

# Numerical Considerations for the Sparse Interpolation Problem

George Labahn

Symbolic Computation Group  
Cheriton School of Computer Science  
University of Waterloo, Canada

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## Sparse Interpolation

Given a black box representation of a multivariate polynomial  $p$  :

$$\vec{x} \longrightarrow \blacksquare \longrightarrow p(\vec{x})$$

Find powers  $\vec{m}_1, \dots, \vec{m}_n$  and coefficients  $c_1, \dots, c_n$  so that

$$p(x_1, \dots, x_d) = c_1 \mathbf{x}^{\vec{m}_1} + \dots + c_n \mathbf{x}^{\vec{m}_n}$$

where  $\mathbf{x}^{\vec{m}_i} = x_1^{m_{1,i}} \dots x_d^{m_{d,i}}$ .

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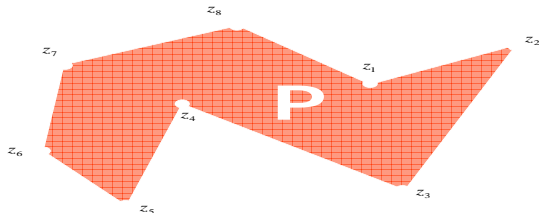
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where  $\mathbf{x}^{\vec{m}_i} = x_1^{m_{1,i}} \dots x_d^{m_{d,i}}$ .

- Determine terms by *special evaluations* :  $h_0, h_1, \dots, h_{2n-1}$
- In case of exact arithmetic : Ben-Or/Tiwari (1988).
- In numeric setting : Giesbrecht-Labahn-Lee (2006 & 2009).

## Shape from Moments

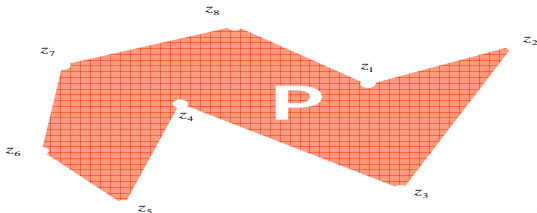
Shape reconstruction from moments (Golub, Milanfar, Varah)



- Given polygon  $P$  with unknown vertices  $z_1, \dots, z_n$ . Find the vertices  $z_i$ .

## Shape from Moments

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- Given polygon  $P$  with unknown vertices  $z_1, \dots, z_n$ . Find the vertices  $z_i$ .
- Determine vertices from **moments** :  $h_0, \dots, h_{2n-1}$
- Solve via generalized Hankel eigenvalue problem

## Sparse interpolation :

- ★ **Special evaluations** :  $h_k = p(\omega_1^k, \dots, \omega_d^k)$ ,
  - Over integers :  $\omega_\ell = p_\ell$ , **prime**.
  - Over complex :  $\omega_\ell = e^{\frac{2\pi i r_\ell}{p_\ell}}$  ( $p_\ell$  relatively prime).
  - Powers for each term determined later (Division or CRT)

## Shape from Moments :

- ★ Integration formula (Davis)

$$\int \int_D f''(z) \, dx dy = c_1 f(z_1) + \dots + c_n f(z_n)$$

- **Moments** (for  $k = 0, \dots, 2n - 1$ ) :

$$h_k = k(k-1) \int \int_P z^{k-2} \, dx dy = c_1 z_1^k + \dots + c_n z_n^k$$

## Prony's Method (1795)

Given a sequence  $h_0, h_1, \dots, h_{2n-1}$  of  $2n$  values, with

$$\begin{aligned} h_0 &= c_1 & + & c_2 & + \dots & + c_n \\ h_1 &= c_1 \lambda_1 & + & c_2 \lambda_2 & + \dots & + c_n \lambda_n \\ h_2 &= c_1 \lambda_1^2 & + & c_2 \lambda_2^2 & + \dots & + c_n \lambda_n^2 \\ \vdots & & & \vdots & & \vdots \\ h_{2n-1} &= c_1 \lambda_1^{2n-1} & + & c_2 \lambda_2^{2n-1} & + \dots & + c_n \lambda_n^{2n-1} \end{aligned}$$

- $\lambda_i$ 's are roots of  $a(x) = a_0 + a_1 x + \dots + a_{n-1} x^{n-1} + x^n$ .

