

Fraction-free Computation of Simultaneous Padé Approximants

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Outline

Basic definitions

- Simultaneous Padé Approximants
- Fraction-free computations
- Vector Hermite-Padé approximants
- The work of Mahler
- Other known algorithms

The Algorithm

- Cramer solutions
- Mahler-Cramer systems

Algorithm Details

- Pivot column
- The other columns
- Example

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Simultaneous Padé Approximants

Given vector of power series $(f_1(z), f_2(z), \dots, f_m(z)) \in \mathbb{K}[[z]]^{1 \times m}$
and a multi-index of integers $\vec{n} = (n_1, \dots, n_m)$,

Find polynomials $p_1(z), \dots, p_m(z)$ with :

- ▶ for $k = 2, 3, \dots, m$: approximation $\frac{p_k(z)}{p_1(z)} \approx \frac{f_k(z)}{f_1(z)}$,
- ▶ for $k = 1, \dots, m$: degree bounds for $p_k(z)$,

Simultaneous Padé Approximants (more precise)

Given vector of power series $(f_1(z), f_2(z), \dots, f_m(z)) \in \mathbb{K}[[z]]^{1 \times m}$
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- ▶ for $k = 2, 3, \dots, m$: approximation $\frac{p_k(z)}{p_1(z)} \approx \frac{f_k(z)}{f_1(z)}$, namely

$$f_k(z)p_1(z) - f_1(z)p_k(z) = \mathcal{O}(z^{|\vec{n}|+1}), \quad |\vec{n}| = n_1 + n_2 + \dots + n_m,$$

- ▶ for $k = 1, \dots, m$: degree bounds for $p_k(z)$, namely

$$\deg p_k(z) \leq |\vec{n}| - n_k.$$

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- ▶ for $k = 1, \dots, m$: degree bounds for $p_k(z)$, namely

$$\deg p_k(z) \leq |\vec{n}| - n_k.$$

- homogeneous system with $(m-1)(|\vec{n}|+1)$ equations and $(m-1)(|\vec{n}|+1) + 1$ unknowns $\implies \exists$ solution.

- $p(\vec{n}, z) = p(z) = (p_1(z), \dots, p_m(z))^t$ is also called type 2 Hermite-Padé approximant or "german polynomials".

History and Applications

- ▶ Introduced by Hermite (transcendence of e), used by Lindemann (transcendence of π)
- ▶ formalized by Padé (see book [Baker & Graves-Morris]), especially the case $m = 2$ of Padé approximants.
- ▶ landmark paper of Mahler "Perfect systems" (1968)

- ▶ Applications for inversion formulas of striped Toeplitz matrices [Labahn 92]
- ▶ Vector rational reconstruction in symbolic computation:

$$\mathbb{K}^n(z) \rightarrow \mathbb{K}^n[[z]] - \text{solve problem} - \mathbb{K}^n[[z]] \rightarrow \mathbb{K}^n(z)$$

(e.g. linear system solving) [c.f. Olesh & Storjohann 07]

Fraction-Free ?

Imagine that $\mathbb{K} = \mathbb{Q}$ (or $\mathbb{K} = \mathbb{C}(a_1, \dots, a_n)$).

- ▶ Computation is easier in integral domain \mathbb{Z} (or $\mathbb{C}[a_1, \dots, a_n]$), no expensive GCD computations for simplifying fractions.
- ▶ Want to divide only through predictable factors in order to remain in integral domain, *and* control coefficient growth.

In what follows: all data in integral domain, of size $\mathcal{O}(\kappa)$.

Example : Fraction-Free Gaussian Elimination

$$A = \begin{bmatrix} a & b & c & \dots & \dots \\ d & e & f & \dots & \dots \\ g & h & i & \dots & \dots \\ \vdots & \vdots & \vdots & & \end{bmatrix} \approx \begin{bmatrix} a & b & c & \dots & \dots \\ 0 & \tilde{e} & \tilde{f} & \dots & \dots \\ 0 & \tilde{h} & \tilde{i} & \dots & \dots \\ \vdots & \vdots & \vdots & & \end{bmatrix}$$

- ▶ Cross multiplication gives exponential growth of coeffs
- ▶ Fraction-free Gaussian elimination (FFGE)

$$A \approx \begin{bmatrix} a & b & c & \dots & \dots \\ 0 & \tilde{e} & \tilde{f} & \dots & \dots \\ 0 & 0 & a(..) & \dots & a(...) \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & a(..) & \dots & a(...) \end{bmatrix}.$$

Allows for linear growth of coefficient size.

