# Alternate Polynomial Bases

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# Approximation and Reduction in Alternate Bases

# The Problem

- Given $f_1, \ldots, f_m \in \mathcal{V}$,

  $n_1, \ldots, n_m, \ \sigma \in \mathbb{Z}$

  Find *all* solutions of :

$$p_1(z) \cdot f_1 + \cdots + p_m(z) \cdot f_m = r$$

  - Here $p_1(z), \ldots, p_m(z) \in \mathbb{K}[z]$
  - $p_i(z)$ of degree at most $n_i$
  - $r \in \mathcal{V}$ with special properties : Order$(r) \geq \sigma$.

- Here $\mathcal{V} \equiv$ power series given in alternate basis

# The Formal Setting

- ⋆ $\mathcal{V}$ infinite dimensional vector space over a field $\mathbb{K}$.
  - Basis $\{\ \omega_i\ \}_{i=0,1,\ldots}$ and dual basis $\{\ c_i\ \}_{i=0,1,\ldots}$ for $\mathcal{V}$ .

$$f = f_0 \cdot \omega_0 + f_1 \cdot \omega_1 + f_2 \cdot \omega_2 + \cdots \qquad \text{with} \qquad c_k(f) = f_k$$

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- ⋆ $z$ is a ‘special’ element which acts on $\mathcal{V}$ via

$c_k( z \cdot f) =$ previous $k$ coefficients of $f$

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- ★ $z$ is a 'special' element which acts on $\mathcal{V}$ via

$$c_k(\ z \cdot f) = c_{k0} \cdot c_0(f) + c_{k1} \cdot c_1(f) + \cdots + c_{kk} \cdot c_k(f)$$

with each $c_{k,j} \in \mathbb{K}$.

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with each $c_{k,j} \in \mathbb{K}$.

- Sometimes want to extend to :

$$c_k(z \cdot f) = c_{k0} \cdot c_0(f) + c_{k1} \cdot c_1(f) + \cdots + c_{kk} \cdot c_k(f) + c_{k,k+1} \cdot c_{k+1}(f)$$

# What are our goals?

(1) Describe general framework for all these problems

(2) Find *all* solutions to such problems

(3) Give formulas for bases of such problems
(e.g. subresultants)

(4) Find an algorithm for the computation of the basis.
(without fractions)

# Problems Covered

- Rational approximation
- Simultaneous rational approx
- Hermite Padé approximation
- Matrix rational approximation
- Subresultant gcd
- Simplification
- Inversion formula block matrices ( Hankel, Toeplitz, etc )
- GREP ( Generalized Richardson Extrapolation )

- Rational Interpolation
- Simultaneous rational interp
- M-Padé approximation
- Matrix rational interpolation
- Matrix polynomial gcd
- Ore identities

# References

(1)* B. Beckermann and G. Labahn, Fraction-free Computation of Matrix Rational Interpolants and Matrix GCD's, SIAM J. Matrix Analysis and Applications, 22(1) (2000) 114-144.

(2) H. Cheng and G. Labahn, On Computing Polynomial GCDs in Alternate Bases, Proceedings of ISSAC'06, Genoa, Italy, ACM Press, (2006) 47-54.

# References

(1)* B. Beckermann and G. Labahn, Fraction-free Computation of Matrix Rational Interpolants and Matrix GCD's, SIAM J. Matrix Analysis and Applications, 22(1) (2000) 114-144.

(2) H. Cheng and G. Labahn, On Computing Polynomial GCDs in Alternate Bases, Proceedings of ISSAC'06, Genoa, Italy, ACM Press, (2006) 47-54.

(3) C. D'Andrea, T. Krick and A. Szantos, Subresultants, Sylvester sums and the Rational Interpolation Problem (November 2012)

# Key Ideas

1. Use of linear functionals (*linearly independent*)
2. Linear algebra with structured matrices
3. Mahler Systems as a module basis for all solutions.
    - Closest normal point
    - Shifted Popov form
4. Structured solve of structured linear system
    - Fraction-free and fast

# Examples

## $\mathcal{V}$ and special rule

# Example : Newton basis (distinct points)

$\mathcal{V}$ is space of formal power series, interpolation points $x_0, \ x_1, \ \ldots$

(i) Newton basis : $1, \ x - x_0, \ (x - x_0)(x - x_1), \ \ldots$.

- $c_k$ is $k^{th}$ divided difference $[x_0, \ldots, x_k]$
- Special element $z = x$.

Special rule

$$c_k( z \cdot f) = x_k \cdot c_{k-1}(f) + c_k(f)$$

since

$$z \cdot (x - x_0) \quad \cdots \quad (x - x_{k-1})$$

$$= \quad x_k(x - x_0) \cdots (x - x_{k-1}) \quad + \quad (x - x_0) \cdots (x - x_k)$$

# Example : Hermite basis (confluent interpolation)

$\mathcal{V}$ is space of formal power series,

– e.g. repeated interpolation points $x_0,\ x_0,\ x_1,\ x_1,\ \ldots$.

