Linear algebra over a field

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Recent Trends in Computer Algebra, Institut Henri Poincaré,

September 2023

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- We will focus on worst case complexity and use big-O notation
- Analysis will reflect reality when K is a finite field

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- **Right Nullspace**: a full column rank matrix $N \in \mathsf{K}^{m \times (m-r)}$ such that $AN = 0_{n \times (m-r)}$

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- Can transform input matrix to row echelon form
  Example is for an input matrix $A \in \mathsf{K}^{6\times 9}$:

$$A = \begin{bmatrix} * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \end{bmatrix} \longrightarrow \begin{bmatrix} \bullet & * & * & * & * & * & * & * & * \\ & & & \bullet & * & * & * & * & * \\ & & & & \bullet & * & * & * & * \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \end{bmatrix} = R.$$

Convention: $*$ possibly nonzero; $\bullet$ necessarily nonzero; blank is zero

Example: Computing an echelon form

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  - Leading constant if $n = m = r$ is $1/3$

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- What about linear solving?

Reduced row echelon form: Gauss-Jordan elimination

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- Now suppose $A \in \mathsf{K}^{n \times n}$ is nonsingular
- The RREF of $\left[\begin{array}{c|c} A & I_n \end{array}\right] \in \mathsf{K}^{n \times 2n}$ is $\left[\begin{array}{c|c} I_n & A^{-1} \end{array}\right]$

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1. $[\,A\,|\,b\,] \longrightarrow \left[\begin{array}{cccc|c} 1 & * & & * & * \\ & & 1 & * & * \\ & & & & \bullet \end{array}\right] \Longrightarrow$ "inconsistent"

2. $[\,A\,|\,b\,] \longrightarrow \left[\begin{array}{cccc|c} 1 & * & & * & c_1 \\ & & 1 & * & c_2 \\ & & & & \end{array}\right] \Longrightarrow x = \begin{bmatrix} -c_1 \\ \\ -c_2 \\ \\ \end{bmatrix}$

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- Recall previous example (consistent): $A \in \mathsf{K}^{3\times 4}$ with rank 2

$$[\, A \,|\, b \,] \longrightarrow \left[\begin{array}{cccc|c} 1 & *_{12} & & *_{14} & c_1 \\ & & 1 & *_{24} & c_2 \\ & & & & \end{array}\right] \Longrightarrow x_p = \begin{bmatrix} -c_1 \\ \\ -c_2 \\ \\ \end{bmatrix}$$

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$$[\, A \,|\, b \,] \longrightarrow \left[\begin{array}{cccc|c} 1 & *_{12} & & *_{14} & c_1 \\ & & 1 & *_{24} & c_2 \\ & & & & \end{array}\right] \Longrightarrow x_p = \begin{bmatrix} -c_1 \\ \\ -c_2 \\ \\ \end{bmatrix}$$

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- General solution to $Ax = b$ is given by

$$x = \left\{ x_p + Nw \mid w \in \mathsf{K}^{2\times 1} \right\}$$

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- Our primary building block is the multiplication of two $n \times n$ matrices: $AB = C$
- Incorporating matrix multiplication gives an improvement both in practice (implementation) and theory (better asymptotic complexity)
- Standard matrix multiplication uses n^2 dot products costing exactly $2n^3 - n^2 \in O(n^3)$ operations in total

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- Number of operations from $\mathbb{Z}/(p)$ is exactly $2n^3 - n^2$
- Wall clock time on my Mac Mini is 9198 seconds

