

Rational Invariants of Finite Abelian Groups

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- construct rational invariants of action
 - * rational invariant : $f \in \mathbb{K}(\mathbf{x}) : f(g \cdot \mathbf{x}) = f(\mathbf{x}) \quad \forall g \in \mathcal{G}$
 - determine rewrite rules for this action

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 - determine rewrite rules for this action

- (2) Given system of polynomial equations
 - if have group action then 'reduce' polynomial system
 - conversely : determine finite abelian group action
(if possible)

References

This talk is a report on paper

- Hubert & Labahn, [Rational invariants of Finite Abelian Groups](#),
To appear in *Mathematics of Computation*

Relevant other publications

- K. Gatermann (ISSAC 1990)
 - : Using group actions to reduce Gröbner bases comp.
- J-C Faugère and J. Svartz (ISSAC 2013)
 - : Using abelian group actions to reduce polynomial systems.
- E. Hubert and G. Labahn (ISSAC 2012, FoCM 2013)
 - : Scaling symmetries

Example : Invariant Polynomial System

Consider the following system of polynomial equations

$$x_1 + x_2 + x_3 - x_1x_2 - x_1x_3 - x_2x_3 + 12 = 0$$

$$x_1x_2 + x_2x_3 + x_1x_3 - 15 = 0$$

$$x_1x_2x_3 - 13 = 0$$

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$$x_1x_2x_3 - 13 = 0$$

Solution space of system is **invariant** under the order 3 permutation

$$(x_1, x_2, x_3) \rightarrow (x_2, x_3, x_1).$$

Goal : work “modulo” this order 3 permutation.

Example : Invariant Polynomial System

(i) Find invariants

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} (3x_1^2x_2 + 3x_1x_3^2 + 3x_2^2x_3)\eta^2 + (3x_1^2x_3 + 3x_1x_2^2 + 3x_2x_3^2)\eta + (x_1^3 + 6x_1x_2x_3 + x_2^3 + x_3^3) \\ (x_1x_2 + x_1x_3 + x_2x_3)\eta^2 + (x_1x_2 + x_1x_3 + x_2x_3)\eta + (x_1^2 + x_2^2 + x_3^2) \\ x_1 + x_2 + x_3 \end{bmatrix}.$$

(η primitive cube root of unity)

(ii) Rewrite system in terms of invariants

$$\begin{aligned} 3y_2 + 3y_3 - 3y_3^2 + 12 &= 0, \\ -3y_2 + 3y_3^2 - 15 &= 0, \\ y_1 + \frac{y_2^3}{y_1} + y_3^3 - 3y_2y_3 - 13 &= 0. \end{aligned}$$

(iii) Solve invariant system for (y_1, y_2, y_3) (2 solutions)

(iv) Work back to get (x_1, x_2, x_3) (6 solutions)

The Process

- Change coordinates
- Scaling actions
- Arithmetic with exponents
- Polynomial system : solve and work back

The Process : \mathbb{Z}_3

- Change to 'Fourier' coordinates $(x_1, x_2, x_3) \rightarrow (z_1, z_2, z_3)$:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \eta & \eta^2 & 1 \\ \eta^2 & \eta & 1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

where η is a primitive cube root of unity.

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- Polynomial system now looks like :

$$0 = 3z_1z_2 + 3z_3 - 3z_3^2 + 12$$

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where η is a primitive cube root of unity.

- Group action now looks like :

$$(z_1, z_2, z_3) \rightarrow (\eta \cdot z_1, \eta^2 \cdot z_2, z_3)$$

Note : Action looks like 'rescaling' of coordinates

The Process : \mathbb{Z}_3

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where η is a primitive 3 root of unity.

- Group action now looks like :

$$(z_1, z_2, z_3) \rightarrow (\eta \cdot z_1, \eta^2 \cdot z_2, z_3)$$

Notice : $\frac{z_2}{z_1}, z_2^3, z_1 z_2 z_3$ all rational invariant functions

The Process : \mathbb{Z}_3

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- Group action now looks like :

$$(z_1, z_2, z_3) \rightarrow (\eta \cdot z_1, \eta^2 \cdot z_2, z_3)$$

Notice : $z_1^3, z_1 \cdot z_2, z_3$ all rational invariant functions

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How to rewrite polynomial system in terms of invariants?

