

# Recent Progress with Order Bases

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# Outline

- 1 Introduction
  - Purpose
  - General Setting
- 2 Rational Approximation
  - Linear Systems
- 3 Order Bases
  - Background
  - Computation
  - Fraction-Free Computation
  - Recursive Computation
  - Matrix Normal Forms
  - Current Activities

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# Purpose

We give a bit of progress on the following problem :

*Find **all** solutions of rational approximation problems*

- Examples : Padé, Hermite-Padé, Simultaneous Padé, . . . ,
- Extensions : matrix versions, rational interpolation versions, . . .

# Hermite-Padé Approximation

Given power series  $A_1(z), \dots, A_m(z)$  and integers  $n_0, \dots, n_m$

Find  $P_1(z), \dots, P_m(z)$  with  $\deg P_i(z) \leq n_i$  and

$$A_1(z)P_1(z) + \dots + A_m(z)P_m(z) \approx 0 .$$

Formally:

$$A_1(z)P_1(z) + \dots + A_m(z)P_m(z) = r_0z^{N+1} + r_1z^{N+2} + \dots$$

with  $N = n_1 + \dots + n_m + m - 1$ .

(  $m = 2$  and  $A_1(z) = -1$  gives Padé approximation )

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## Examples

Given power series  $y(z)$  find  $P_0(z), P_1(z), P_2(z)$  such that

- $P_0(z)y(z) + P_1(z)y'(z) + P_2(z)y''(z) \approx 0$
- $P_0(z) + P_1(z)y(z) + P_2(z)y^2(z) \approx 0$

(generalized rational reconstruction)

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# Rational Approximation Problems

Given  $m \times m$  matrix  $\mathbf{A}(z)$ , orders,  $\vec{\sigma} = (\sigma_1, \dots, \sigma_m)$ , find **basis** of solutions of

$$\mathbf{A}(z) \cdot \mathbf{Q}(z) = z^{\vec{\sigma}} \mathbf{R}(z).$$

with some added degree constraints  $\vec{n} = (n_1, \dots, n_m)$

$$\deg Q_i(z) \leq n_i.$$

$\mathbf{R}(z)$  called residual.

Question : Basis in what sense?

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# Applications

Rational approximation problems appear in:

- Transcendence of  $e$  and other famous numbers
- Inversion formulae for structured matrices (scalar and block)
- Linear diophantine equations (and hence to GCDs)
- Guessing recurrence formulae (e.g. Gfun)
- Reconstruction of power series to polynomial problems (e.g. DFactor)
- Matrix normal forms (Popov, etc)
- Fast polynomial matrix arithmetic
- ...

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# Associated Linear System

$$A(z)V_n(z) - U_m(z) = z^{m+n+1}W(z)$$

$$(a_0 + a_1z + \dots)(v_0 + \dots + v_nz^n) - (u_0 + \dots + u_mz^m) = z^{m+n+1}w_0 + z^{m+n+2}w_1 + \dots$$

Same as

$$\begin{bmatrix} a_{m-n+1} & \cdots & \cdots & a_{n-1} & a_n \\ a_{m-n+2} & \cdots & \cdots & a_n & a_{n+1} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ a_{n-1} & \cdots & \cdots & a_{m+n-2} & a_{m+n-1} \\ a_n & \cdots & \cdots & a_{m+n-1} & \end{bmatrix} \cdot \begin{bmatrix} v_n \\ v_{n-1} \\ \vdots \\ \vdots \\ v_2 \\ v_1 \end{bmatrix} = -v_0 \begin{bmatrix} a_{m+1} \\ a_{m+2} \\ \vdots \\ \vdots \\ a_{m+n-1} \\ a_{m+n} \end{bmatrix}$$

Similarly for  $a_i$  square matrices.

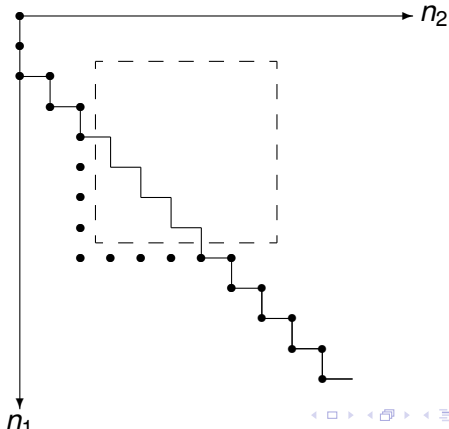
Similarly have structured linear system for other approx problems.

Nice when coefficient matrix is nonsingular.

## Additional Information

- **All** Padé approximants known in scalar case
  - Padé table in scalar case has a type of block structure
- Padé approximants related to diophantine equations
  - There are algorithms corresponding to Euclidean algorithm
  - Fast way to compute Padé approximants in scalar case
- Nothing known about structure of matrix Padé case or Hermite-Padé case by 1990
- Use in inversion formulas for Hankel and Toeplitz matrices gives rise to efficient numerically stable algorithms.