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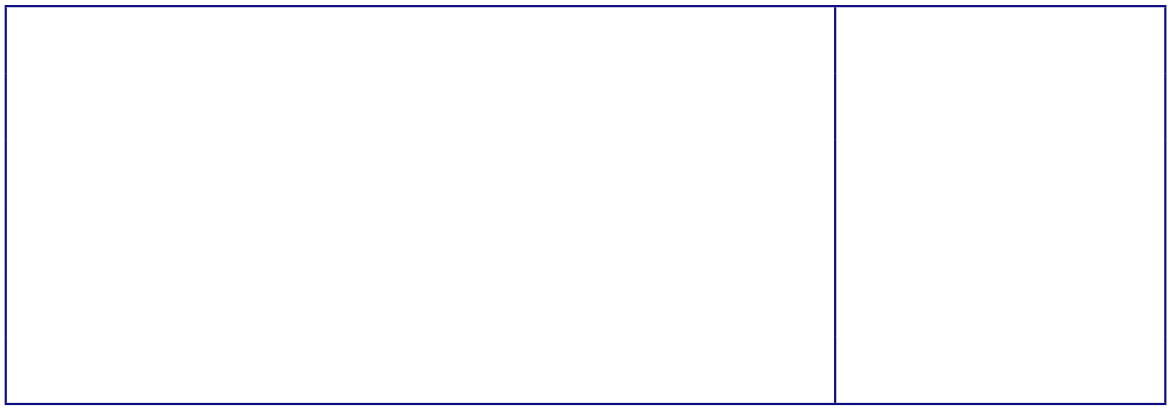

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  - Computing AB costs 12 multiplications of $k \times k$ matrices (plus some additions which do not dominate)

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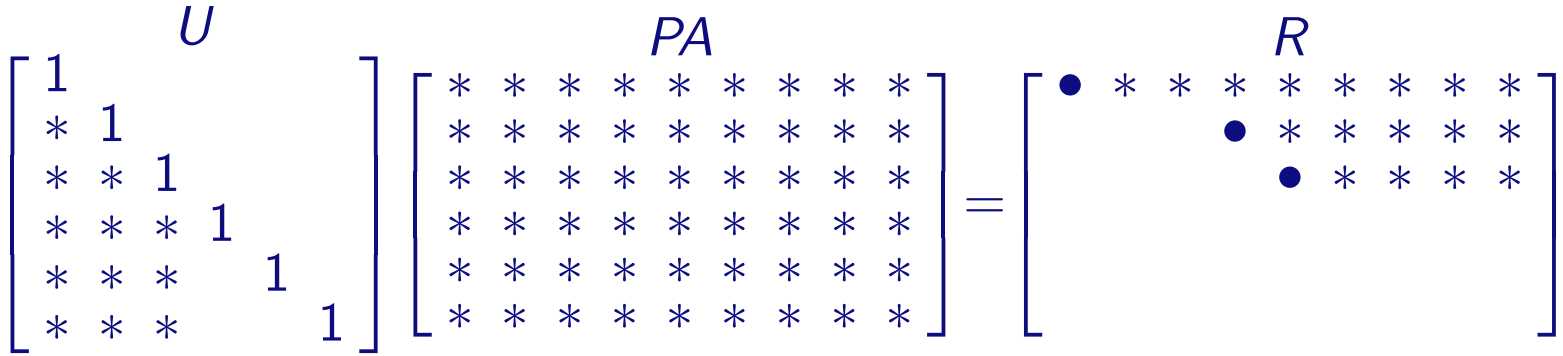

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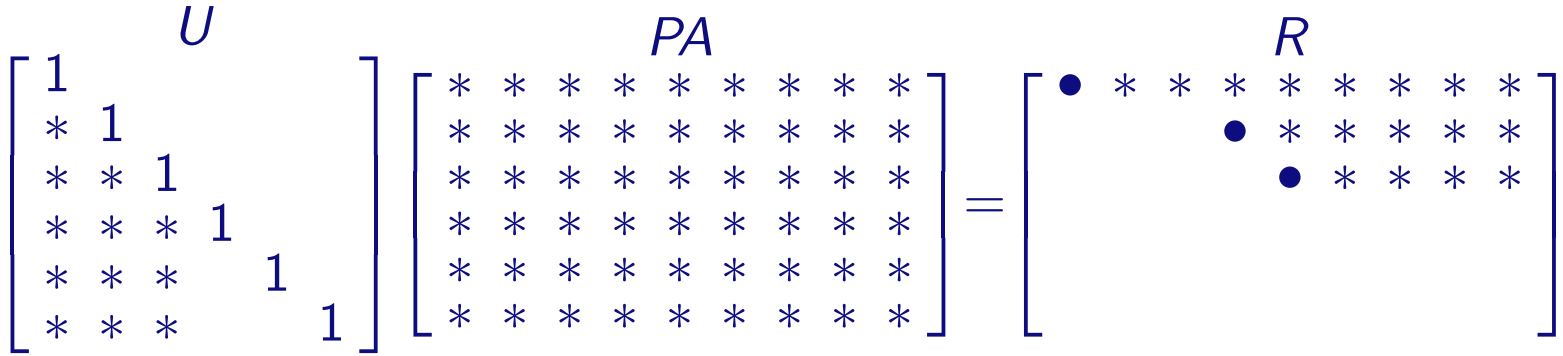