The Process : \mathbb{Z}_3

- Group action now looks like :

$$(z_1, z_2, z_3) \rightarrow (\eta \cdot z_1, \eta^2 \cdot z_2, z_3)$$

Notice :

$$y = z_1^a \cdot z_2^b \cdot z_3^c = \eta^{a+2b} z_1^a \cdot z_2^b \cdot z_3^c$$

is a rational invariant function iff

$$a + 2b \equiv 0 \pmod{3}$$

- Kernel determined via integer linear algebra on exponents.
- Rewrite rules reverse such kernel operations

Original transformed polynomial system:

$$0 = 3z_1z_2 + 3z_3 - 3z_3^2 + 12$$

$$0 = -3z_1z_2 + 3z_3^2 - 15$$

$$0 = z_1^3 + z_2^3 + z_3^3 - 3z_1z_2z_3 - 13$$

* $y_1 = z_1^3$, $y_2 = z_1 \cdot z_2$, $y_3 = z_3$ – rational invariants

* $z_1 = y_1^{1/3}$, $z_2 = \frac{y_2}{y_1^{1/3}}$, $z_3 = y_3$ – rewrite rules

- Rational invariant system

$$0 = 3y_2 + 3y_3 - 3y_3^2 + 12$$

$$0 = -3y_2 + 3y_3^2 - 15$$

$$0 = y_1 + \frac{y_2^3}{y_1} + y_3^3 - 3y_2y_3 - 13$$

General Process : $\mathcal{G} \subset GL_n(\mathbb{K})$

- (1) Fourier step \equiv matrix diagonalization
- (2) Finite group + diagonalization \equiv scaling
 - order and exponent matrices
- (3) Rational invariants \equiv kernel of exponents + order
 - integer linear algebra
- (4) Rewrite rules \equiv 'inverting' kernels

'Fourier Step = Diagonalization'

\mathcal{G} : finite abelian subgroup $GL_n(\mathbb{K})$ (order $p = p_1 \cdots p_s$)

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- \exists matrix R such that $\mathcal{D} = R^{-1} \cdot \mathcal{G} \cdot R$ all diagonal matrices

'Fourier Step = Diagonalization'

\mathcal{G} : finite abelian subgroup $GL_n(\mathbb{K})$ (order $p = p_1 \cdots p_s$)

(i) \mathcal{G} is diagonalizable .

- \exists matrix R such that $\mathcal{D} = R^{-1} \cdot \mathcal{G} \cdot R$ all diagonal matrices
- Let $\mathbf{x} = R \cdot \mathbf{z}$. Then have diagonal action

$$\begin{aligned} \mathcal{D} \times \mathbb{K}^n &\rightarrow \mathbb{K}^n \\ (\text{diag}(d_1, \dots, d_n), (z_1, \dots, z_n)) &\mapsto (d_1 \cdot z_1, \dots, d_n \cdot z_n) \end{aligned}$$

'Exponents = Finite Direct Sum'

\mathcal{G} : finite abelian subgroup $GL_n(\mathbb{K})$ (order $p = p_1 \cdots p_s$)

(ii) Group isomorphism : $\mathcal{D} \leftrightarrow \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_s}$

Explicit via exponents :

$$\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_s} \rightarrow \mathcal{D}$$

$$(m_1, \dots, m_s) \mapsto D_1^{m_1} \cdots D_s^{m_s}$$

'Exponents = Finite Direct Sum'

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$$(m_1, \dots, m_s) \mapsto D_1^{m_1} \cdots D_s^{m_s}$$