(ii) Interpolation basis : $\omega_0(x),\ \ldots,\ \omega_i(x),\ \ldots$ (Hermite basis)

- Linear functional :

$$\begin{aligned} c_{2k}(f) &= f(x_k) \\ c_{2k+1}(f) &= f'(x_k) \end{aligned}$$

- Special element : $z = x$.

$$\begin{aligned} c_{2k}(\,z \cdot f) &= x_k c_{2k}(f) \\ c_{2k+1}(\,z \cdot f) &= x_k c_{2k+1}(f) + c_{2k}(f) \end{aligned}$$

# Linear Algebra

# Big Picture

- Express
  $$p_1(z) \cdot f_1 + \cdots + p_m(z) \cdot f_m = r$$
  as linear system in unknowns (the coefficients of the $p_i(z)$)

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- For any polynomial $q(z) = q_0 + q_1 z + \cdots + q_k z^k$:

  $$\begin{aligned} q(z) \cdot f &= q_0 \cdot (f) + q_1 \cdot (zf) + q_2 \cdot \left(z^2 f\right) + \cdots \\ c_n(\, q(z) \cdot f \,) &= q_0 \cdot c_n(f) + q_1 \cdot c_{n-1}(zf) + q_2 \cdot c_{n-2}(z^2 f) + \cdots \end{aligned}$$

  Matrix of linear system depends on coeffs of $f$, $zf$, $z^2 f, \ldots$

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  $$q(z) \cdot f \quad = \quad q_0 \cdot (f) + q_1 \cdot (zf) + q_2 \cdot \left(z^2 f\right) + \cdots$$

  $$c_n(\ q(z) \cdot f\ ) \quad = \quad q_0 \cdot c_n(f) + q_1 \cdot c_{n-1}(zf) + q_2 \cdot c_{n-2}(z^2 f) + \cdots$$

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  Matrix of linear system depends on coeffs of $f$, $zf$, $z^2 f, \ldots$

- Matrix of linear system has *special structure*

- Matrix of linear system always square.

# Structured Linear System

Identity

$$p^{(1)}(z)\cdot f_1 + \cdots + p^{(m)}(z)\cdot f_m = r$$

same as linear system in unknowns (the coefficients of the $p_i(z)$) :

$$\left[\begin{array}{cccc|cc|cccc} c_0(f_1) & c_0(zf_1) & \cdots & c_0(z^{n_1-1}f_1) & \cdots & \cdots & c_0(f_m) & c_0(zf_m) & \cdots & c_0(z^{n_m-1}f_m) \\ c_1(f_1) & c_1(zf_1) & \cdots & c_1(z^{n_1-1}f_1) & \cdots & \cdots & c_1(f_m) & c_1(zf_m) & \cdots & c_1(z^{n_m-1}f_m) \\ \vdots & \vdots & & \vdots & & & \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & & \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & & \vdots & \vdots & & \vdots \\ c_\sigma(f_1) & c_\sigma(zf_1) & \cdots & c_\sigma(z^{n_1-1}f_1) & \cdots & \cdots & c_\sigma(f_m) & c_\sigma(zf_m) & \cdots & c_\sigma(z^{n_m-1}f_m) \end{array}\right] \left[\begin{array}{c} p_0^{(1)} \\ \vdots \\ p_{n_1-1}^{(1)} \\ \hline \vdots \\ p_0^{(m)} \\ \vdots \\ p_{n_m-1}^{(m)} \end{array}\right] = \left[\begin{array}{c} c_0(r) \\ c_1(r) \\ \vdots \\ \vdots \\ \vdots \\ c_\sigma(r) \end{array}\right]$$

When $\sigma + 1 = n_1 + \cdots + n_m = |\vec{n}|$ then system is square.

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$$p^{(1)}(z)\cdot f_1+\cdots+p^{(m)}(z)\cdot f_m=r$$

when order is $\sigma$ same as linear system in unknowns (coeffs of $p_i(z)$) :

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c_0(f_1) & c_0(zf_1) & \cdots & c_0(z^{n_1-1}f_1) & \cdots & \cdots & c_0(f_m) & c_0(zf_m) & \cdots & c_0(z^{n_m-1}f_m)\\
c_1(f_1) & c_1(zf_1) & \cdots & c_1(z^{n_1-1}f_1) & \cdots & \cdots & c_1(f_m) & c_1(zf_m) & \cdots & c_1(z^{n_m-1}f_m)\\
\vdots & \vdots & & \vdots & & & \vdots & \vdots & & \vdots\\
\vdots & \vdots & & \vdots & & & \vdots & \vdots & & \vdots\\
\vdots & \vdots & & \vdots & & & \vdots & \vdots & & \vdots\\
c_\sigma(f_1) & c_\sigma(zf_1) & \cdots & c_\sigma(z^{n_1-1}f_1) & \cdots & \cdots & c_\sigma(f_m) & c_\sigma(zf_m) & \cdots & c_\sigma(z^{n_m-1}f_m)
\end{array}\right]
\left[\begin{array}{c}
p_0^{(1)}\\ \vdots\\ p_{n_1-1}^{(1)}\\ \hline \vdots\\ p_0^{(m)}\\ \vdots\\ p_{n_m-1}^{(m)}
\end{array}\right]
=
\left[\begin{array}{c}
0\\ 0\\ \vdots\\ \vdots\\ \vdots\\ \vdots\\ 0\\ c_\sigma(r)
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When $c_{|\vec{n}|}(r) =$ det $\mathbf{K}(\vec{n},\mathbf{F})$ then no fractions.

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\vdots & \vdots & & \vdots & & & \vdots & & \vdots \\
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When $c_{|\vec{n}|}(r) =$ det $\mathbf{K}(\vec{n},\mathbf{F})$ then no fractions.