- ▶ Computes Cramer solution of linear system

Faster Fraction-Free Computation

- ▶ For our problems we would want to take advantage of special structure of associated linear system
- ▶ Has been successfully done for Hermite-Padé case by B-L [SIMAX 2000]
 - ▶ Computes a sequence of Cramer solutions for growing sequence of linear systems
 - ▶ Also works for case of vector Hermite-Padé
- ▶ Gives an algorithm for Simultaneous-Padé approximants
 - ▶ Even gives fraction-free way of finding Order Bases.
(Order bases \equiv finding *all* solutions of SP problem)
- ▶ Basically our goal is to do better than above.

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Vector Hermite-Padé approximants

For $F(z) \in \mathbb{K}[[z]]^{m \times m}$ find $P(z) \in \mathbb{K}[z]^{m \times 1}$ with certain degree constraints such that $F(z)P(z) = \mathcal{O}(z^{\vec{\sigma}})$, i.e.,

j th row of $F(z)P(z)$ is $\mathcal{O}(z^{\sigma_j})$ for $j = 1, \dots, m$.

Example 1:

$$F(z) = \begin{bmatrix} f_1(z) & f_2(z) & \cdots & \cdots & f_m(z) \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}, \quad \deg P_k \leq n_k - 1, \quad \vec{\sigma} = (|\vec{n}| - 1, 0, \dots, 0)$$

Hermite-Padé approximants (of type 1) or "latin polynomials".

Example 2:

$$\mathfrak{F}(z) = \begin{bmatrix} f_1(z) & 0 & \cdots & \cdots & 0 \\ -f_2(z) & f_1(z) & 0 & \cdots & 0 \\ -f_3(z) & 0 & f_1(z) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -f_m(z) & 0 & \cdots & 0 & f_1(z) \end{bmatrix}, \quad \vec{\sigma} = (0, |\vec{n}| + 1, \dots, |\vec{n}| + 1)$$

Simultaneous Padé approximants as before.

The work of Mahler (using strong assumptions)

Arrange m neighboring HP-approximants in $m \times m$ matrix

$$M(\vec{n}, z) = [P(\vec{n} + \vec{e}_1, z), \dots, P(\vec{n} + \vec{e}_m, z)], \text{ degrees } \begin{bmatrix} = n_1 & < n_1 & \dots & \dots & < n_1 \\ < n_2 & = n_2 & < n_2 & \dots & < n_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ < n_m & \dots & \dots & < n_m & = n_m \end{bmatrix}$$

Normalized : entries on diagonal monic.

Arrange m neighboring SP-approximants in $m \times m$ matrix

$$\mathfrak{M}(\vec{n}, z) = [p(\vec{n} - \vec{e}_1, z), \dots, \mathfrak{P}(\vec{n} - \vec{e}_m, z)],$$

$$\text{degrees } \begin{bmatrix} = |\vec{n}| - n_1 & < |\vec{n}| - n_1 & \dots & \dots & < |\vec{n}| - n_1 \\ < |\vec{n}| - n_2 & = |\vec{n}| - n_2 & < |\vec{n}| - n_2 & \dots & < |\vec{n}| - n_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ < |\vec{n}| - n_m & \dots & \dots & < |\vec{n}| - n_m & = |\vec{n}| - n_m \end{bmatrix}$$

Normalized : entries on diagonal monic.

Duality $M(\vec{n}, z)\mathfrak{M}(\vec{n}, z)^t = z^{|\vec{n}|}I_m.$

Ingredient $F(z)\mathfrak{F}(z)^t = f_1(z)I_m$ [B-L, JCAM 97]

The work of Mahler (continued)

Mahler also shows how to determine $U_j(\vec{n}, z) \in \mathbb{K}[z]^{m \times m}$ from $M(\vec{n}, z)$ such that

$$M(\vec{n} + \vec{e}_j, z) = M(\vec{n}, z) U_j(\vec{n}, z).$$

Since $M(\vec{0}, z) = I_m$, this gives a recursive way to compute $M(\vec{n}, z)$ for any multi-index \vec{n} .

By duality, one may deduce $\mathfrak{U}_j(\vec{n}, z) \in \mathbb{K}[z]^{m \times m}$ such that

$$\mathfrak{M}(\vec{n} + \vec{e}_j, z) = \mathfrak{M}(\vec{n}, z) \mathfrak{U}_j(\vec{n}, z).$$

Dependency on $\mathfrak{M}(\vec{n}, z)$ only implicit in paper of Mahler...

Other Algorithms : σ -basis algorithm

Idea: Generalize work of Mahler to vector HP, but drop all regularity assumptions [B-L, SIMAX 94].