Diagonal action:

$$(\text{diag}(d_1, \dots, d_n), (z_1, \dots, z_n)) \mapsto (d_1 \cdot z_1, \dots, d_n \cdot z_n)$$

with each $d_j = D_1^{m_{1j}} \cdots D_s^{m_{sj}}$

Notation

- Diagonal action : $(\alpha, \beta) \in \mathbb{Z}_7 \times \mathbb{Z}_5$:

$$(z_1, z_2, z_3, z_4, z_5) \rightarrow \left(\alpha^6 z_1, \beta^3 z_2, \frac{\beta}{\alpha^4} z_3, \frac{\alpha}{\beta^4} z_4, \alpha^3 \beta^3 z_5 \right).$$

- Exponent and Order matrices:

$$A := \begin{bmatrix} 6 & 0 & -4 & 1 & 3 \\ 0 & 3 & 1 & -4 & 3 \end{bmatrix} \quad P := \begin{bmatrix} 7 & \\ & 5 \end{bmatrix}$$

- Notation:

$$(\alpha, \beta)^A = \left(\alpha^6, \beta^3, \frac{\beta}{\alpha^4}, \frac{\alpha}{\beta^4}, \alpha^3 \beta^3 \right)$$

$$(z_1, z_2, z_3, z_4, z_5) \rightarrow (\alpha, \beta)^A \star (z_1, z_2, z_3, z_4, z_5)$$

Finite Abelian Group Actions

- Rational invariants
- Integer linear algebra
- Rewrite rules

Rational Invariants $\mathbb{K}(z)^A$

$F(z)$ is *invariant* under $\mathbf{z} \mapsto \lambda^A \star \mathbf{z}$ if $F(\lambda^A \star \mathbf{z}) = F(\mathbf{z})$

Lemma

Laurent monomials: $\mathbf{z}^v = z_1^{v_1} \cdots z_n^{v_n}$, $v \in \mathbb{Z}^n$. *Invariant iff*

$$(\lambda^A \star \mathbf{z})^v = \mathbf{z}^v \Leftrightarrow A \cdot v = 0 \pmod{P}$$

Lemma

Rational Invariants: $F(z) \in \mathbb{K}(z)^A$:

$$F(z) = \frac{\sum_{v \in \ker_{\mathbb{Z}} A \pmod{P}} a_v z^v}{\sum_{v \in \ker_{\mathbb{Z}} A \pmod{P}} b_v z^v}$$

Kernel? Use Hermite Normal Form

Diagonal action : $(\alpha, \beta) \in \mathbb{Z}_6 \times \mathbb{Z}_3$:

$$\begin{bmatrix} 4 & -1 & -3 & -6 & 0 \\ -1 & 4 & -3 & 0 & -3 \end{bmatrix}$$

$$[A, -P]$$

exponent
matrix

Kernel? Use Hermite Normal Form

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$$\begin{bmatrix} 4 & -1 & -3 & -6 & 0 \\ -1 & 4 & -3 & 0 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$[A, -P]$$

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→

$$[H_i \ 0]$$

Hermite
normal form

Kernel? Use Hermite Normal Form

Diagonal action : $(\alpha, \beta) \in \mathbb{Z}_6 \times \mathbb{Z}_3$:

$$\begin{bmatrix} 4 & -1 & -3 & -6 & 0 \\ -1 & 4 & -3 & 0 & -3 \end{bmatrix} \left[\begin{array}{cc|ccc} 1 & 1 & 3 & 2 & 1 \\ 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 2 & 1 & 0 \\ 1 & 0 & -1 & 2 & 0 \end{array} \right] = \begin{bmatrix} 3 & 2 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 \end{bmatrix}$$

$[A, -P]$	$\left[\begin{array}{cc} V_i & V_n \\ P_i & P_n \end{array} \right]$	$[H_i \ 0]$
exponent matrix	unimodular multiplier	Hermite normal form

(Unimodular means $W = V^{-1} \in \mathbb{Z}^{5 \times 5}$)