# Structured Linear System

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with order $\sigma$ satisfied by $p(\vec{n}, z) = (p^{(1)}(z), \cdots, p^{(m)}(z))$ where :

$$p^{(1)}(z) = \left[ \begin{array}{cccc|cc|ccc} c_0(f_1) & c_0(zf_1) & \cdots & c_0(z^{n_1-1}f_1) & \cdots & \cdots & c_0(f_m) & \cdots & c_0(z^{n_m-1}f_m) \\ c_1(f_1) & c_1(zf_1) & \cdots & c_1(z^{n_1-1}f_1) & \cdots & \cdots & c_1(f_m) & \cdots & c_1(z^{n_m-1}f_m) \\ \vdots & \vdots & & \vdots & & & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & & \vdots & & \vdots \\ c_{\sigma-1}(f_1) & c_{\sigma-1}(zf_1) & \cdots & c_{\sigma-1}(z^{n_1-1}f_1) & \cdots & \cdots & c_{\sigma-1}(f_m) & \cdots & c_{\sigma-1}(z^{n_m-1}f_m) \\ f_1 & zf_1 & \cdots & z^{n_1-1}f_1 & \cdots & \cdots & 0 & \cdots & 0 \end{array} \right]$$

and similarly for $p^{(2)}(z)$, $p^{(3)}(z)$, etc.

When $c_{|\vec{n}|}(r) =$ det $\mathbf{K}(\vec{n}, \mathbf{F})$ then no fractions.

# Structured System

Multi-gradients, Cramer’s rule

Determinant Polynomials and Mahler Systems

# Matrix of Structured Linear System

- ⋆ Let $\mathbb{K}(\vec{n}, c, f)$ denotes square coefficient matrix of identity

$$p^{(1)}(z) \cdot f_1 + \cdots + p^{(m)}(z) \cdot f_m = r$$

- ⋆ Let $\mathbf{F} = [c_0(f), \ldots, c_\sigma(f)]^T$ denotes coefficent vector.
- ⋆ Let

$$\mathbf{C} = \begin{bmatrix} c_{00} & & \\ & & \\ \vdots & \ddots & \\ c_{\sigma 0} & \cdots & c_{\sigma\sigma} \end{bmatrix}$$

where

$$c_k(z \cdot f) = c_{k0}c_0(f) + \cdots + c_{kk}c_k(f)$$

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Then

$$\mathbb{K}(\vec{n}, c, f) = \left[ \begin{array}{cccc|c|ccc} \mathbf{F}^{(1)} & \mathbf{C} \cdot \mathbf{F}_\sigma^{(1)} & \cdots & \mathbf{C}^{\vec{n}1-1} \cdot \mathbf{F}^{(1)} & \cdots & \mathbf{F}^{(m)} & \cdots & \mathbf{C}^{\vec{n}m-1} \cdot \mathbf{F}_\sigma^{(m)} \end{array} \right]_{|\vec{n}| \times |\vec{n}|}$$

# Example : Standard Basis : $c_n(z \cdot f) = c_{n-1}(f)$.

$$\mathbf{F} = (A, B, C), \qquad \mathbf{C} = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}, \qquad \vec{n} = (2, 3, 3) \qquad |\vec{n}| = 8$$

$$\mathbf{K}(\ \vec{n},\ \mathbf{C},\ \mathbf{F}) = \left[\begin{array}{cc|ccc|ccc} A_\sigma & \mathbf{C} \cdot A_\sigma & B_\sigma & \mathbf{C} \cdot B_\sigma & \mathbf{C}^2 \cdot B_\sigma & C_\sigma & \mathbf{C} \cdot C_\sigma & \mathbf{C}^2 \cdot C_\sigma \end{array}\right]$$

$$= \left[\begin{array}{cc|ccc|ccc} a_0 & & b_0 & & & c_0 & & \\ a_1 & a_0 & b_1 & b_0 & & c_1 & c_0 & \\ a_2 & a_1 & b_2 & b_1 & b_0 & c_2 & c_1 & c_0 \\ a_3 & a_2 & b_3 & b_2 & b_1 & c_3 & c_2 & c_1 \\ a_4 & a_3 & b_4 & b_3 & b_2 & c_4 & c_3 & c_2 \\ a_5 & a_4 & b_5 & b_4 & b_3 & c_5 & c_4 & c_3 \\ a_6 & a_5 & b_6 & b_5 & b_4 & c_6 & c_5 & c_4 \\ a_7 & a_6 & b_7 & b_6 & b_5 & c_7 & c_6 & c_5 \end{array}\right]$$

# Example : Confluent Basis : $c_n(f) = \frac{f^{(n)}(x_0)}{n!}$.

$\mathbf{F} = (A, B, C)$, Points : $x_0 = x_1$; $x_2 = x_3 = x_4$; $x_5 = x_6$, etc $\vec{n} = (2, 3, 3)$ $\sigma = 7$

$$\mathbf{C} = \begin{bmatrix} x_0 & 0 & & & \cdots & \\ 1 & x_0 & & & \cdots & \\ & & x_1 & 0 & 0 & \cdots \\ & & 1 & x_1 & 0 & \\ & & & 1 & x_1 & \end{bmatrix}$$