Structure of scalar Padé table helpful for algorithms. For example, a staircase path of computation in a Padé table:



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## People

- K. Mahler(1925-1969), J. Coates(1965) , J. Della Dora (1980),  
- ( strong conditions always assumed )
- [B. Beckermann](#) and G. Labahn; A. Bultheel and M. van Barel
- B. Salvy and P. Zimmermann (gfun), (and H. Derksen) ; M. Rubey
- M. van Hoeij (use in differential factorization)
- G. Villard; Beckermann, Labahn, Villard (matrix normal forms)
- P. Giorgi, C-P. Jeannerod, G. Villard (fast polynomial matrix arithmetic)
- B. Beckermann, H. Cheng, G. Labahn ( Noncommutative domains ); . . .

## **Bernhard Beckermann**



# Order Bases

Idea : look at order condition independently of degree bounds,

$$R_\sigma = \{\mathbf{Q}(z) \in F^{(m)}[z] \mid \mathbf{A}(z) \cdot \mathbf{Q}(z) = O(z^\sigma)\}$$

Find basis of  $R_\sigma$  as a *module* over  $F[z]$ .

- Basis always has  $m$  elements
- Write as columns of an  $m \times m$  matrix polynomial  $\mathbf{M}(z)$ .

# Example

Let

$$\mathbf{A}(z) = \begin{bmatrix} \frac{1}{2} + z^2 - z^4 & 1 + \sin(z^2)^4 & \frac{1}{\sqrt{1+z^2}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and  $\vec{\sigma} = (z^8, 1, 1)$ . Then a **basis for all solutions** given by

$$\mathbf{M}(z) = \begin{bmatrix} z^4 + \frac{11z^2}{2} & -\frac{10z^2}{19} + \frac{2}{19} & \frac{9z^2}{19} - \frac{11}{76} \\ -\frac{59z^2}{4} & z^2 - \frac{33}{19} & -\frac{5z^2}{4} + \frac{59}{152} \\ 12z^2 & \frac{32}{19} & z^2 - \frac{6}{19} \end{bmatrix}$$

with  $\det \mathbf{M}(z) = z^8$ . In this case the first 4 terms of the order residual  $\mathbf{R}$  of  $\mathbf{M}$  are given by

$$\mathbf{R}(z) = \begin{bmatrix} -\frac{19}{4} - \frac{367}{32}z^2 - \frac{189}{64}z^4 + O(z^6) & -\frac{97}{76} + \frac{89}{152}z^2 + O(z^4) & -\frac{13}{1216} - \frac{1093}{1216}z^2 + O(z^4) \\ -\frac{59z^2}{4} & -\frac{33}{19} + z^2 & \frac{59}{152} - \frac{5z^2}{4} \\ 12z^2 & \frac{32}{19} & -\frac{6}{19} + z^2 \end{bmatrix}.$$

# Order Bases

Degree bounds? Given  $\vec{n} = (n_1, \dots, n_m)$ :

Then

$$\mathbf{Q}(z) = \alpha_1(z)\mathbf{M}_1(z) + \dots + \alpha_m(z)\mathbf{M}_m(z)$$

with

$$\deg \alpha_i(z) \leq \text{defect } \mathbf{Q}(z) - \text{defect } \mathbf{M}_i(z)$$

Here **defect** is a measure of the difference between degrees and bounds  $n_i$ .

Implies  $\mathbf{M}(z)$  describes **all** solutions of  $\mathbf{A}(z)\mathbf{Q}(z) = O(z^\sigma)$

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# Computation Complexity

Hermite-Padé approx. with degree bounds  $(n_1, \dots, n_m)$

Set  $N = n_1 + \dots + n_m$ . Then :

- Linear algebra :  $O(N^3)$
- Sigma basis :  $O(N \log^2 N)$  i.e. -  $O(N^{1+\epsilon})$  in scalar case (BL - 1994)
- MBasis :  $O(m^\omega N^{1+\epsilon})$  - in case of matrix input (GJV - 2003)
- Generating set :  $O(m^\omega (N/m)^{1+\epsilon})$  (Storjohann. 2006)
- Order basis :  $O(m^\omega (N/m)^{1+\epsilon})$  Zhou (2008)

# Sigma Basis Algorithm [SIMAX - BL]

- 1 Start with Order basis =  $\mathbf{I}$  and order = 0.
- 2 Of all the columns that need to have order increased:
  - pick one with minimal defect.
  - use to eliminate other columns needing order increase.
- 3 Multiply pivot column by  $z$ . Continue. Quadratic complexity.

Double order everytime : obtain superfast version.