Hermite Normal Form

Diagonal action : $(\alpha, \beta) \in \mathbb{Z}_6 \times \mathbb{Z}_3$:

$$\begin{bmatrix} 4 & -1 & -3 & -6 & 0 \\ -1 & 4 & -3 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 & 2 & 1 \\ 1 & 0 & & 2 & 1 \\ 0 & 0 & & & 1 \\ 1 & 1 & 2 & 1 & 0 \\ 1 & 0 & -1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$[A, -P]$$

exponent
matrix

$$\begin{bmatrix} V_i & V_n \\ P_i & P_n \end{bmatrix}$$

unimodular
multiplier

$$[H_i \ 0]$$

Hermite
normal form

Note : V not unique but can be normalized. Implies V_n is special

Rational Invariants and Rewrite Rules

Theorem

$$A \in \mathbb{Z}^{s \times n}, \quad [A, -P] \cdot V = [H, 0],$$

$$V = \begin{bmatrix} V_i & V_n \\ P_i & P_n \end{bmatrix}, \quad W = V^{-1} = \begin{bmatrix} W_u & P_u \\ W_d & P_d \end{bmatrix}$$

- (a) $y = [z_1, \dots, z_n]^{V_n}$ form generating set of rational invariants.
- (b) V normalized : components of $y = [z_1, \dots, z_n]^{V_n}$ are polynomials.
- (c) Rewrite rule : $F \in \mathbb{K}(z)^A \implies F(z) = F(y^{(W_d - P_d P_u^{-1} W_u)})$

Why?

Rational Invariants and Rewrite Rules

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Why? $v = V_n(W_d - P_d P_u^{-1} W_u)v$. any term z^v with $v \in \text{colspan}_{\mathbb{Z}} V_n$:

$$\begin{aligned} z^v &= z^{V_n(W_d - P_d P_u^{-1} W_u)v} \\ &= (z^{V_n})^{(W_d - P_d P_u^{-1} W_u)v} \\ &= (y^{(W_d - P_d P_u^{-1} W_u)})^v \end{aligned}$$

Then use Lemma.

Example : Rational Invariants for \mathbb{Z}_n

\mathcal{G} be cyclic group of permutations $(1, 2, \dots, n)$.

- Diagonalizing matrix $R(\eta) = [\eta^{ij}]$, (η n -root of unity)

$$\mathcal{D} : A = \begin{bmatrix} 1 & 2 & 3 & \dots & n-1 & 0 \end{bmatrix} \text{ and } P = \begin{bmatrix} n \end{bmatrix}.$$

$$V = \left[\begin{array}{c|cccccc} 1 & n & n-2 & \dots & \dots & 1 & 0 \\ 0 & 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \ddots & 0 \\ \vdots & 0 & 0 & \dots & \dots & 0 & 1 \\ \hline 0 & 1 & 1 & \dots & \dots & 1 & 0 \end{array} \right] \text{ and } W = \left[\begin{array}{c|cccccc} 1 & 2 & 3 & \dots & n-1 & 0 & -n \\ 0 & -1 & -1 & \dots & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & 1 & 0 \\ \vdots & & & & & 0 & 1 \\ \hline 0 & \dots & \dots & \dots & \dots & 0 & 0 \end{array} \right].$$

- Generating invariants are

$$g = z^{Vn} = (z_1^n, z_1^{n-2}z_2, z_1^{n-3}z_3, \dots, z_1z_{n-1}, z_n),$$

- Associated rewrite rules are

$$z \rightarrow g^{Wd^{-1}Pd^{-1}Wu} = \left(g_1^{\frac{1}{n}}, \frac{g_2}{g_1^{\frac{n-2}{n}}}, \dots, \frac{g_{n-1}}{g_1^{\frac{1}{n}}}, g_n \right), \quad \text{that is, } z_k \rightarrow \frac{g_k}{g_1^{\frac{n-k}{n}}}$$