( made up of Jordan blocks )

$$\mathbf{K}(\,\vec{n},\ \mathbf{C},\ \mathbf{F}) = \left[\begin{array}{cc|ccc|ccc} A_\sigma & \mathbf{C}\cdot A_\sigma & B_\sigma & \mathbf{C}\cdot B_\sigma & \mathbf{C}^2\cdot B_\sigma & C_\sigma & \mathbf{C}\cdot C_\sigma & \mathbf{C}^2\cdot C_\sigma \end{array}\right]$$

$$= \left[\begin{array}{cc|ccc|ccc} a_0 & x_0a_0 & b_0 & x_0b_0 & x_0^2b_0 & c_0 & x_0c_0 & x_0^2c_0 \\ a_1 & x_0a_1 + a_0 & b_1 & x_0b_1 + b_0 & x_0^2b_1 + 2x_0b_0 & c_1 & x_0c_1 + c_0 & x_0^2c_1 + 2x_0c_0 \\ a_2 & x_1a_2 & b_2 & x_1b_2 & x_1^2b_2 & c_2 & x_1c_2 & x_1^2c_2 \\ a_3 & x_1a_3 + a_2 & b_3 & x_1b_3 + b_2 & x_1^2b_3 + 2x_1b_2 & c_3 & x_1c_3 + c_2 & x_1^2c_3 + 2x_1c_2 \\ a_4 & x_1a_4 + a_3 & b_4 & x_1b_4 + b_3 & x_1^2b_4 + 2x_1b_3 + b_2 & c_4 & x_1c_4 + c_3 & x_1^2c_4 + 2x_1c_3 + c_2 \\ a_5 & x_2a_5 & b_5 & x_2b_5 & x_2^2b_5 & c_5 & x_2c_5 & x_1^2c_5 \\ a_6 & x_2a_6 + a_5 & b_6 & x_2b_6 + b_5 & x_1^2b_6 + 2x_1b_5 & c_6 & x_2c_6 + c_5 & x_2^2c_6 + 2x_2c_5 \\ a_7 & x_2a_7 + a_6 & b_7 & x_2b_7 + b_6 & x_1^2b_7 + 2x_1b_6 & c_7 & x_2c_7 + c_6 & x_2^2c_7 + 2x_2c_6 \end{array}\right]$$

# Multi-gradients

Finding a solution $\mathbf{p}$ of order $|\vec{n}| - 1$ and degree bound $\vec{n}$ equivalent to solving

$$\mathbf{K}(\vec{n}, |\vec{n}|) \cdot \mathbf{P} = [0, \dots, 0, 1]^T \qquad \text{( Note : we drop } \mathbf{C} \text{ and } \mathbf{F} \text{ from notation )}$$

Cramer's rule solution :

$$\mathbf{K}(\ \vec{n}, |\vec{n}|\ ) \cdot \mathbf{P} = d(\vec{n}) \cdot [0, \dots, 0, 1]^T, \quad \text{where} \quad d(\vec{n}) = \det(\ \mathbf{K}(\vec{n}, |\vec{n}|)\ ).$$

Equivalent to:

$$p^{(1)}(z) \cdot f^{(1)} + \cdots + p^{(m)}(z) \cdot f^{(m)} = r$$

where

$$p^{(\ell)}(z) \quad = \quad \det \left[ \begin{array}{ccc|c|ccc|c|ccc} \mathbf{F}^{(1)}_{|\vec{n}|-1} & \cdots & \mathbf{C}^{\vec{n}_1-1}_{|\vec{n}|-1} \mathbf{F}^{(1)}_{|\vec{n}|-1} & \cdots & \mathbf{F}^{(\ell)}_{|\vec{n}|-1} & \cdots & \mathbf{C}^{\vec{n}_\ell-1}_{|\vec{n}|-1} \mathbf{F}^{(\ell)}_{|\vec{n}|-1} & \cdots & \mathbf{F}^{(m)}_{|\vec{n}|-1} & \cdots & \mathbf{C}^{\vec{n}_m-1}_{|\vec{n}|-1} \mathbf{F}^{(m)}_{|\vec{n}|-1} \\ \hline 0 & \cdots & 0 & & 1 & \cdots & z^{\vec{n}_\ell-1} & & 0 & \cdots & 0 \end{array} \right]$$

$$r \quad = \quad \det \left[ \begin{array}{ccc|cc|ccc} \mathbf{F}^{(1)}_{|\vec{n}|-1} & \cdots & \mathbf{C}^{\vec{n}_1-1}_{|\vec{n}|-1} \mathbf{F}^{(1)}_{|\vec{n}|-1} & \cdots & \cdots & \mathbf{F}^{(m)}_{|\vec{n}|-1} & \cdots & \mathbf{C}^{\vec{n}_m-1}_{|\vec{n}|-1} \mathbf{F}^{(m)}_{|\vec{n}|-1} \\ \hline f^{(1)} & \cdots & z^{n_1-1} f^{(1)} & \cdots & \cdots & f^{(m)} & \cdots & z^{n_m-1} f^{(m)} \end{array} \right]$$

Note similarity to approach used with subresultants.