Here coefficients  $a_i$  are determined by solving

$$a_0 h_i + a_1 h_{i+1} + \dots + a_{n-1} h_{i+n-1} = -h_{i+n} .$$

- $c_i$ 's solved via Vandermonde system
- $\lambda_i$ 's give the terms (SI) and vertices (SM)

### Questions :

- (1) How many terms (or vertices)?
- (2) What are the terms (or vertices)?  
( in a numeric environment )

### Answers:

- M. Giesbrecht, G. Labahn and W-s Lee,  
*Symbolic-numeric Sparse Interpolation of Multivariate Polynomials*, Journal of Symbolic Computation, 44(8) (2009) 943-959
- B. Beckermann, G. Golub and G. Labahn,  
*On the Numerical Condition of a Generalized Hankel Eigenvalue Problem*, Numerische Mathematik, 106(1) (2007) 41-68.
- S. Cabay and R. Meleshko,  
*A weakly stable algorithm for Padé approximants and the inversion of Hankel matrices*, SIAM Journal on Matrix Analysis and Applications, 14(3) (1993) 735-765
- E. Kaltofen, W-s Lee and Z. Yang.  
*Fast estimates of Hankel matrix condition numbers and numeric sparse interpolation*. In SNC'11 Proceedings of 2011 International Workshop on Symbolic-Numeric Computation, pages 130-136.

# Part I

## Finding the Number of Terms (or vertices)

## Finding number $n$

Matrix form in step 2 of Prony : Coefficients  $a_i$  determined by

$$\begin{bmatrix} h_0 & h_1 & \cdots & h_{n-1} \\ h_1 & h_2 & \cdots & h_n \\ \vdots & \vdots & & \vdots \\ h_{n-1} & h_n & \cdots & h_{2n-2} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} = - \begin{bmatrix} h_n \\ h_{n+1} \\ \vdots \\ h_{2n-1} \end{bmatrix}$$

From [Kaltofen-Lee](#) :

- High probability all principal Hankel submatrices nonsingular (exact)

From [Kaltofen-Lee-Yang](#) :

- High probability principal Hankel submatrices well conditioned (numeric)

Key point :  $n$  determined from first “bad” principal Hankel matrix

- ★ Looking to solve numeric Hankel linear system efficiently

$$\begin{bmatrix} h_0 & h_1 & h_2 & h_3 & h_4 \\ h_1 & h_2 & h_3 & h_4 & h_5 \\ h_2 & h_3 & h_4 & h_5 & h_6 \\ h_3 & h_4 & h_5 & h_6 & h_7 \\ h_4 & h_5 & h_6 & h_7 & h_8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

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## Aside : 1993

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- ★ Previous to 1993 - use look ahead and hope things work
- ★ Cabay-Meleshko 1993 - weakly-stable, usually fast, solver
- ★ Key idea: estimate condition numbers via inverse formulae.  
(this generated lots of excitement at the time)

# Inversion Formulas for Hankel Matrices

$H_n = [h_{i+j-1}]_{i,j=1\dots n}$ .      Inverse of  $H_n$  given by (Gohberg-Semencul)

$$\frac{1}{f_0} \left( \begin{bmatrix} f_{n-1} & \cdots & f_0 \\ \vdots & \ddots & \\ f_0 & & \end{bmatrix} \begin{bmatrix} q_{n-1} & \cdots & q_0 \\ & \ddots & \\ & & q_{n-1} \end{bmatrix} - \begin{bmatrix} q_{n-2} & \cdots & q_0 & 0 \\ \vdots & \ddots & & \\ q_0 & & & 0 \end{bmatrix} \begin{bmatrix} 0 & f_{n-1} & \cdots & f_1 \\ & \ddots & & \vdots \\ & & & f_{n-1} \\ & & & 0 \end{bmatrix} \right)$$

where

$$H_n \cdot \begin{bmatrix} f_{n-1} \\ \vdots \\ f_1 \\ f_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{and} \quad H_n \cdot \begin{bmatrix} q_{n-1} \\ \vdots \\ q_1 \\ q_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

with  $f_0 \neq 0$ .

