# Computing Popov Form of General Ore Polynomial Matrices

Howard Cheng[1] Patrick Davies[1] George Labahn[2]

[1]Department of Mathematics and Computer Science
University of Lethbridge
Canada

[2]Symbolic Computation Group
David R. Cheriton School of Computer Science
University of Waterloo
Canada

# Ore Polynomials

- The ring of Ore polynomials $\mathbb{Q}_{\mathbb{D}}[Z; \sigma, \delta]$
  - $\sigma$: automorphism over $\mathbb{Q}_{\mathbb{D}}$
  - $\delta$: additive homomorphism on $\mathbb{Q}_{\mathbb{D}}$
  - Polynomial multiplication: $Z \cdot a = \sigma(a)Z + \delta(a)$

| | $\sigma(a(t))$ | $\delta(a(t))$ |
|---|---|---|
| Polynomials | $a(t)$ | $0$ |
| Differential operator | $a(t)$ | $a'(t)$ |
| Difference operator | $a(t+1)$ | $0$ |

- Matrices of Ore polynomials represent systems of linear differential equations, difference equations, etc.

# Example

The system of differential equations

$$\begin{array}{rcrcrcl} y_1''(t) + (t+2)y_1(t) & + & y_2''(t) + y_2(t) & + & y_3'(t) + y_3(t) & = & 0 \\ y_1'(t) + 3y_1(t) & + & y_2'''(t) + 2y_2'(t) - y_2(t) & + & y_3'''(t) - 2t^2 y_3(t) & = & 0 \\ y_1'(t) + y_1(t) & + & y_2''(t) + 2ty_2'(t) - y_2(t) & + & y_3''''(t) & = & 0. \end{array}$$

can be represented by the Ore polynomial matrix

$$\begin{bmatrix} Z^2 + (t+2) & Z^2 + 1 & Z + 1 \\ Z + 3 & Z^3 + 2Z - 1 & Z^3 - 2t^2 \\ Z + 1 & Z^2 + 2tZ + 1 & Z^4 \end{bmatrix} \cdot \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \mathbf{0}.$$

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Any input matrix $\mathbf{F}(Z)$ can be transformed into a unique Popov form by row operations.

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2. multiply a row by a nonzero constant
3. add a polynomial multiple of one row to another

# Problem Statement

Given an $m \times n$ Ore polynomial matrix $\mathbf{F}(Z)$, we wish to compute:

- its Popov Form $\mathbf{T}(Z)$
- the associated unimodular transformation matrix $\mathbf{U}(Z)$

so that

$$\mathbf{U}(Z) \cdot \mathbf{F}(Z) = \mathbf{T}(Z).$$

We do not know if $\mathbf{F}(Z)$ has full row rank *a priori*.

# Applications

- Obtain normal form: canonical representation of the system of equations represented;
- greatest common right divisors (GCRD) and least common left multiples (LCLM)

  i.e. intersection and union of systems;
- reduce order of systems of equations;
- isolate highest powers.

  e.g. convert DAE systems to first order.

# Applications

Let $Z$ be the differentiation operator on $t$. If the system of equations is represented by:

$$\begin{bmatrix} Z^2 + (t+2) & Z^2 + 1 & Z + 1 \\ Z + 3 & Z^3 + 2Z - 1 & Z^3 - 2t^2 \\ Z + 1 & Z^2 + 2tZ + 1 & Z^4 \end{bmatrix} \cdot \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \mathbf{0},$$

then

$$\begin{aligned} y_1''(t) &= -(t+2)y_1(t) - y_2''(t) - y_2(t) - y_3'(t) - y_3(t) \\ y_2'''(t) &= -y_1'(t) - 3y_1(t) - 2y_2'(t) + y_2(t) - y_3'''(t) + 2t^2 y_3(t) \\ y_3''''(t) &= -y_1'(t) - y_1(t) - y_2''(t) - 2ty_2'(t) - y_2(t) \end{aligned}$$

# Previous Works

- Elimination-based approaches for Ore Popov form (Giesbrecht, Labahn, Zhang).

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- The FFreduce algorithm is fraction-free. i.e. No fractions are introduced while controlling coefficient growth.
- A modular algorithm (Cheng, Labahn) is available for the same computations.

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- Popov form algorithm for polynomial matrices:
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  - Beckermann, Labahn, Villard
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- A number of other algorithms for row/column-reduced form of polynomial matrices:
  - Beelen, van den Hurk, Praagman
  - Neven and Praagman
  - ...

