpha < \beta \quad \text{or } \alpha = \beta \text{ and } i < j.$$

If M is a submodule of $\mathbb{K}^{1\times n}[z]$ then we can now speak of Gröbner bases for the module M.

Popov form vs Hermite form : Gröbner Bases

(Kojima, Rapisarda, Takaba [System & Control Letters 2007])

Let M be a submodule of $\mathbb{K}^{1\times m}[z]$ with a *term over position* ordering. Then

$\{f_i\}_{i=1,..,s}$ is a reduced Gröbner basis for the module $M \iff$:

(a) $M = \langle f_1, \ldots, f_s \rangle$;

(b) The matrix row($f_1, \ldots, f_s$) is in Popov form.

If TOP is replaced by *position over term* ordering then Popov form in (b) is replaced by Hermite form.

Some references

- T. Kailath, *Linear Systems*, Prentice Hall Englewood Cliffs, NJ, 1980.
- G. Villard, Computing Popov and Hermite forms of polynomial matrices, in *Proceedings of ISSAC'96*, Zurich, ACM Press, New York, 1996, pp. 250-258.
- B. Beckermann and G. Labahn, Fraction-free Computation of Matrix Rational Interpolants and Matrix GCD's, *SIAM Journal of Matrix Analysis and Applications*, 22(1) (2000) 114-144.
- C Kojima, P Rapisarda, K Takaba, Canonical forms for polynomial and quadratic differential operators. *Systems & Control Letters* 56 (11-12), 678-684, 2007
- V. Neiger, J. Rosenkilde, G. Solomatov, Computing Popov and Hermite forms of rectangular matrices *Proceedings of ISSAC 2018*