- ▶ For fixed $F(z)$ and fixed order vector $\vec{\sigma}$, consider the $\mathbb{K}[z]$ module of all $P(z) \in \mathbb{K}[z]^m$ having order $\mathcal{O}(z^{\vec{\sigma}})$.
- ▶ determine basis $\in \mathbb{K}[z]^{m \times m}$ of this module with particular degree constraints allowing to parametrize all vectors $P(z) \in \mathbb{K}[z]^m$ in module with degree $\leq \vec{n} + (d, \dots, d)$
- ▶ compute such $\vec{\sigma}$ -bases recursively by increasing in each step one component of the order by 1.

$M(\vec{n} + (d, \dots, d), z)$ for $d \in \mathbb{Z}$ does the job, but does not always exist! Also, it is cheaper to weaken a bit the degree constraints for our bases.

Summary of algorithms

- ▶ (Vector) Hermite-Padé :
 - ▶ Bultheel-van Barel (Num Alg - 1992)
 - ▶ Beckermann-Labahn (Num Alg - 1992)
 - ▶ P. Giorgi, C-P. Jeannerod, G. Villard (ISSAC 2003)
 - ▶ fast polynomial matrix arithmetic
 - ▶ Beckermann-Labahn (SIMAX 2000)
 - ▶ fraction-free arithmetic
 - ▶ Zhou-Labahn (ISSAC 2009)
 - ▶ fast arithmetic
- ▶ Simultaneous Padé
 - ▶ Olesky and Storjohann (2007)

Cost for SP fraction-free algorithms

If input has size $O(\kappa)$ and $N = |\vec{n}|$ then :

- ▶ Fraction-Free Gaussian Elimination (FFGE) of Bareiss :
 - Bit complexity of operations: $O(\kappa^2 m^6 N^5)$
- ▶ B-L [SIMAX 2000] :
 - Bit complexity of operations: $O(\kappa^2 m^5 N^4)$
 - Size of objects : $O(\kappa m N)$
- ▶ B-L [ISAAC 2009] : **Today's talk**
 - **Bit complexity of operations: $O(\kappa^2 m^2 N^4)$**
 - **Size of objects : $O(\kappa N)$**

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Computation Process

Input : $F(z)$, \vec{n} .

Process for Hermite-Padé : Computes Order bases of type

$$\vec{v}^{(0)}, \vec{v}^{(1)}, \dots, \dots, \vec{v}^{(N+1)}$$

$$I = M(\vec{v}^{(0)}, z) \rightarrow M(\vec{v}^{(1)}, z) \rightarrow \dots \rightarrow M(\vec{v}^{(N+1)}, z)$$

Computation Process

Input : $F(z)$, \vec{n} .

Process for Hermite-Padé : Computes Order bases of type

$$\vec{v}^{(0)}, \vec{v}^{(1)}, \dots, \dots, \vec{v}^{(N+1)}$$

$$I = M(\vec{v}^{(0)}, z) \rightarrow M(\vec{v}^{(1)}, z) \rightarrow \dots \rightarrow M(\vec{v}^{(N+1)}, z)$$

$$I = \mathfrak{M}(\vec{v}^{(0)}, z) \rightarrow \mathfrak{M}(\vec{v}^{(1)}, z) \rightarrow \dots \rightarrow \mathfrak{M}(\vec{v}^{(N+1)}, z)$$

Computation Process

Input : $F(z)$, \vec{n} .

Process for Hermite-Padé : Computes Order bases of type

$$\vec{v}^{(0)}, \vec{v}^{(1)}, \dots, \dots, \vec{v}^{(N+1)}$$

$$\begin{array}{ccccccc} I = M(\vec{v}^{(0)}, z) & \rightarrow & M(\vec{v}^{(1)}, z) & \rightarrow & \dots & \rightarrow & M(\vec{v}^{(N+1)}, z) \\ & & \uparrow & & & & \uparrow \\ & & \downarrow & & & & \downarrow \\ I = \mathfrak{M}(\vec{v}^{(0)}, z) & \rightarrow & \mathfrak{M}(\vec{v}^{(1)}, z) & \rightarrow & \dots & \rightarrow & \mathfrak{M}(\vec{v}^{(N+1)}, z) \end{array}$$

The Recursion (H-P)

$$M(\vec{v}^{(i+1)}, z) = M(\vec{v}^{(i)}, z)U(z)V(z)$$

$$\begin{bmatrix} m_{11}^{(i+1)}(z) & \cdots & m_{1,m}^{(i+1)}(z) \\ \vdots & & \vdots \\ m_{m1}^{(i+1)}(z) & \cdots & m_{m,m}^{(i+1)}(z) \end{bmatrix} = \begin{bmatrix} m_{11}^{(i)}(z) & \cdots & m_{1,m}^{(i)}(z) \\ \vdots & & \vdots \\ m_{m1}^{(i)}(z) & \cdots & m_{m,m}^{(i)}(z) \end{bmatrix} \begin{bmatrix} * & & \\ * & * & \\ & \ddots & \\ * & & * \end{bmatrix} \begin{bmatrix} z & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