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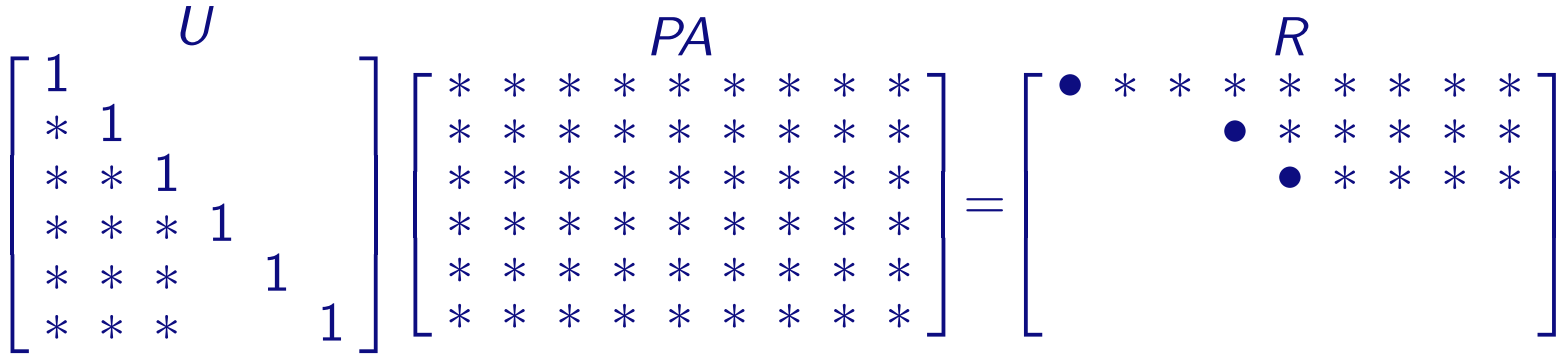

- Call the pair (U, P) a Gauss-transform for A
- To divide and conquer we split the matrix into half according to columns

Recursive echelon form: first subproblem

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  Let A_1 be the first 4 columns

$$A = \overset{A_1}{\left[\begin{array}{cccc|ccccc} * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \end{array}\right]}$$

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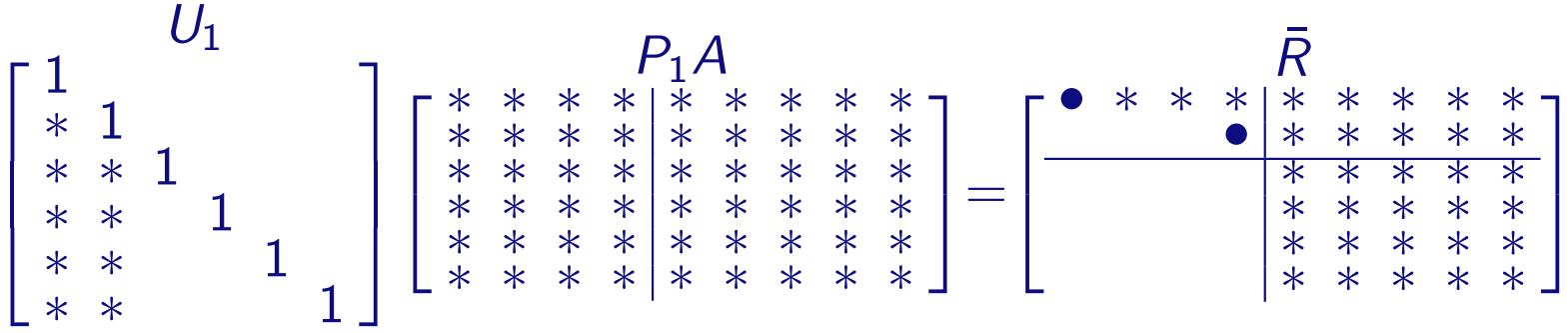