## Alternatively (GJV)

To get  $(\sigma + 1)$ -basis from a  $\sigma$ -basis do:

- 1 we compute the terms in  $z^\sigma$  in the residue  $\mathbf{R}(z)$ . This gives us a matrix  $\Delta$
- 2 we compute a row echelon form of  $\Delta$
- 3 we apply some transformations according to the row echelon form. These transformations are of two types :
  - Either  $\mathbf{M}_j$  is replaced by a linear combination of some  $\mathbf{M}_j$
  - Or all the polynomials in  $\mathbf{M}_j$  are multiplied by  $z$

## Example : calculation of a 1-basis

$$\mathbf{A}(z) = \begin{bmatrix} z+1 & z^2-1 & 5 \\ 2z-1 & 1 & z \end{bmatrix} \text{ and } \mathbf{M}(z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Then: } \Delta = \begin{bmatrix} 1 & -1 & 5 \\ -1 & 1 & 0 \end{bmatrix} \xrightarrow{\text{row echelon form}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So we replace the 2<sup>nd</sup> column by the 2<sup>nd</sup> column minus the 1<sup>st</sup> column, and we shift column 1 and 3 because they were used

$$\text{as pivots to obtain the 1-basis : } \mathbf{M}(z) = \begin{bmatrix} z & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z \end{bmatrix}$$

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# Fraction-free Computation

Fraction-free computation for given  $\vec{n}$ .

- set up linear system (structured Krylov matrix) for each order
- find so-called Cramer's solutions
- eliminate known divisors by a type of Sylvester's identity
- Order basis called Mahler system

Also version for Ore case. Also modular versions.

**Goal:** Try to find Cramer solutions

- e.g. Hermite-Padé problem

$$a(x) \cdot p(x) + b(x) \cdot q(x) + c(x) \cdot r(x) = O(x^6)$$

with  $\deg p(x) \leq 2$ ,  $\deg q(x) \leq 1$ ,  $\deg r(x) \leq 1$

$$\begin{bmatrix} a_0 & 0 & 0 & | & b_0 & 0 & | & c_0 & 0 \\ a_1 & a_0 & 0 & | & b_1 & b_0 & | & c_1 & c_0 \\ a_2 & a_1 & a_0 & | & b_2 & b_1 & | & c_2 & c_1 \\ a_3 & a_2 & a_1 & | & b_3 & b_2 & | & c_3 & c_2 \\ a_4 & a_3 & a_2 & | & b_4 & b_3 & | & c_4 & c_3 \\ a_5 & a_4 & a_3 & | & b_5 & b_4 & | & c_5 & c_4 \\ a_6 & a_5 & a_4 & | & b_6 & b_5 & | & c_6 & c_5 \end{bmatrix} \cdot \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \hline q_0 \\ q_1 \\ \hline r_0 \\ r_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ d \end{bmatrix}$$

where  $d$  is determinant of coefficient matrix.

- Solution has determinant representation in nonsingular case:

$$\text{e.g. } p(z) = \det \left[ \begin{array}{ccc|ccc|c} a_0 & 0 & 0 & b_0 & 0 & c_0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & c_1 & c_0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & c_2 & c_1 \\ a_3 & a_2 & a_1 & b_3 & b_2 & c_3 & c_2 \\ a_4 & a_3 & a_2 & b_4 & b_3 & c_4 & c_3 \\ 1 & z & z^2 & 0 & 0 & 0 & 0 \end{array} \right]$$

- Unique in nonsingular case.
- Recursively build Cramer solutions from Cramer solutions of smaller problems along offdiagonal of associated table.

- Matrix  $\mathbf{M}(z)$  of determinantal polynomials with degrees

$$\begin{bmatrix} n_1 & n_1 - 1 & \cdots & n_1 - 1 \\ n_2 - 1 & n_2 & \cdots & n_2 - 1 \\ \vdots & & \ddots & \vdots \\ n_m - 1 & \cdots & \cdots & n_m \end{bmatrix}$$

and lcoeff of diagonal = determinant of coeff matrix.

- Unique in nonsingular case
- Basic building block of recursions.
- Method 1: via modified Schur complements
  - nonsingular location to nonsingular location in table
  - similar to look ahead
- Method 2: via determinantal identities
  - works in singular cases by computing at closest nonsingular locations (look around)

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## Recursive Computation

Order bases problems with degree and order having equal status.

Find  $\mathbf{M}(z)$  satisfying:

$$\mathbf{A}(z) \cdot \mathbf{M}(z) = z^{\vec{\sigma}} \mathbf{R}(z)$$

$$\mathbf{H}(z^{-1}) \cdot \mathbf{M}(z) = O(z^0)_{z \rightarrow \infty}$$

- Order bases if  $\mathbf{R}(0)$  nonsingular. Of  $\mathbf{H}$ -degree if the second residual starts with nonsingular matrix.
- B-L (1997) show how this can be solved recursively.  
Advantage : allows one to specify paths of computation via order or degrees.

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## Shifted Popov Normal Forms

Shifted Popov form problem : gives

$$\mathbf{A}(z) \cdot \mathbf{U}(z) = \mathbf{P}(z)$$

Embed normal form problem inside part of a Mahler system for

$$[\mathbf{A}(z), -\mathbf{I}] \begin{bmatrix} \mathbf{V}(z) & \mathbf{U}(z) \\ \mathbf{Q}(z) & \mathbf{P}(z) \end{bmatrix} = [O(z^{\vec{n}}), 0].$$

Fraction-free computation of normal forms.

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Currently we are working on:

- Fast algorithms in differential case
- Better fraction-free algorithms
- Use with differential-algebraic problems
  - Popov forms
  - Invariants and algorithms
- Alternate bases
- Multivariate Order Bases