The Recursion (S-P)

$$\mathfrak{M}(\vec{v}^{(i+1)}, z) = \mathfrak{M}(\vec{v}^{(i)}, z)\mathfrak{A}(z)\mathfrak{B}(z)$$

$$\begin{bmatrix} m_{11}^{(i+1)}(z) & \cdots & m_{1,m}^{(i+1)}(z) \\ \vdots & & \vdots \\ m_{m1}^{(i+1)}(z) & \cdots & m_{m,m}^{(i+1)}(z) \end{bmatrix} = \begin{bmatrix} m_{11}^{(i)}(z) & \cdots & m_{1,m}^{(i)}(z) \\ \vdots & & \vdots \\ m_{m1}^{(i)}(z) & \cdots & m_{m,m}^{(i)}(z) \end{bmatrix} \begin{bmatrix} * & & * \\ 0 & * & * \\ \vdots & & \vdots \\ 0 & * & * \end{bmatrix} \begin{bmatrix} 1 & & & \\ & z & & \\ & & \ddots & \\ & & & z \end{bmatrix}$$

Computation Process

Input : $\vec{n}, f_1(z), f_2(z), \dots, f_m(z), f_1(0) \neq 0$

Output : Closest normal multi-indices $\vec{v}^{(0)}, \vec{v}^{(1)}, \dots, |\vec{v}^{(k)}| = k,$

$$I_m = \mathfrak{M}(\vec{v}^{(0)}, z) \rightarrow \mathfrak{M}(\vec{v}^{(1)}, z) \rightarrow \mathfrak{M}(\vec{v}^{(2)}, z) \rightarrow \dots$$

Here $\vec{v}^{(k+1)} = \vec{v}^{(k)} + \vec{e}_\pi$ s.t. $K(\vec{v}^{(k+1)}) \neq 0$, and

$$\vec{n}_\pi - \vec{v}_\pi^{(k)} = \max \left\{ \vec{n}_\pi - \vec{v}_j^{(k)} : K(\vec{v}^{(k)} + \vec{e}_j) \neq 0 \right\}.$$

Order bases in shifted Popov form gives a parametrized solution for all $\vec{n} + (d, d, \dots, d)$ SP approximants for $d \in \mathbb{Z}$.

[B-L 2009] versus [B-L 2000]

At each iteration for Hermite-Padé [B-L 2000]

- ▶ Increase order of other columns using pivot column π
- ▶ Increase order of pivot column π
- ▶ Normalize order basis to get special shifted *Popov* form

At each iteration for Simultaneous Padé [B-L 2009]

- ▶ Increase order of pivot column using fraction-free Gaussian elimination on first terms of residual
- ▶ Increase order of the other columns using the new pivot column
- ▶ Normalize order basis to get special shifted *Popov* form

How to solve homog. systems without fractions?

For the system

$$\begin{bmatrix} c_{2,1} & c_{2,2} & \cdots & c_{2,m} \\ c_{3,1} & c_{3,2} & \cdots & c_{3,m} \\ \vdots & \vdots & & \vdots \\ c_{m,1} & c_{m,2} & \cdots & c_{m,m} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

we compute the Cramer solution via FFGE

$$c_{j,\ell}^{k+1} = \frac{1}{c_{k,k-1}^{k-1}} \left(c_{j,\ell}^k c_{k+1,k}^k - c_{j,k}^k c_{k+1,\ell}^k \right),$$
$$c_{j,\ell}^1 = c_{j,\ell}, \quad c_{1,0}^0 = 1$$

(Sylvester identity), gives

Cramer solution for homogeneous systems

Cramer solution

$$y_1 = \det \begin{bmatrix} c_{2,1} & c_{2,2} & c_{2,3} & \cdots & c_{2,m} \\ c_{3,1} & c_{3,2} & c_{3,3} & \cdots & c_{3,m} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{m,1} & c_{m,2} & c_{m,3} & \cdots & c_{m,m} \\ \mathbf{1} & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Cramer solution for homogeneous systems

Cramer solution

$$y_2 = \det \begin{bmatrix} C_{2,1} & C_{2,2} & C_{2,3} & \cdots & C_{2,m} \\ C_{3,1} & C_{3,2} & C_{3,3} & \cdots & C_{3,m} \\ \vdots & \vdots & \vdots & & \vdots \\ C_{m,1} & C_{m,2} & C_{m,3} & \cdots & C_{m,m} \\ 0 & \mathbf{1} & 0 & \cdots & 0 \end{bmatrix}$$

Cramer solution for homogeneous systems

Cramer solution

$$y_m = \det \begin{bmatrix} c_{2,1} & c_{2,2} & c_{2,3} & \cdots & c_{2,m} \\ c_{3,1} & c_{3,2} & c_{3,3} & \cdots & c_{3,m} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{m,1} & c_{m,2} & c_{m,3} & \cdots & c_{m,m} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

Each component of Cramer solution is of size $\mathcal{O}(m\kappa)$.
Unique solution up to normalization iff Cramer solution is non-trivial.