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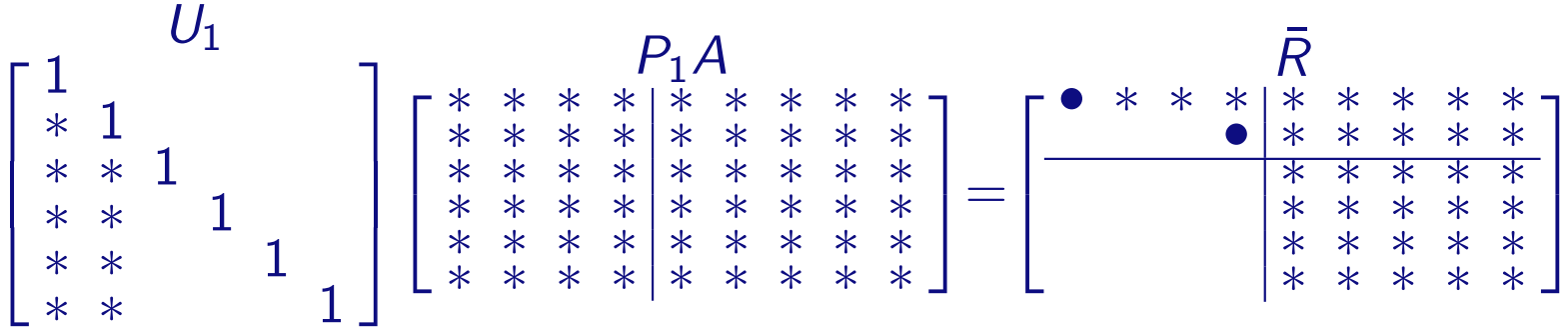

- Compute a Gauss-transform for last five columns of transformed matrix $\bar{R}$ (below horizontal line only)

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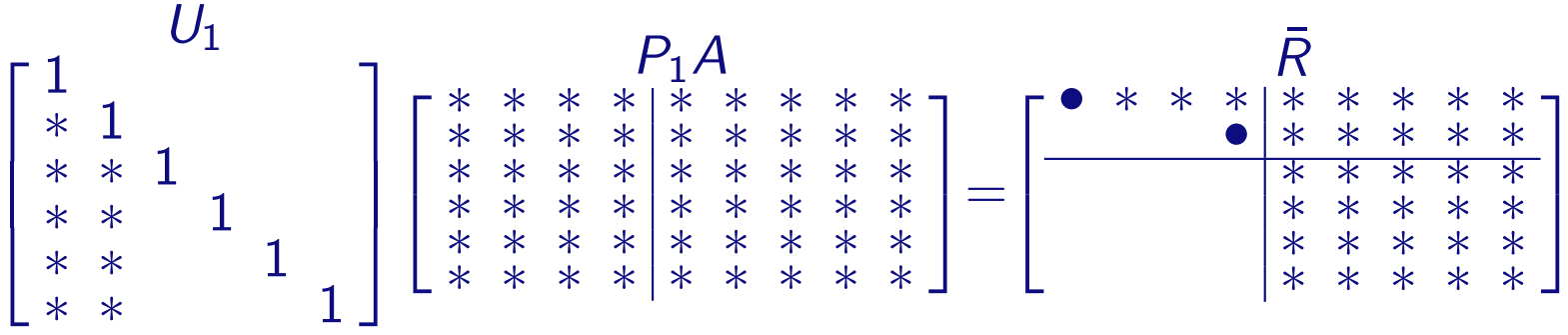