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with  $f_0 \neq 0$ . Very elegant, but extra condition ( $f_0 \neq 0$ ) needed.

Extra condition : Invertibility of  $(n - 1) \times (n - 1)$  submatrix.

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where

$$H_n \cdot \begin{bmatrix} v_n \\ \vdots \\ v_2 \\ v_1 \end{bmatrix} = -v_0 \begin{bmatrix} h_{n+1} \\ \vdots \\ h_{2n+1} \end{bmatrix} \quad \text{and} \quad H_n \cdot \begin{bmatrix} q_{n-1} \\ \vdots \\ q_1 \\ q_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ r_0 \end{bmatrix}$$

with  $r_0 \neq 0, v_0 \neq 0$ .

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with  $r_0 \neq 0, v_0 \neq 0$ . If and only if formula. More useful than G-S.

# Condition Numbers of $H_n$ (informally)

Exact

$$H_n^{-1} = \frac{1}{r_0 v_0} \left( \begin{bmatrix} v_{n-1} & \cdots & v_0 \\ \vdots & & \\ v_0 & & \end{bmatrix} \begin{bmatrix} q_{n-1} & \cdots & q_0 \\ \vdots & & \\ q_{n-1} & & \end{bmatrix} - \begin{bmatrix} q_{n-2} & \cdots & q_0 & 0 \\ \vdots & & \\ q_0 & & 0 \end{bmatrix} \begin{bmatrix} v_n & \cdots & v_1 \\ \vdots & & \\ v_n & & \end{bmatrix} \right)$$

where

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Approximate: [Cabay-Meleshko]

$$\begin{aligned} \kappa(H_n) &= \|H_n\| \cdot \|H_n^{-1}\| \\ &= \|H_n^{-1}\| && \text{rescale } h_i \\ &\approx \frac{c}{r_0 v_0} && \text{rescale } q_0, \dots, q_{n-1}, v_0, \dots, v_n. \end{aligned}$$

- (1) Iterative scheme to compute inversion components
  - similar to : numerical sigma basis algorithm
  - Usually quadratic
  
- (2) Scheme provably **weakly stable**
  - Use first order error analysis

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- (3) Key point : Quadratic and provable error analysis for finding **first** ill conditioned principal Hankel submatrix

## Aside : Cabay-Meleshko (weak stable solver)

Used condition number estimate for **weakly-stable** solver for Hankel systems

$$\begin{bmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 & h_8 & h_9 \\ h_2 & h_3 & h_4 & h_5 & h_6 & h_7 & h_8 & h_9 & h_{10} \\ h_3 & h_4 & h_5 & h_6 & h_7 & h_8 & h_9 & h_{10} & h_{11} \\ h_4 & h_5 & h_6 & h_7 & h_8 & h_9 & h_{10} & h_{11} & h_{12} \\ h_5 & h_6 & h_7 & h_8 & h_9 & h_{10} & h_{11} & h_{12} & h_{13} \\ h_6 & h_7 & h_8 & h_9 & h_{10} & h_{11} & h_{12} & h_{13} & h_{14} \\ h_7 & h_8 & h_9 & h_{10} & h_{11} & h_{12} & h_{13} & h_{14} & h_{15} \\ h_8 & h_9 & h_{10} & h_{11} & h_{12} & h_{13} & h_{14} & h_{15} & h_{16} \\ h_9 & h_{10} & h_{11} & h_{12} & h_{13} & h_{14} & h_{15} & h_{16} & h_{17} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ \vdots \\ x_9 \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ \vdots \\ \vdots \\ b_9 \end{bmatrix}$$