Cramer solution for Hermite-Padé

$$P_1(\vec{n}, z) = \det$$

$$\left[\begin{array}{ccc|ccc|ccc} f_{1,0} & & & f_{2,0} & & & f_{3,0} & & \\ \vdots & \ddots & & \vdots & \ddots & & \vdots & \ddots & \\ \vdots & & & \vdots & & & \vdots & & \\ \vdots & & f_{1,0} & \vdots & & f_{2,0} & \vdots & & f_{3,0} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ f_{1,|\vec{n}|-2} & \cdots & f_{1,|\vec{n}|-n_1-1} & f_{2,|\vec{n}|-2} & \cdots & f_{2,|\vec{n}|-n_2-1} & f_{3,|\vec{n}|-2} & \cdots & f_{3,|\vec{n}|-n_3-1} \\ \hline & z^0 & \cdots & & 0 & \cdots & & 0 & \cdots & 0 \end{array} \right]$$

$$= \pm K(\vec{n} - \vec{e}_1) z^{n_1-1} + \text{lower degrees}$$

Cramer solution for Hermite-Padé

$$P_2(\vec{n}, z) = \det$$

$$\left[\begin{array}{ccc|ccc|ccc} f_{1,0} & & & f_{2,0} & & & f_{3,0} & & \\ \vdots & \ddots & & \vdots & \ddots & & \vdots & \ddots & \\ \vdots & & & \vdots & & & \vdots & & \\ \vdots & & & \vdots & & & \vdots & & \\ \vdots & & & \vdots & & & \vdots & & \\ f_{1,|\vec{n}|-2} & \cdots & f_{1,|\vec{n}|-n_1-1} & f_{2,|\vec{n}|-2} & \cdots & f_{2,|\vec{n}|-n_2-1} & f_{3,|\vec{n}|-2} & \cdots & f_{3,|\vec{n}|-n_3-1} \\ \hline & 0 & \cdots & 0 & & & 0 & \cdots & 0 \end{array} \right]$$

$z^0 \quad \dots \quad z^{n_2-1}$

$$= \pm K(\vec{n} - \vec{e}_2) z^{n_2-1} + \text{lower degrees}$$

Residual for Cramer solution for Hermite-Padé

$$f_1(z)P_1(\vec{n}, z) + f_2(z)P_2(\vec{n}, z) + f_3(z)P_3(\vec{n}, z) = \det$$

$$\left[\begin{array}{ccc|ccc|ccc} f_{1,0} & & & f_{2,0} & & & f_{3,0} & & \\ \vdots & \ddots & & \vdots & \ddots & & \vdots & \ddots & \\ \vdots & & & \vdots & & & \vdots & & \\ \vdots & & f_{1,0} & \vdots & & f_{2,0} & \vdots & & f_{3,0} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ f_{1,|\vec{n}|-2} & \cdots & f_{1,|\vec{n}|-n_1-1} & f_{2,|\vec{n}|-2} & \cdots & f_{2,|\vec{n}|-n_2-1} & f_{3,|\vec{n}|-2} & \cdots & f_{3,|\vec{n}|-n_3-1} \\ \hline f_1(z)z^0 & \cdots & f_1(z)z^{n_1-1} & f_2(z)z^0 & \cdots & f_2(z)z^{n_2-1} & f_3(z)z^0 & \cdots & f_3(z)z^{n_3-1} \end{array} \right]$$

$$= K(\vec{n})z^{|\vec{n}|-1} + \text{higher order}$$

Cramer solution for SP seen as vector HP, $f_1(z) = 1$

$$p_1(\vec{n}, z) =$$

$$\det \begin{bmatrix} -f_{2,|\vec{n}|-n_2+1} & -f_{2,|\vec{n}|-n_2+0} & \cdots & -f_{2,n_1-n_2+1} \\ -f_{2,|\vec{n}|-n_2+2} & -f_{2,|\vec{n}|-n_2+1} & \cdots & -f_{2,n_1-n_2+2} \\ \vdots & \vdots & & \vdots \\ -f_{2,|\vec{n}|} & -f_{2,|\vec{n}|-1} & \cdots & -f_{2,n_1} \\ \hline -f_{3,|\vec{n}|-n_3+1} & -f_{3,|\vec{n}|-n_3+0} & \cdots & -f_{3,n_1-n_3+1} \\ -f_{3,|\vec{n}|-n_3+2} & -f_{3,|\vec{n}|-n_3+1} & \cdots & -f_{3,n_1-n_3+2} \\ \vdots & \vdots & & \vdots \\ -f_{3,|\vec{n}|} & -f_{3,|\vec{n}|-1} & \cdots & -f_{3,n_1} \\ \hline z^0 & \dots & \dots & z^{|\vec{n}|-n_1} \end{bmatrix}$$