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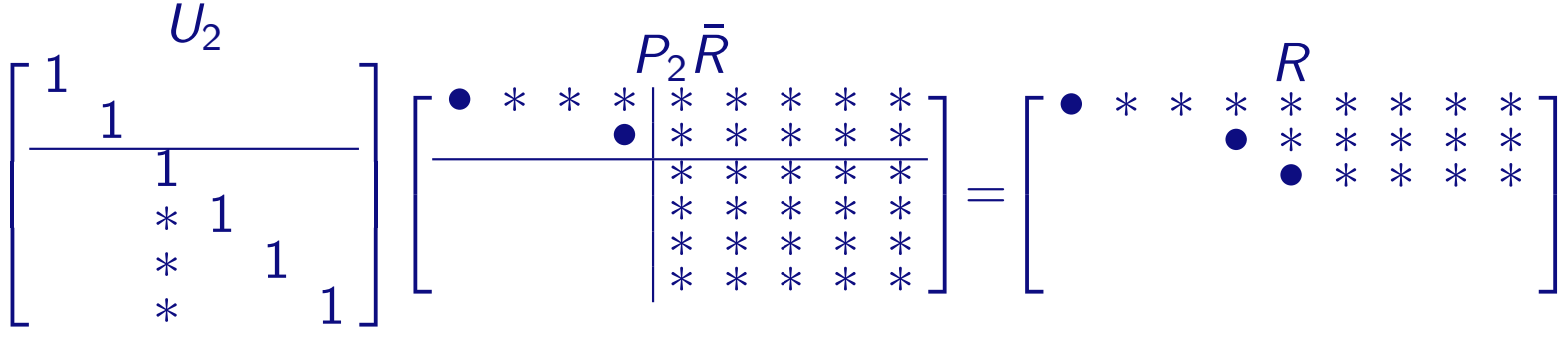

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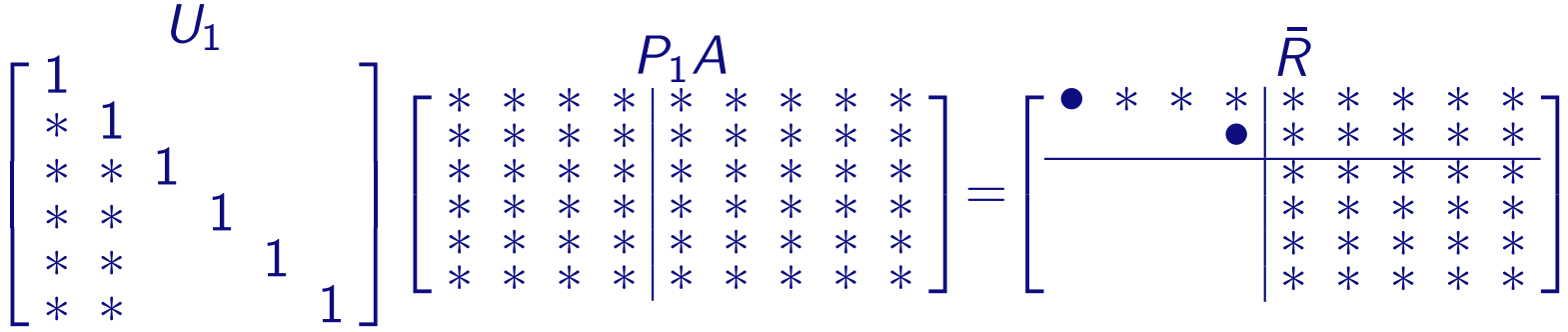