# Cramer's Rule revisited

$$\left[\begin{array}{ccc|c|ccc|c|ccc} \mathbf{F}^{(1)}_{|\vec{n}|} & \cdots & \mathbf{C}^{\vec{n}_1-1}_{|\vec{n}|}\mathbf{F}^{(1)}_{|\vec{n}|} & \cdots & \mathbf{F}^{(\ell)}_{|\vec{n}|} & \cdots & \mathbf{C}^{\vec{n}_\ell-1}_{|\vec{n}|}\mathbf{F}^{(\ell)}_{|\vec{n}|} & \cdots & \mathbf{F}^{(m)}_{|\vec{n}|} & \cdots & \mathbf{C}^{\vec{n}_m-1}_{|\vec{n}|}\mathbf{F}^{(m)}_{|\vec{n}|} \end{array}\right]\cdot\mathbf{P} = d(\vec{n})\cdot[0,\ldots,0,1]^T,$$

# Cramer's Rule revisited

$$\left[\begin{array}{ccc|c|ccc|c|ccc} \mathbf{F}^{(1)}_{|\vec{n}|} & \cdots & \mathbf{C}^{\vec{n}_1-1}_{|\vec{n}|}\mathbf{F}^{(1)}_{|\vec{n}|} & \cdots & \mathbf{F}^{(\ell)}_{|\vec{n}|} & \cdots & \mathbf{C}^{\vec{n}_\ell-1}_{|\vec{n}|}\mathbf{F}^{(\ell)}_{|\vec{n}|} & \cdots & \mathbf{F}^{(m)}_{|\vec{n}|} & \cdots & \mathbf{C}^{\vec{n}_m-1}_{|\vec{n}|}\mathbf{F}^{(m)}_{|\vec{n}|} \end{array}\right]\cdot\mathbf{P} = d(\vec{n})\cdot[0,\ldots,0,1]^T,$$

$$p^{(\ell)}(z) = \det\left[\begin{array}{ccc|c|ccc|c|ccc} \mathbf{F}^{(1)}_{|\vec{n}|-1} & \cdots & \mathbf{C}^{\vec{n}_1-1}_{|\vec{n}|-1}\mathbf{F}^{(1)}_{|\vec{n}|-1} & \cdots & \mathbf{F}^{(\ell)}_{|\vec{n}|-1} & \cdots & \mathbf{C}^{\vec{n}_\ell-1}_{|\vec{n}|-1}\mathbf{F}^{(\ell)}_{|\vec{n}|-1} & \cdots & \mathbf{F}^{(m)}_{|\vec{n}|-1} & \cdots & \mathbf{C}^{\vec{n}_m-1}_{|\vec{n}|-1}\mathbf{F}^{(m)}_{|\vec{n}|-1} \\ \hline 0 & \cdots & 0 & & 1 & \cdots & z^{\vec{n}_\ell-1} & & 0 & \cdots & 0 \end{array}\right]$$

$$r = \det\left[\begin{array}{ccc|cc|ccc} \mathbf{F}^{(1)}_{|\vec{n}|-1} & \cdots & \mathbf{C}^{\vec{n}_1-1}_{|\vec{n}|-1}\mathbf{F}^{(1)}_{|\vec{n}|-1} & \cdots & \cdots & \mathbf{F}^{(m)}_{|\vec{n}|-1} & \cdots & \mathbf{C}^{\vec{n}_m-1}_{|\vec{n}|-1}\mathbf{F}^{(m)}_{|\vec{n}|-1} \\ \hline f^{(1)} & \cdots & z^{n_1-1}f^{(1)} & \cdots & \cdots & f^{(m)} & \cdots & z^{n_m-1}f^{(m)} \end{array}\right]$$

# Cramer’s Rule revisited

$$\left[\begin{array}{cc|c|cc|c|cc} \mathbf{F}^{(1)}_{|\vec{n}|} & \cdots\ \mathbf{C}^{\vec{n}1-1}_{|\vec{n}|}\mathbf{F}^{(1)}_{|\vec{n}|} & \cdots & \mathbf{F}^{(\ell)}_{|\vec{n}|} & \cdots\ \mathbf{C}^{\vec{n}\ell-1}_{|\vec{n}|}\mathbf{F}^{(\ell)}_{|\vec{n}|} & \cdots & \mathbf{F}^{(m)}_{|\vec{n}|} & \cdots\ \mathbf{C}^{\vec{n}m-1}_{|\vec{n}|}\mathbf{F}^{(m)}_{|\vec{n}|} \end{array}\right]\cdot\mathbf{P} = d(\vec{n})\cdot[0,\ldots,0,1]^T,$$

$$p^{(\ell)}(z) = \det\left[\begin{array}{cc|c|cc|c|cc} \mathbf{F}^{(1)}_{|\vec{n}|-1} & \cdots\ \mathbf{C}^{\vec{n}1-1}_{|\vec{n}|-1}\mathbf{F}^{(1)}_{|\vec{n}|-1} & \cdots & \mathbf{F}^{(\ell)}_{|\vec{n}|-1} & \cdots\ \mathbf{C}^{\vec{n}\ell-1}_{|\vec{n}|-1}\mathbf{F}^{(\ell)}_{|\vec{n}|-1} & \cdots & \mathbf{F}^{(m)}_{|\vec{n}|-1} & \cdots\ \mathbf{C}^{\vec{n}m-1}_{|\vec{n}|-1}\mathbf{F}^{(m)}_{|\vec{n}|-1} \\ \hline 0 & \cdots\ 0 & & 1 & \cdots\ z^{\vec{n}\ell-1} & & 0 & \cdots\ 0 \end{array}\right]$$

$$\begin{aligned} r &= \det\left[\begin{array}{cc|cc|cc} \mathbf{F}^{(1)}_{|\vec{n}|-1} & \cdots\ \mathbf{C}^{\vec{n}1-1}_{|\vec{n}|-1}\mathbf{F}^{(1)}_{|\vec{n}|-1} & \cdots & \cdots & \mathbf{F}^{(m)}_{|\vec{n}|-1} & \cdots\ \mathbf{C}^{\vec{n}m-1}_{|\vec{n}|-1}\mathbf{F}^{(m)}_{|\vec{n}|-1} \\ \hline f^{(1)} & \cdots\ z^{n1-1}f^{(1)} & \cdots & \cdots & f^{(m)} & \cdots\ z^{nm-1}f^{(m)} \end{array}\right] \\ &= p^{(1)}(z)\cdot f^{(1)} + \cdots + p^{(m)}(z)\cdot f^{(m)} \end{aligned}$$