Example : Rational Invariants for $\mathbb{Z}_n \times \mathbb{Z}_n$

- $\mathcal{D} : A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 3 & \cdots & n-1 & 0 \end{bmatrix}$ and $P = \begin{bmatrix} n & 0 \\ 0 & n \end{bmatrix}$. V and W then:

$$\left[\begin{array}{cc|cccccc} 2 & -1 & n & 0 & 1 & 2 & \cdots & n-2 \\ -1 & 1 & 0 & n & n-2 & n-3 & \cdots & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & & 0 & \\ \vdots & \vdots & & & & & \ddots & \vdots & \vdots \\ \vdots & \vdots & & & & & & \vdots & \vdots \\ \vdots & \vdots & & & & & & 1 & 0 \\ \vdots & \vdots & & & & & & & 1 \\ \hline 0 & 0 & 1 & 1 & 1 & \cdots & \cdots & 1 & 1 \\ 0 & 0 & 1 & 2 & 2 & \cdots & \cdots & 2 & 1 \end{array} \right] \& \left[\begin{array}{cccccc|cc} 1 & 1 & 1 & \cdots & 1 & 1 & -n & 0 \\ 1 & 2 & 3 & \cdots & n-1 & 0 & 0 & -n \\ \hline 0 & 0 & 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & \cdots & -1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & & & & \vdots & \vdots \\ \vdots & \vdots & & & & & \vdots & \vdots \\ \vdots & \vdots & & & & & \vdots & \vdots \\ \vdots & \vdots & & & & & \vdots & \vdots \\ \vdots & \vdots & & & & & \vdots & \vdots \\ \hline 0 & 0 & & & & 1 & 0 & 0 \end{array} \right]$$

- Generating invariants are

$$g = z^V n = (z_1^n, z_2^n, z_1 z_2^{n-2} z_3, z_1^2 z_2^{n-3} z_4, \dots, z_1^{n-3} z_2^2 z_{n-1}, z_1^{n-2} z_2 z_n),$$

- Associated rewrite rules are

$$z \rightarrow g^{W_d^{-1} P_d P_u^{-1} W_u} = \left(g_1^{\frac{1}{n}}, g_2^{\frac{1}{n}}, \frac{g_3}{g_1^{\frac{1}{n}} g_2^{\frac{n-2}{n}}}, \dots, \frac{g_{n-1}}{g_1^{\frac{n-3}{n}} g_2^{\frac{2}{n}}}, \frac{g_n}{g_1^{\frac{n-2}{n}} g_2^{\frac{1}{n}}} \right).$$

Solving Polynomial Systems

- Using invariants and rewrite rules
- (A, P) -degree and (A, P) -homogeneous
- Solving invariant systems

Example : Invariant Dynamic System

Consider system of polynomial equations (c parameter)¹

$$1 - cx_1 - x_1x_2^2 - x_1x_3^2 = 0$$

$$1 - cx_2 - x_2x_1^2 - x_2x_3^2 = 0$$

$$1 - cx_3 - x_3x_1^2 - x_3x_2^2 = 0$$

¹Steady state for Neural network model [Noonburg SIAM Num Anal 1989]

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$$1 - cx_3 - x_3x_1^2 - x_3x_2^2 = 0$$

Solution space of system is invariant under the permutation

$$(x_1, x_2, x_3) \rightarrow (x_2, x_3, x_1).$$

However no polynomial is invariant under the permutation.

¹Steady state for Neural network model [Noonburg SIAM Num Anal 1989]

(A, P) -homogeneity (Faugère and Svartz)

Definition

- (i) $\deg_{(A,P)}(z^u) = A \cdot u \pmod P$
- (ii) $f \in \mathbb{K}[z, z^{-1}]$ can be written as

$$f = \sum_{d \in \mathcal{Z}} f_d$$

terms in f_d $\deg_{(A,P)} = d$ (homogeneous of (A, P) -degree d)

Lemma

$f \in \mathbb{K}[z, z^{-1}]$ is (A, P) -homogeneous of (A, P) -degree d iff

$$f(\lambda^A \star z) = \lambda^d f(z)$$

for all $\lambda \in \mathcal{U}$.

Solving via (A, P) -homogenous components

Theorem

Let $F \subset \mathbb{K}[z, z^{-1}]$ and $F^h = \{f_d \mid f \in F, d \in \mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_s}\}$ set of homogeneous components of F .

If set of toric zeros of F is invariant by the diagonal action of \mathcal{U} defined by A then it is equal to toric zeros of F^h .

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If set of toric zeros of F is invariant by the diagonal action of \mathcal{U} defined by A then it is equal to toric zeros of F^h .