Coefficients of size $\mathcal{O}(|\vec{n}|\kappa)$ for $f_1(z) = 1$ versus size $\mathcal{O}(m|\vec{n}|\kappa)$ for general $f_1(z)$.

New formula always of size $\mathcal{O}(|\vec{n}|\kappa)$...

ℓ th residual for new Cramer solution for SP

$$\ell = 2: -f_2(z)p_1(\vec{n}, z) + f_1(z)p_2(\vec{n}, z) = \det$$

$$\left[\begin{array}{ccc|ccc|ccc|c} f_{1,0} & & & f_{2,0} & & & f_{3,0} & & & \\ \vdots & \ddots & & \vdots & \ddots & & \vdots & \ddots & & \\ \vdots & & & \vdots & & & \vdots & & & \\ \vdots & & f_{1,0} & \vdots & & f_{2,0} & \vdots & & f_{3,0} & \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \\ f_{1,|\vec{n}|} & \cdots & f_{1,|\vec{n}|-n_1} & f_{2,|\vec{n}|} & \cdots & f_{2,|\vec{n}|-n_2} & f_{3,|\vec{n}|} & \cdots & f_{3,|\vec{n}|-n_3} & \\ \hline z^0 f_1(z) & \cdots & z^{n_1} f_1(z) & z^0 f_2(z) & \cdots & z^{n_2} f_2(z) & z^0 f_3(z) & \cdots & z^{n_3} f_3(z) & 0 \\ z^0 f_2(z) & \cdots & z^{n_2} f_2(z) & z^0 f_2(z) & \cdots & z^{n_2} f_2(z) & & & & 1 \\ & & & z^0 f_2(z) & \cdots & z^{n_2} f_2(z) & & & & 0 \\ & & & & & & z^0 & \cdots & z^{n_3} & \end{array} \right]$$

$$= O(z^{|\vec{n}|+1}), \text{ more precisely}$$

l th residual for new Cramer solution for SP

$$l = 2: -f_2(z)p_1(\vec{n}, z) + f_1(z)p_2(\vec{n}, z) = \pm z^{|\vec{n}|+1} \det$$

$f_{1,0}$	$f_{1,0}$	$f_{2,0}$	$f_{2,0}$	$f_{3,0}$	$f_{3,0}$
$f_{1,1}$	$f_{1,1}$	$f_{2,1}$	$f_{2,1}$	$f_{3,1}$	$f_{3,1}$
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
$f_{1, \vec{n} +1}$	$f_{1, \vec{n} }$	$f_{2, \vec{n} +1}$	$f_{2, \vec{n} }$	$f_{3, \vec{n} }$	$f_{3, \vec{n} }$
	\cdots	\cdots	\cdots	\cdots	\cdots
	$f_{1, \vec{n} -n_1+1}$	$f_{2, \vec{n} -n_2+1}$	$f_{2, \vec{n} -n_2+1}$	$f_{3, \vec{n} -n_3+1}$	$f_{3, \vec{n} -n_3+1}$

$$+ O(z^{|\vec{n}|+2}).$$

Mahler-Cramer systems (provided that $K(\vec{n}) \neq 0$)

Up to normalization there exist unique m neighboring HP-approximants in $m \times m$ matrix

$$M(\vec{n}, z) = [P(\vec{n} + \vec{e}_1, z), \dots, P(\vec{n} + \vec{e}_m, z)], \text{ degrees } \begin{bmatrix} = n_1 & < n_1 & \cdots & \cdots & < n_1 \\ < n_2 & = n_2 & < n_2 & \cdots & < n_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ < n_m & \cdots & \cdots & < n_m & = n_m \end{bmatrix},$$

with coefficients of entries in integral domain if leading coefficient of diagonal entries = $d(\vec{n}) = \pm K(\vec{n})$.