# Cramer's Rule revisited

$$\left[\begin{array}{ccc|c|ccc|c|ccc} \mathbf{F}^{(1)}_{|\vec{n}|} & \cdots & \mathbf{C}^{\vec{n}_1-1}_{|\vec{n}|}\mathbf{F}^{(1)}_{|\vec{n}|} & \cdots & \mathbf{F}^{(\ell)}_{|\vec{n}|} & \cdots & \mathbf{C}^{\vec{n}_\ell-1}_{|\vec{n}|}\mathbf{F}^{(\ell)}_{|\vec{n}|} & \cdots & \mathbf{F}^{(m)}_{|\vec{n}|} & \cdots & \mathbf{C}^{\vec{n}_m-1}_{|\vec{n}|}\mathbf{F}^{(m)}_{|\vec{n}|} \end{array}\right]\cdot\mathbf{P} = d(\vec{n})\cdot[0,\ldots,0,1]^T,$$

$$\begin{aligned} p^{(\ell)}(z) &= \det\left[\begin{array}{ccc|c|ccc|c|ccc} \mathbf{F}^{(1)}_{|\vec{n}|-1} & \cdots & \mathbf{C}^{\vec{n}_1-1}_{|\vec{n}|-1}\mathbf{F}^{(1)}_{|\vec{n}|-1} & \cdots & \mathbf{F}^{(\ell)}_{|\vec{n}|-1} & \cdots & \mathbf{C}^{\vec{n}_\ell-1}_{|\vec{n}|-1}\mathbf{F}^{(\ell)}_{|\vec{n}|-1} & \cdots & \mathbf{F}^{(m)}_{|\vec{n}|-1} & \cdots & \mathbf{C}^{\vec{n}_m-1}_{|\vec{n}|-1}\mathbf{F}^{(m)}_{|\vec{n}|-1} \\ \hline 0 & \cdots & 0 & & 1 & \cdots & z^{\vec{n}_\ell-1} & & 0 & \cdots & 0 \end{array}\right] \\ &= \pm\ \det\left(\left[\begin{array}{c|ccc|c} \cdots & \mathbf{F}^{(\ell)}_{|\vec{n}|-1} & \cdots & \mathbf{C}^{\vec{n}_\ell-2}_{|\vec{n}|-1}\mathbf{F}^{(\ell)}_{|\vec{n}|-1} & \cdots \end{array}\right]\right)\cdot z^{\vec{n}_\ell-1} + \text{lower order terms} \end{aligned}$$

$$\begin{aligned} r &= \det\left[\begin{array}{ccc|cc|ccc} \mathbf{F}^{(1)}_{|\vec{n}|-1} & \cdots & \mathbf{C}^{\vec{n}_1-1}_{|\vec{n}|-1}\mathbf{F}^{(1)}_{|\vec{n}|-1} & \cdots & \cdots & \mathbf{F}^{(m)}_{|\vec{n}|-1} & \cdots & \mathbf{C}^{\vec{n}_m-1}_{|\vec{n}|-1}\mathbf{F}^{(m)}_{|\vec{n}|-1} \\ \hline f^{(1)} & \cdots & z^{n_1-1}f^{(1)} & \cdots & \cdots & f^{(m)} & \cdots & z^{n_m-1}f^{(m)} \end{array}\right] \\ &= p^{(1)}(z)\cdot f^{(1)} + \cdots + p^{(m)}(z)\cdot f^{(m)} \\ &= \det\left(\left[\begin{array}{c|ccc|c} \cdots & \mathbf{F}^{(\ell)}_{|\vec{n}|-1} & \cdots & \mathbf{C}^{\vec{n}_\ell-1}_{|\vec{n}|-1}\mathbf{F}^{(\ell)}_{|\vec{n}|-1} & \cdots \end{array}\right]\right)\cdot \omega_{|\vec{n}|} + \text{higher order terms} \end{aligned}$$

# Cramer's Rule revisited

$$\left[\begin{array}{ccc|c|ccc|c|ccc} \mathbf{F}_{|\vec{n}|}^{(1)} & \cdots & \mathbf{C}_{|\vec{n}|}^{\vec{n}1-1}\mathbf{F}_{|\vec{n}|}^{(1)} & \cdots & \mathbf{F}_{|\vec{n}|}^{(\ell)} & \cdots & \mathbf{C}_{|\vec{n}|}^{\vec{n}\ell-1}\mathbf{F}_{|\vec{n}|}^{(\ell)} & \cdots & \mathbf{F}_{|\vec{n}|}^{(m)} & \cdots & \mathbf{C}_{|\vec{n}|}^{\vec{n}m-1}\mathbf{F}_{|\vec{n}|}^{(m)} \end{array}\right]\cdot\mathbf{P} = d(\vec{n})\cdot[0,\ldots,0,1]^T,$$