Why:

Since $f(\lambda^A \star z) = \sum_d \lambda^d f_d(z)$ for all $\lambda \in \mathcal{U}$ we have a square linear system

$$(f(\lambda^A \star z))_{\lambda \in \mathcal{U}} = (\lambda^d)_{\lambda \in \mathcal{U}, d \in \mathcal{Z}} (f_d)_{d \in \mathcal{Z}}.$$

With an appropriate ordering of the elements of \mathcal{U} and \mathcal{Z} the square matrix $(\lambda^d)_{\lambda \in \mathcal{U}, d \in \mathcal{Z}}$ is the Kronecker product of the Vandermonde matrices $(\xi_i^{(k-1)(l-1)})_{1 \leq k, l \leq p_i}$, for $1 \leq i \leq s$ and ξ_i a primitive p_i th root of unity. So it is invertible.

Example : Neural Network

Recall Neural Network system (c is a parameter):

$$\begin{aligned}1 - cx_1 - x_1x_2^2 - x_1x_3^2 &= 0 \\1 - cx_2 - x_2x_1^2 - x_2x_3^2 &= 0 \\1 - cx_3 - x_3x_1^2 - x_3x_2^2 &= 0\end{aligned}\tag{1}$$

- (i) Zeros invariant under permutation $\sigma = (321)$.
- (ii) Diagonal action : exponents $A = [1 \ 2 \ 0]$; order $P = [3]$.

Example : Neural Network

(iii) Change coordinates via $x = R \cdot z$ gives

$$0 = f_0 = \hat{f}_0 - \xi \hat{f}_1 - \xi^2 \hat{f}_2$$

$$0 = f_1 = \hat{f}_0 - \xi^2 \hat{f}_1 - \xi \hat{f}_2$$

$$0 = f_2 = \hat{f}_0 - \hat{f}_1 - \hat{f}_2$$

where

$$\hat{f}_0 = 1 - cz_3 + z_1^3 + z_2^3 - 2z_3^3$$

$$\hat{f}_1 = cz_1 + 3z_1^2 z_2 - 3z_2^2 z_3$$

$$\hat{f}_2 = cz_2 + 3z_1 z_2^2 - 3z_1^2 z_3.$$

(iv) Each \hat{f}_i is (A, P) -homogeneous of degree i , for $0 \leq i \leq 2$.

Example (cont.)

(i) What about non-toric zeros? Localize at z_1

(ii) The reduced system corresponding to $\left\{f_0, \frac{f_1}{z_1}, \frac{f_2}{z_1}\right\}$ is

$$0 = 1 + y_1 - c y_3 - 2 y_3^2 + \frac{y_2^3}{y_1}, \quad 0 = c + 3 y_2 - 3 \frac{y_2^2 y_3}{y_1}, \quad 0 = -3 y_3 + c \frac{y_2}{y_1} + 3 \frac{y_2^2}{y_1}.$$

(iii) This system has $6 = 2 + 4$ zeros : union of triangular sets

$$y_3 = 0, \quad y_2 = \frac{c}{3}, \quad y_1^2 + y_1 - \frac{c^3}{27} = 0;$$

and

$$162 c y_3^4 - 54 y_3^3 + 81 c^2 y_3^2 - 108 c y_3 + 4 c^3 + 27 = 0,$$

$$y_2 = -\frac{81 c}{49 c^3 - 27} y_3^3 - \frac{14 c^3}{49 c^3 - 27} y_3^2 - \frac{93 c^2}{2(49 c^3 - 27)} y_3 - \frac{c(70 c^3 - 243)}{6(49 c^3 - 27)}$$

$$y_1 = y_3^3 + \frac{c}{2} y_3 - \frac{1}{2}.$$

Example (cont.)

Original system has 6 orbits of zeros, that is 18 solutions, where $z_1 = \xi^2 x_1 + \xi x_2 + x_3 \neq 0$.