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- (1) Estimate condition numbers
  - Use Gohberg-Semencul formula
  - Extra condition not a problem (no jumping)
- (2) Iterate using Gohberg and Koltracht [1989]
- (3) Quadratic complexity for first ill-conditioned Hankel submatrix

# Part II

## Finding the Terms (or Vertices)

## Finding the Terms (or Vertices)

**Given**  $h_0, h_1, \dots, h_{2n-1}$ ,

**Find** :  $c_1, \dots, c_n$  and  $\lambda_1, \dots, \lambda_n$  such that

$$h_0 = c_1 + c_2 + \dots + c_n$$

$$h_1 = c_1 \lambda_1 + c_2 \lambda_2 + \dots + c_n \lambda_n$$

$$\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots$$

$$h_{2n-1} = c_1 \lambda_1^{2n-1} + c_2 \lambda_2^{2n-1} + \dots + c_n \lambda_n^{2n-1}$$

## Generalized Hankel eigenvalue problem

Generalized Hankel eigenvalue problem (GHEP)

**Given :**  $h_0, h_1, \dots, h_{2n-1}$ , solve

$$\begin{bmatrix} h_1 & h_2 & \cdots & h_n \\ h_2 & h_3 & \cdots & h_{n+1} \\ \vdots & \vdots & & \vdots \\ h_n & h_{n+1} & \cdots & h_{2n-1} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \lambda \begin{bmatrix} h_0 & h_1 & \cdots & h_{n-1} \\ h_1 & h_2 & \cdots & h_n \\ \vdots & \vdots & & \vdots \\ h_{n-1} & h_n & \cdots & h_{2n-2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

for eigenvalues  $\lambda$  and eigenvectors  $\vec{y} = [y_1, y_2, \dots, y_n]^T$ .

We also write this as

$$\tilde{H}_n \cdot \vec{y} = \lambda \cdot H_n \cdot \vec{y}.$$

## Numerical GHEP

Assume the  $h_0, h_1, \dots, h_{2n-1}$  are all **approximately** known.

Our question : What about the sensitivity of

$$\begin{aligned} G_n & : \mathbb{C}^{2n} \mapsto \mathbb{C}^n \\ & (h_0, \dots, h_{2n-1}) \mapsto (\lambda_1, \dots, \lambda_n)? \end{aligned}$$

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★ Error Analysis :

- BGL : Structured error  $\approx$  Unstructured error

★ Sensitivity Analysis :

- BGL : In general conditioning of GHEP is **bad**
- GLL : GHEP for Sparse interpolation is **good**:

## Structured vs Unstructured Error Analysis

If  $H_n$  has some noise then

$$H_n = \begin{bmatrix} h_0 & h_1 & \cdots & h_{n-1} \\ h_1 & h_2 & \cdots & h_n \\ \vdots & \vdots & & \vdots \\ h_{n-1} & h_n & \cdots & h_{2n} \end{bmatrix}$$

## Structured vs Unstructured Error Analysis

If  $H_n$  has some noise then

$$H_n + E_n = \begin{bmatrix} h_0 & h_1 & \cdots & h_{n-1} \\ h_1 & h_2 & \cdots & h_n \\ \vdots & \vdots & & \vdots \\ h_{n-1} & h_n & \cdots & h_{2n} \end{bmatrix} + \begin{bmatrix} e_{1,1} & e_{1,2} & \cdots & e_{1,n} \\ e_{2,1} & e_{2,2} & \cdots & e_{2,n} \\ \vdots & \vdots & & \vdots \\ e_{n,1} & e_{n,2} & \cdots & e_{n,n} \end{bmatrix}$$

i.e. unstructured error analysis

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If  $h_i$  have some noise then

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## Structured vs Unstructured Error Analysis

If  $H_n$  has some noise then

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i.e. structured error analysis

## Error Analysis

Do usual first order error arithmetic.