$$\begin{aligned} p^{(\ell)}(z) &= \det\left[\begin{array}{ccc|c|ccc|c|ccc} \mathbf{F}_{|\vec{n}|-1}^{(1)} & \cdots & \mathbf{C}_{|\vec{n}|-1}^{\vec{n}1-1}\mathbf{F}_{|\vec{n}|-1}^{(1)} & \cdots & \mathbf{F}_{|\vec{n}|-1}^{(\ell)} & \cdots & \mathbf{C}_{|\vec{n}|-1}^{\vec{n}\ell-1}\mathbf{F}_{|\vec{n}|-1}^{(\ell)} & \cdots & \mathbf{F}_{|\vec{n}|-1}^{(m)} & \cdots & \mathbf{C}_{|\vec{n}|-1}^{\vec{n}m-1}\mathbf{F}_{|\vec{n}|-1}^{(m)} \\ \hline 0 & \cdots & 0 & & 1 & \cdots & z^{\vec{n}\ell-1} & & 0 & \cdots & 0 \end{array}\right] \\ &= \pm\ \det\left(\left[\begin{array}{c|ccc|c} \cdots & \mathbf{F}_{|\vec{n}|-1}^{(\ell)} & \cdots & \mathbf{C}_{|\vec{n}|-1}^{\vec{n}\ell-2}\mathbf{F}_{|\vec{n}|-1}^{(\ell)} & \cdots \end{array}\right]\right)\cdot z^{\vec{n}\ell-1} + \text{ lower order terms} \\ &= \pm\, d(\vec{n}-\vec{e}_\ell)\cdot z^{\vec{n}\ell-1} + \text{ lower order terms} \end{aligned}$$

$$\begin{aligned} r &= \det\left[\begin{array}{ccc|cc|ccc} \mathbf{F}_{|\vec{n}|-1}^{(1)} & \cdots & \mathbf{C}_{|\vec{n}|-1}^{\vec{n}1-1}\mathbf{F}_{|\vec{n}|-1}^{(1)} & \cdots & \cdots & \mathbf{F}_{|\vec{n}|-1}^{(m)} & \cdots & \mathbf{C}_{|\vec{n}|-1}^{\vec{n}m-1}\mathbf{F}_{|\vec{n}|-1}^{(m)} \\ \hline f^{(1)} & \cdots & z^{n1-1}f^{(1)} & \cdots & \cdots & f^{(m)} & \cdots & z^{nm-1}f^{(m)} \end{array}\right] \\ &= p^{(1)}(z)\cdot f^{(1)} + \cdots + p^{(m)}(z)\cdot f^{(m)} \\ &= \det\left(\left[\begin{array}{c|ccc|c} \cdots & \mathbf{F}_{|\vec{n}|-1}^{(\ell)} & \cdots & \mathbf{C}_{|\vec{n}|-1}^{\vec{n}\ell-1}\mathbf{F}_{|\vec{n}|-1}^{(\ell)} & \cdots \end{array}\right]\right)\cdot\omega_{|\vec{n}|} + \text{ higher order terms} \\ &= d(\vec{n})\cdot\omega_{|\vec{n}|} + \text{ higher order terms} \end{aligned}$$

# When $d(\vec{n}) \neq 0$

## Theorem

*For* $K(\vec{n}, |\vec{n}|)$ *denoting*

$$\left[\begin{array}{ccc|c|ccc|c|ccc} \mathbf{F}^{(1)}_{|\vec{n}|} & \cdots & \mathbf{C}^{\vec{n}_1 - 1}_{|\vec{n}|}\mathbf{F}^{(1)}_{|\vec{n}|} & \cdots & \mathbf{F}^{(\ell)}_{|\vec{n}|} & \cdots & \mathbf{C}^{\vec{n}_\ell - 1}_{|\vec{n}|}\mathbf{F}^{(\ell)}_{|\vec{n}|} & \cdots & \mathbf{F}^{(m)}_{|\vec{n}|} & \cdots & \mathbf{C}^{\vec{n}_m - 1}_{|\vec{n}|}\mathbf{F}^{(m)}_{|\vec{n}|} \end{array}\right]$$

*Then* $d(\vec{n}) =$ *det* $K(\vec{n}, |\vec{n}|) \neq 0$ *iff we can solve:*

$$K(\vec{n}, |\vec{n}|) \cdot \mathbf{P}^{(\ell)} \quad = \quad -d(\vec{n}) \cdot \mathbf{C}^{\vec{n}_\ell}_{|\vec{n}|}\mathbf{F}^{(\ell)}_{|\vec{n}|}, \qquad \ell = 1, \ldots, m.$$

# When $d(\vec{n}) \neq 0$

## Theorem

*For* $K(\vec{n}, |\vec{n}|)$ *denoting*

$$\left[ \begin{array}{ccc|c|ccc|c|ccc} \mathbf{F}_{|\vec{n}|}^{(1)} & \cdots & \mathbf{C}_{|\vec{n}|}^{\vec{n}_1 - 1}\mathbf{F}_{|\vec{n}|}^{(1)} & \cdots & \mathbf{F}_{|\vec{n}|}^{(\ell)} & \cdots & \mathbf{C}_{|\vec{n}|}^{\vec{n}_\ell - 1}\mathbf{F}_{|\vec{n}|}^{(\ell)} & \cdots & \mathbf{F}_{|\vec{n}|}^{(m)} & \cdots & \mathbf{C}_{|\vec{n}|}^{\vec{n}_m - 1}\mathbf{F}_{|\vec{n}|}^{(m)} \end{array} \right]$$