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  - growth from Gaussian elimination
  - growth from multiplication by $Z$
- Nullspace algorithms (fraction-free and modular) control growth but cannot be used to directly compute Popov form.
- Straightforward and efficient modular algorithms are difficult:
  - evaluation of $t$ is not an Ore ring homomorphism

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- When input matrix does not have full row rank, unimodular multiplier is not unique.
- Many algorithms and/or proofs for polynomial matrices require tools such as:
  - fractions of polynomials
  - determinant
  - adjoint
  - Cramer's rule
  - matrix fractions

# General Approach

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$$\mathbf{M}(Z) = \begin{bmatrix} \mathbf{U}(Z) & \mathbf{T}(Z) \cdot Z^b \end{bmatrix}$$

so that

$$\begin{bmatrix} \mathbf{U}(Z) & \mathbf{T}(Z) \cdot Z^b \end{bmatrix} \cdot \begin{bmatrix} \mathbf{F}(Z) \cdot Z^b \\ -\mathbf{I} \end{bmatrix} = \mathbf{0}.$$

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- In other words:

$$\mathbf{U}(Z) \cdot \mathbf{F}(Z) \cdot Z^b = \mathbf{T}(Z) \cdot Z^b$$

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- $\mathbf{M}(Z)$ in Popov form $\Leftrightarrow$ $\mathbf{T}(Z)$ in Popov form.
- Old idea, but proofs do not work when matrix entries are not commutative and do not form a field.

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- These algorithms control coefficient growth by formulating the problems as linear systems of equations.
- Modular approach (Cheng and Labahn): can be reduced to systems of linear equations over $\mathbb{Z}_p[Z]$—simpler and more efficient.

# Main Result

Let $\vec{\mu} = \text{rdeg}\, \mathbf{F}(Z)$ and $b > |\vec{\mu}| - \min_j\{\mu_j\}$.

Suppose that $[\mathbf{U}(Z) \;\; \mathbf{R}(Z)]$ is a minimal polynomial basis in Popov form of the left nullspace of $\begin{bmatrix} \mathbf{F}(Z) \cdot Z^b \\ -\mathbf{I} \end{bmatrix}$.

Let $\mathbf{T}(Z) = \mathbf{R}(Z) \cdot Z^{-b}$.

1. $\mathbf{U}(Z)$ is unimodular;
2. $\mathbf{T}(Z) = \mathbf{U}(Z) \cdot \mathbf{F}(Z)$ is an Ore polynomial matrix in Popov form.

This is true even when $\mathbf{F}(Z)$ does not have full row rank.

# Key Tools

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  - do not use commutativity of matrix elements
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- Predictable Degree Property
- Minimial Multiplier

# Predictable Degree Property

Let $\vec{\mu} = \text{rdeg}\,\mathbf{F}(Z)$. Then $\mathbf{F}(Z)$ is row-reduced if and only if, for all $\mathbf{Q}(Z)$,

$$\deg\left(\mathbf{Q}(Z)\cdot\mathbf{F}(Z)\right) = \max_j \left(\deg \mathbf{Q}(Z)_{1,j} + \mu_j\right).$$

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- central in our proofs

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- The minimal multiplier is computed by the nullspace algorithms.
- All unimodular multipliers can be transformed to the minimal multiplier by row operations.
- Polynomial matrix case: Beckermann, Labahn, and Villard.

# Minimal Multiplier

Informally, the unimodular minimal multiplier $\mathbf{U}(Z)$ looks like:

$$\begin{bmatrix} \mathbf{U}(Z)_{J_c,K} & \mathbf{U}(Z)_{J_c,K_c} \\ \mathbf{U}(Z)_{J,K} & \mathbf{U}(Z)_{J,K_c} \end{bmatrix} \cdot \mathbf{F}(Z) = \begin{bmatrix} \mathbf{T}(Z)_{J_c,*} \\ \mathbf{0} \end{bmatrix}.$$

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1. $\begin{bmatrix} \mathbf{U}(Z)_{J,K} & \mathbf{U}(Z)_{J,K_c} \end{bmatrix}$ is the minimal polynomial basis in Popov form, with leading entries in column set $K$.
2. The entries in $\mathbf{U}(Z)_{J_c,K}$ are reduced by the leading entries in $\mathbf{U}(Z)_{J,K}$.

# Degree Bound on Minimal Multiplier

$$\begin{bmatrix} \mathbf{U}(Z)_{J_c,K} & \mathbf{U}(Z)_{J_c,K_c} \\ \mathbf{U}(Z)_{J,K} & \mathbf{U}(Z)_{J,K_c} \end{bmatrix} \cdot \mathbf{F}(Z) = \begin{bmatrix} \mathbf{T}(Z)_{J_c,*} \\ \mathbf{0} \end{bmatrix}.$$

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  - Predictable degree property with $\mathbf{T}(Z)_{J_c,*}$ and $\mathbf{U}(Z)_{J,K}$ (both in Popov form) $\rightarrow$ degree bounds on $\mathbf{V}(Z)$
  - Row-reduced form of $\mathbf{V}(Z)$ is $\mathbf{I}$ with unique transformation matrix $\mathbf{U}(Z)$ $\rightarrow$ degree bound of $\mathbf{U}(Z)$ (Beckermann, Cheng, Labahn)

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- For the special case of polynomial matrices, our result specializes to that of Beckermann, Labahn, and Villard with a much more “elementary” proof...
- ...but less precise if additional information is available *a priori* (e.g. row degrees of the Popov form, row degrees of the minimal polynomial basis).