Mahler-Cramer systems (provided that $K(\vec{n}) \neq 0$)

Also, up to normalization there exist unique m neighboring SP-approximants in $m \times m$ matrix

$$\mathfrak{M}(\vec{n}, z) = [\mathfrak{p}(\vec{n} - \vec{e}_1, z), \dots, \mathfrak{p}(\vec{n} - \vec{e}_m, z)],$$
$$\text{degrees} \begin{bmatrix} = |\vec{n}| - n_1 & < |\vec{n}| - n_1 & \cdots & \cdots & < |\vec{n}| - n_1 \\ < |\vec{n}| - n_2 & = |\vec{n}| - n_2 & < |\vec{n}| - n_2 & \cdots & < |\vec{n}| - n_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ < |\vec{n}| - n_m & \cdots & \cdots & < |\vec{n}| - n_m & = |\vec{n}| - n_m \end{bmatrix},$$

with coefficients of entries in integral domain if leading coefficient of diagonal entries $d(\vec{n}) = \pm K(\vec{n})$.

All coefficients are of size $\mathcal{O}(|\vec{n}|^\kappa)$.

Outline

Basic definitions

- Simultaneous Padé Approximants

- Fraction-free computations

- Vector Hermite-Padé approximants

- The work of Mahler

- Other known algorithms

The Algorithm

- Cramer solutions

- Mahler-Cramer systems

Algorithm Details

- Pivot column

- The other columns

- Example

Computing SP Mahler-Cramer systems recursively

One has $\mathfrak{M}(\vec{0}, z) = I_m$. Suppose that $d(\vec{n}) = \pm K(\vec{n}) \neq 0$.

- ▶ How to check that $K(\vec{n} + \vec{e}_\pi) \neq 0$?
This is true for at least one $\pi \in \{1, 2, \dots, m\}$ since $f_1(0) \neq 0$.
- ▶ How to compute $\mathfrak{M}(\vec{n} + \vec{e}_\pi, z)$ from $\mathfrak{M}(\vec{n}, z)$? Column-wise
 - ▶ $d(\vec{n})p(\vec{n}, z) = \sum_{j=1}^m y_j p(\vec{n} - \vec{e}_j, z)$
and $d(\vec{n} + \vec{e}_\pi) = y_\pi$.
 - ▶ for $j \neq \pi$: $d(\vec{n})p(\vec{n} + \vec{e}_\pi - \vec{e}_j, z) = y_\pi z p(\vec{n} - \vec{e}_j, z) - b_j p(\vec{n}, z)$.

How to find the y_j ? Step 1

In order to obtain the correct degrees/order conditions for SP approximants of type \vec{n} , we require that

$$\begin{bmatrix} c_{2,1} & c_{2,2} & \cdots & c_{2,m} \\ c_{3,1} & c_{3,2} & \cdots & c_{3,m} \\ \vdots & \vdots & & \vdots \\ c_{m,1} & c_{m,2} & \cdots & c_{m,m} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where

$$-f_\ell(z)p_1(\vec{n} - \vec{e}_j, z) + f_1(z)p_\ell(\vec{n} - \vec{e}_j, z) = z^{|\vec{n}|}c_{\ell,j} + \mathcal{O}(z^{|\vec{n}|+1})$$

How to find the y_j ? Step 2

We have seen that there exist $\delta_\ell, \epsilon_j \in \{\pm 1\}$ such that $\delta_\ell \epsilon_j c_{\ell,j} = \det$

$$\begin{array}{c|c|c|c|c}
 f_{1,0} & & f_{2,0} & & f_{3,0} \\
 f_{1,1} & f_{1,0} & f_{2,1} & f_{2,0} & f_{3,0} \\
 \vdots & \vdots & \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots \\
 f_{1,|\vec{n}|+1} & f_{1,|\vec{n}|} & f_{2,|\vec{n}|+1} & f_{2,|\vec{n}|} & f_{3,|\vec{n}|} \\
 0 & 0 & 0 & 0 & 0
 \end{array}$$

(here $m = 3$, $\ell = 2$, $j = 3$) and thus Cramer solution

$$y_j = \det \begin{bmatrix} c_{2,1} & c_{2,2} & c_{2,3} & \cdots & c_{2,m} \\ c_{3,1} & c_{3,2} & c_{3,3} & \cdots & c_{3,m} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{m,1} & c_{m,2} & c_{m,3} & \cdots & c_{m,m} \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} = \pm [f_1(0)K(\vec{n})]^{m-2} K(\vec{n} + \vec{e}_j).$$

How to find the y_j ? Step 3

Thus solution of full rank system

$$\begin{bmatrix} c_{2,1} & c_{2,2} & \cdots & c_{2,m} \\ c_{3,1} & c_{3,2} & \cdots & c_{3,m} \\ \vdots & \vdots & & \vdots \\ c_{m,1} & c_{m,2} & \cdots & c_{m,m} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

with FFGE and first pivot $c_{1,0}^0 = f_1(0)d(\vec{n})$ gives components $y_j = \pm K(\vec{n} + \vec{e}_j)$.

At least one $y_\pi \neq 0$.