*Then* $d(\vec{n}) =$ *det* $K(\vec{n}, |\vec{n}|) \neq 0$ *iff we can solve:*

$$K(\vec{n}, |\vec{n}|) \cdot \mathbf{P}^{(\ell)} \quad = \quad -d(\vec{n}) \cdot \mathbf{C}_{|\vec{n}|}^{\vec{n}_\ell} \mathbf{F}_{|\vec{n}|}^{(\ell)}, \qquad \ell = 1, \ldots, m.$$

Same as solving $p^{(1)}(z) \cdot f^{(1)} + \cdots + p^{(m)}(z) \cdot f^{(m)} = r^{(\ell)}$

with : deg $p^{(\ell)}(z) = \vec{n}_\ell$, deg $p^{(i)}(z) \leq \vec{n} - \vec{e}_i$, : order $r^{(\ell)} \geq |\vec{n}| + 1$.

and : coeff( $p^{(\ell)}(z), z, n_\ell) = d(\vec{n})$) and $c_{|\vec{n}|+1}(r^{(\ell)} = d(\vec{n} + \vec{e}_i)$

# When $d(\vec{n}) \neq 0$

## Theorem

*For* $K(\vec{n}, |\vec{n}|)$ *denoting*

$$\left[\begin{array}{ccc|c|ccc|c|ccc} \mathbf{F}_{|\vec{n}|}^{(1)} & \cdots & \mathbf{C}_{|\vec{n}|}^{\vec{n}_1-1}\mathbf{F}_{|\vec{n}|}^{(1)} & \cdots & \mathbf{F}_{|\vec{n}|}^{(\ell)} & \cdots & \mathbf{C}_{|\vec{n}|}^{\vec{n}_\ell-1}\mathbf{F}_{|\vec{n}|}^{(\ell)} & \cdots & \mathbf{F}_{|\vec{n}|}^{(m)} & \cdots & \mathbf{C}_{|\vec{n}|}^{\vec{n}_m-1}\mathbf{F}_{|\vec{n}|}^{(m)} \end{array}\right]$$

*Then* $d(\vec{n}) =$ *det* $K(\vec{n}, |\vec{n}|) \neq 0$ *iff we can solve:*

$$K(\vec{n}, |\vec{n}|) \cdot \mathbf{P}^{(\ell)} \quad = \quad -d(\vec{n}) \cdot \mathbf{C}_{|\vec{n}|}^{\vec{n}_\ell}\mathbf{F}_{|\vec{n}|}^{(\ell)}, \qquad \ell = 1, \ldots, m.$$

Same as solving $p^{(1)}(z) \cdot f^{(1)} + \cdots + p^{(m)}(z) \cdot f^{(m)} = r^{(\ell)}$

with : deg $p^{(\ell)}(z) = \vec{n}_\ell$, deg $p^{(i)}(z) \leq \vec{n} - \vec{e}_i$, : order $r^{(\ell)} \geq |\vec{n}| + 1$.

and : coeff( $p^{(\ell)}(z), z, n_\ell) = d(\vec{n})$) and $c_{|\vec{n}|+1}(r^{(\ell)} = d(\vec{n} + \vec{e}_i)$

Notation : $p(\vec{n}, z) = [p^{(1)}(z), \ldots, p^{(m)}(z)]$

# Mahler Systems

## Definition

Given $\sigma$ and $\vec{n}$ a Mahler System is

$$\mathbf{M}_\sigma = [M_\sigma^{(\cdot,\ 1)}, \ldots,\ \mathbf{M}_\sigma^{(\cdot,\ m)}]$$

where the $i$th column is $\pm\, p(\vec{n} + \vec{e}_i, z)$.

**Note :** Mahler systems have degrees bounded by

$$\deg \mathbf{M}_\sigma = \begin{bmatrix} n_1 & n_1 - 1 & \cdots & n_1 - 1 \\ n_2 - 1 & n_2 & & n_2 - 1 \\ \vdots & & \ddots & \\ n_m - 1 & \cdots & n_m - 1 & n_m \end{bmatrix}$$

NOTE: Leading coefficents of diagonals : $d(\vec{n})$.

# Finding *all* Solutions of Order Problems

## Theorem

*Let*

- $\vec{n}^{\,(k)}$ *defined by* $\vec{n}^{\,(k)} = \vec{n}^{\,(k-1)} + \vec{e}_{j_k}$
- $\vec{v}^{\,(k)}$ *denote the closest normal point to* $\vec{n}^{\,(k)}$
- $\mathbf{M}_\sigma^{(\cdot,\,1)}, \ldots, \mathbf{M}_\sigma^{(\cdot,\,m)}$ *denote the columns of Mahler system* $\mathbf{M}_\sigma$*.*

*Then any of solution* $\mathbf{P}$ *of order* $\sigma$ *and degree bound* $\vec{n}^{\,(k)}$ *satisfies*

$$\mathbf{P} = \alpha_1(z) \cdot \mathbf{M}_\sigma^{(\cdot,\,1)} \;+\; \ldots \;+\; \alpha_m(z)\, \mathbf{M}_\sigma^{(\cdot,\,m)}$$

*where deg* $(\, \alpha_i(z) \,) < n_i^{(k)} - \vec{v}_i^{\,(k)}$*.*