(i) Given sol. (y_1, y_2, y_3) orbit : by solve triangular system:

$$z_1^3 = y_1, \quad z_1 z_2 = y_2, \quad z_3 = y_3.$$

(ii) With $x = R z$ get 18 solutions of the system with 6 orbits.

What about $z_1 = 0$?

(iii) Here, there are three solutions satisfying

$$z_1 = 0, \quad z_2 = 0, \quad 2z_3^3 + c z_3 - 1 = 0.$$

(iv) They each form an orbit. The corresponding solutions :

$$x_1 = x_2 = x_3 = \eta, \quad \text{for } 2\eta^3 + c\eta - 1 = 0.$$

Total number of solutions : 21.

Determining groups of homogeneity

- Matrix of exponents of polynomial system
- Smith Form and finding A and P

Matrix of Exponents of System

Rational functions in $\mathbb{K}(z_1, \dots, z_n)$. K matrix of exponents.

$$f_1 = z_1^2 z_2^2 z_3^2 - z_2^3 - z_1 z_2 z_3 + 8$$

$$f_2 = z_1^2 z_2^2 z_3^2 - z_2^3 + 7$$

$$f_3 = z_1^6 z_2^3 z_3^3 - 3z_1^4 z_2^4 z_3 + z_1^6 + z_2^3 + 32z_1^3$$

Matrix of Exponents :

$$K = \left[\begin{array}{ccc|ccc|ccc} 2 & 0 & 1 & 2 & 0 & 3 & 1 & 3 & -3 \\ 2 & 3 & 1 & 2 & 3 & 3 & 4 & 0 & 3 \\ 2 & 0 & 1 & 2 & 0 & 3 & 1 & 0 & 0 \end{array} \right].$$

Smith Normal Form and finding A and P

Smith normal form $K : U \cdot K \cdot V = [\text{diag}(1, \dots, 1, p_1, \dots, p_s) \ 0]$

Theorem

$$\text{Partition: } U = \begin{bmatrix} C \\ A \end{bmatrix} \quad \text{and} \quad U^{-1} = \begin{bmatrix} U_0 & U_1 \end{bmatrix}$$

(i) F invariants for diagonal action

$$P = \text{diag}(p_1, \dots, p_s), \quad A = \text{the last } s \text{ rows of } U.$$

(ii) $[y_1, \dots, y_n] = z^{\begin{bmatrix} U_0 & U_1 P \end{bmatrix}}$ minimal generating invariants

(iii) Rewrite rule : for any invariant $f \in \mathbb{K}(z)$ of (A, P) :

$$f(z) = f\left(y^{\begin{bmatrix} C \\ P^{-1}A \end{bmatrix}}\right).$$

Matrix from previous example:

$$K = \begin{bmatrix} 2 & 0 & 1 & 2 & 0 & 3 & 1 & 3 & -3 \\ 2 & 3 & 1 & 2 & 3 & 3 & 4 & 0 & 3 \\ 2 & 0 & 1 & 2 & 0 & 3 & 1 & 0 & 0 \end{bmatrix}.$$

Smith normal form:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \cdot K \cdot V = \left[\begin{array}{ccc|cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Matrix from previous example:

$$K = \begin{bmatrix} 2 & 0 & 1 & 2 & 0 & 3 & 1 & 3 & -3 \\ 2 & 3 & 1 & 2 & 3 & 3 & 4 & 0 & 3 \\ 2 & 0 & 1 & 2 & 0 & 3 & 1 & 0 & 0 \end{bmatrix}.$$

Smith normal form:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \cdot K \cdot V = \left[\begin{array}{ccc|cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 3 & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 3 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

The underlying symmetry group is $\mathbb{Z}_3 \times \mathbb{Z}_3$.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

Invariant exponents:

$$V_n = [U_0 \ U_1 P] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -3 \\ 1 & 3 & 3 \end{bmatrix} \approx \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Future Research Directions

- (i) Extend to parameterized and dynamic systems
- (ii) Extend from Finite Abelian to Finite Solvable Group actions
 - e.g. Neural network example invariant under S_3 .
- (iii) Combine scaling symmetries with finite diagonal actions
 - makes use of Smith Normal Form