What to do with the y_j ?

Let $y_\pi = \pm K(\vec{n} + \vec{e}_\pi) \neq 0$.

- ▶ Cramer approximant $p(\vec{n}, z)$ is unique SP approximant of type \vec{n} up to scaling.

It's π th component is of degree $|\vec{n}| - n_\pi$, with leading coefficient $\pm y_\pi$.



$$p(z) = \sum_{j=1}^m y_j p(\vec{n} - \vec{e}_j, z)$$

satisfies the degree/order conditions for an SP approximant of type \vec{n} .

It's π th component is of degree $|\vec{n}| - n_\pi$, with leading coefficient $y_\pi d(\vec{n})$.

Conclusion: $p(z) = d(\vec{n}) p(\vec{n}, z)$.

How to get the other columns of the new Mahler-Cramer system?

Let $y_\pi = \pm K(\vec{n} + \vec{e}_\pi) \neq 0$, and $j \neq \pi$.

- ▶ Cramer approximant $p(\vec{n} + \vec{e}_\pi - \vec{e}_j, z)$ is unique SP approximant of type $\vec{n} + \vec{e}_\pi - \vec{e}_j$ up to scaling. It's j th component is of degree $|\vec{n}| - n_j$, with leading coefficient $\pm y_\pi$.
- ▶ $p(z) = z y_\pi p(\vec{n} - \vec{e}_j, z) - b_j p(\vec{n}, z)$ with

$$b_j = \text{coeff}(p_\pi(\vec{n} - \vec{e}_j, z), z^{|\vec{n}-\vec{e}_j|-n_\pi})$$

satisfies the degree/order conditions for an SP approximant of type $\vec{n} + \vec{e}_\pi - \vec{e}_j$.

It's j th component is of degree $|\vec{n} + \vec{e}_\pi - \vec{e}_j| - (n_j - 1)$, with leading coefficient $y_\pi d(\vec{n})$.

Conclusion: $p(z) = d(\vec{n}) p(\vec{n} + \vec{e}_\pi - \vec{e}_j, z)$.

A small example...

Consider $\vec{n} = (3, 4, 3)$ and

$$f_1(z) = 3 + 3z + 6z^2 + 18z^3 + 72z^4 + 360z^5 + O(z^{11})$$

$$f_2(z) = 1 + 8z^3 + 64z^6 + 512z^9 + O(z^{11})$$

$$f_3(z) = 1 - z + z^2 - z^3 + z^4 - z^5 + O(z^{11})$$

Closest normal indices

$$(0, 0, 0), (0, 1, 0), (1, 1, 0), (1, 2, 0), (1, 2, 1), (2, 2, 1), (2, 3, 1), \dots$$

After step 4 the closest normal index is $\vec{v}^{(4)} = [1, 2, 1]$ with the SP Mahler-Cramer system given by

$$\mathfrak{M}(\vec{v}^{(4)}, z) = \begin{bmatrix} 48z^3 + 33z^2 + 30z + 3 & -9z^2 - 126z - 27 & -54z^2 - 36z - 18 \\ 9z + 1 & 48z^2 - 33z - 9 & -6z - 6 \\ -8z^2 + 8z + 1 & 72z^2 - 24z - 9 & 48z^3 - 6 \end{bmatrix}.$$

Columns SP approximants of index $(0, 2, 1)$, $(1, 1, 1)$ and $(1, 2, 0)$.

To construct $\mathfrak{M}(\vec{v}^{(5)}, z)$ the C matrix for the next step is given by

$$\begin{bmatrix} -6 & 54 & -252 \\ 210 & -738 & -252 \end{bmatrix}, \quad \text{with kernel} \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1386 \\ 378 \\ 48 \end{bmatrix}.$$

New pivot index is $\pi = 1$. After replacing column 1 of $\mathfrak{M}(\vec{v}^{(4)}, z)$ by the linear combination of all three columns, we get a new column 1 which has order 5. Multiplying columns 2 and 3 by z then implies that all the columns now have order 5. In this case the degrees of the resulting matrix polynomial are

$$\begin{bmatrix} 3 & 3 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$$

and the leading coefficient of the 1, 1 term is 48×1386 . Eliminating the highest terms in row 1 of columns 2 and 3 using cross multiplication with the new column 1 then gives degrees of the form

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}.$$

Correct degree bounds for the SP Mahler-Cramer system for the closest normal point $\vec{v}^{(5)} = (2, 2, 1)$, it remains to divide by 48.

Future Research

- ▶ Fraction-Free \rightarrow modular methods
- ▶ Use alternative order basis algorithm for noncommutative case of Ore matrix polynomials

Code in Maple available at
www.cs.uwaterloo.ca/~glabahn/pade-code .