Structured and unstructured errors depend on [BGL]

$$(|\lambda_j| + 1) \|W_n^{-1} e_j\|^2$$

where

$$W_n = \text{diag}(\sqrt{c_1}, \dots, \sqrt{c_n}) \begin{bmatrix} 1 & \dots & \lambda_1^{n-1} \\ 1 & \dots & \lambda_2^{n-1} \\ \vdots & & \vdots \\ 1 & \dots & \lambda_n^{n-1} \end{bmatrix}$$

scaled Vandermonde matrix.

## Sensitivity of GHEP

Sensitivity related to:

- Conditioning of Hankel matrices :  $H_n$  and  $\tilde{H}_n$
- Conditioning of row scaled Vandermonde matrices :  $W_n$
- How  $\lambda_1, \dots, \lambda_n$  are distributed in a simply connected compact subset of  $\mathbb{C}$ .

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Main result : If  $E$  contains the generalized eigenvals, then (usually)

$$\text{Sensitivity} \geq \gamma_n(E)$$

where

$$n^2 \gamma(E)^{n-1} \geq \gamma_n(E) \geq \frac{1}{\sqrt{n}} \left[ \frac{\pi}{2V} \gamma(E)^{n-1} - 1 - \frac{V}{\pi} \right]$$

and  $\gamma(E) \geq 1$  measures which part of unit circle is not part of  $E$ .

## Good vs Bad Conditioning

In fact for numeric sparse interpolation:

- ★ Conditioning is bad
  - there usually exists at least some examples where conditioning is bad
- ★ Bad conditioning is not a problem [GLL]
  - we can always pick another set of interpolation points to get better conditioning
  - with probability bigger than  $\frac{1}{2}$  such a set of interpolation points will get us a good answer.

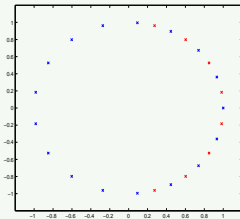
## Extra Slide : Example

### Example: Sparse Interpolation

Let  $E_n = \{\lambda_1, \dots, \lambda_n\}$ , with  $n = 6m + 1$ , be the set

$$\left\{ \exp\left(\frac{2\pi i j}{4m+1}\right) : j = 0, \dots, 4m \right\} \cup \left\{ \exp\left(\frac{2\pi i (2j+1)}{8m+2}\right) : j = -m, -m+1, \dots, m-1 \right\},$$

a subset of the unit circle which has no significant gaps.



$$\text{Then : } \gamma_n(E_n) \geq \frac{\rho^m}{n} \text{ with } \rho = \exp\left(\frac{8\text{Catalan}}{\pi}\right) \approx 10.30 > 1$$

Note : exponential growth.

## Extra Slide : Conditioning via distribution of eigenvalues

### Theorem

*For any compact set  $E \subset \mathbb{C}$  which is regular with respect to the Dirichlet problem we have that*

$$\lim_{n \rightarrow \infty} \gamma_n(E)^{1/n} = \gamma(E),$$

*and  $\gamma_n(E) \leq n^2 \gamma(E)^{n-1}$  for all  $n \geq 1$ . If, in addition  $E$  is simply connected and of bounded variation  $V$ , then we have that*

$$\gamma_n(E) \geq \frac{1}{\sqrt{n}} \left[ \frac{\pi}{2V} \gamma(E)^{n-1} - 1 - \frac{V}{\pi} \right].$$

Note :  $E$  is of *bounded variation*  $V$  if, given  $\beta : [0, 1] \mapsto \partial E$  there exists a tangent at almost every  $\beta(s)$ , forming an angle  $\theta(s)$  with the positive real axis, and if  $\theta$  has a total variation bounded by  $V$ . For instance, convex sets are of bounded variation  $2\pi$ .