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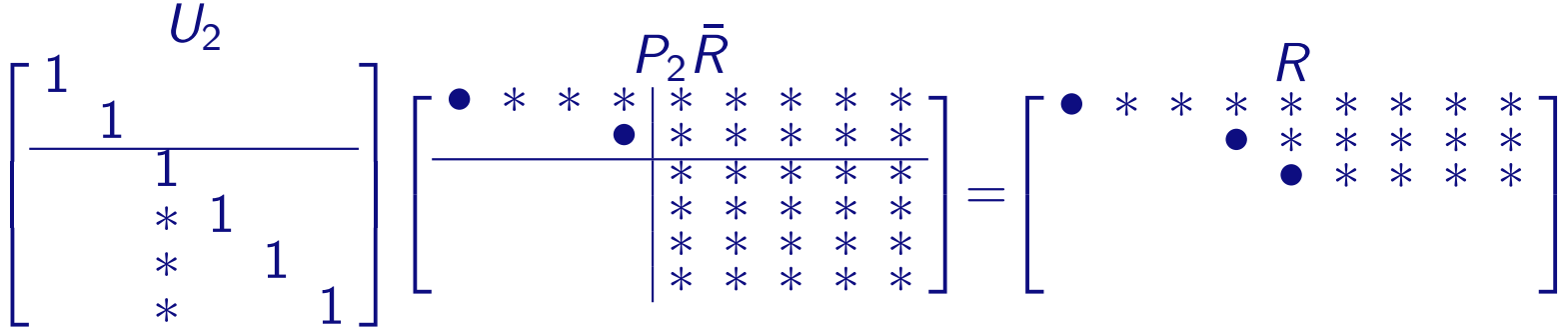

- Combine: Set $U = U_2 P_2 U_1 P_2^{-1}$ and $P = P_2 P_1$.

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- Model cost as binary tree: label at root is cost of combining subproblems. For some constant c we have:

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- Example: CUP decomposition of $A \in \mathrm{K}^{7\times 5}$ with rank 3

$$\underbrace{\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}}_{AP^T} = \underbrace{\begin{bmatrix} \overline{*}_1 & & \\ * & & \\ * & \overline{*}_2 & \\ * & * & \overline{*}_3 \\ * & * & * \\ * & * & * \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} 1 & * & * & * & * \\ & 1 & * & * & * \\ & & 1 & * & * \\ & & & 1 & \\ & & & & 1 \end{bmatrix}}_{U}$$

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- J.-G. Dumas, P. Giorgi, C. Pernet, *Dense Linear Algebra over Word-Size Prime Fields: the FFLAS and FFPACK Packages. ACM Trans. Mathematical. Software*, 2008
  - Highly-optimized implementations over small finite fields

Characteristic polynomial

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- So, A is *similar* to a single companion matrix in this case.

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- V. Neiger and C. Pernet. *Deterministic computation of the characteristic polynomial in the time of matrix multiplication*, J. Complex., 2021
  - $O(n^\omega)$
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- Many applications, eg, compute $A^s = UF^sU^{-1}$ quickly

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- With high probability, the diagonal blocks of $\bar{F}$ will be correct, ie, those of the Frobenius form (this is easy to check)
- "Purifying" the vectors in $\bar{U}$ to get U with $U^{-1}AU$ in Frobenius form is easily done with one more run of Keller-Gehrig's algorithm
- This is the first "nearly-optimal" algorithm for Frobenius form. But it is randomized and requires the field be large enough

Selected references: Frobenius form with transform

- M. Giesbrecht (1993). M. Giesbrect, *Nearly optimal algorithms for canonical matrix forms*, Siam J. Comp., 1995.
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- W. Eberly, *Asymptotically efficient algorithms for the Frobenius form*, 2000, Dept. of Comp. Sci., University of Calgary
  - Las Vegas $O(n^\omega \log n)$ also over small fields
- A. Storjohann, PhD thesis, 2000
  - Deterministic $O(n^\omega \log n \operatorname{loglog} n)$
- C. Pernet and A. Storjohann, *Faster algorithms for the characteristic polynomial*, ISSAC' 07.
  - Las Vegas $O(n^\omega \operatorname{loglog} n)$ when $\#K \geq n^2$