Scaling Symmetries and Integer Linear Algebra

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Outline

References

Report on papers

- E. Hubert and G. Labahn, Rational invariants of scalings from Hermite normal forms, To appear *Proceedings of ISSAC 2012*, Grenoble, France, July 22-25, 2012.
- E. Hubert and G. Labahn, Scaling Invariants and Symmetry Reduction of Dynamical Systems, Submitted 2012.

Polynomial System (Ürgüplü and Lemaire)

$$\begin{cases} 0 & = & z_2{z_4}^2 - z_1 \\ 0 & = & z_1 z_3 - z_2 \end{cases}$$

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where new coordinates are :

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where new coordinates are :

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Solution set is :

$$(z_1, z_2, z_3, z_4) = (\lambda_1, \lambda_2, \frac{\lambda_2}{\lambda_1}, \sqrt{\frac{\lambda_1}{\lambda_2}})$$

with λ_1, λ_2 parameters.

Motivation : Polynomial System II

$$\begin{cases} z_1^4 z_3^6 - 5 z_1^2 z_2 z_3^3 + 6 z_2^2 & = & 0 \\ z_1^2 z_2^5 z_3^4 z_4^4 - 2 z_1^3 z_2^2 z_3^5 z_4^2 - z_1^2 z_3^3 + z_2 & = & 0 \\ z_1 z_2^3 z_3^4 z_4^3 z_5 - z_1^2 z_3^3 - z_2 & = & 0. \end{cases}$$

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- Reduced high dimensional to zero-dimensional problem.
- Solve zero-dimensional system (4 solutions).
- Parameterized solutions determined from 4 points.

Motivation : Dynamical Systems

Predator-prey model [Murray, Mathematical Biology (2002)]

$$\begin{cases} \dot{n} &= \left(\left(1-\frac{n}{k_1}\right) r - k_2 \frac{p}{n+e}\right) n, \\ \dot{p} &= s\left(1-h\frac{p}{n}\right) p. \end{cases}$$

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r, s, e, h, k_1, k_2 parameters.

$\mathfrak{s}, \mathfrak{h}, \mathfrak{k}$ parameters

$$\mathfrak{t} = r\,t,\ \mathfrak{n} = \frac{n}{e},\ \mathfrak{p} = \frac{h\,p}{e},\ \mathfrak{s} = \frac{s}{r},\ \mathfrak{h} = \frac{k_2}{rh}, \mathfrak{k} = \frac{k_1}{e}.$$

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Solutions of reduced system constructs solution to original system.

Common Theme : Invariants under Scalings

Example 1: Coordinates $y_1 = \frac{z_1 z_3}{z_2}$, $y_2 = \frac{z_2 z_4^2}{z_1}$, invariant under 2 parameter group action:

$$\begin{array}{rclrcl} z_1 & \rightarrow & \alpha\, z_1, & z_3 & \rightarrow & \beta^2\, z_3, \\ z_2 & \rightarrow & \alpha\beta^2 z_2 & z_4 & \rightarrow & \frac{1}{\beta}\, z_4, \end{array}$$

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Example 2: Coordinates $y_1 = \frac{z_1^2 z_3^3}{z_2},\ y_2 = z_1 z_2^2 z_3^2 z_4^2,\ y_3 = z_3 z_4 z_5,$ invariant under 2 parameter group action :

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Example 2: Coordinates $y_1 = \frac{z_1^2 z_3^3}{z_2},\ y_2 = z_1 z_2^2 z_3^2 z_4^2,\ y_3 = z_3 z_4 z_5,$ invariant under 2 parameter group action :

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Example 3: Coords $\mathfrak{t} = r\,t,\ \mathfrak{n} = \frac{n}{e},\ \mathfrak{p} = \frac{h\,p}{e},\ \mathfrak{s} = \frac{s}{r},\ \mathfrak{h} = \frac{k_2}{rh}, \mathfrak{k} = \frac{k_1}{e}.$ invariant under 3 parameter group action:

$$\begin{array}{lllllllll} t & \rightarrow & \lambda^{-1}\, t, & r & \rightarrow & \lambda\, r, & h & \rightarrow & \nu\, h, \\ n & \rightarrow & \mu\, n, & s & \rightarrow & \lambda\, s, & k_1 & \rightarrow & \mu\, k_1, \\ p & \rightarrow & \mu\nu^{-1}\, p, & e & \rightarrow & \mu\, e, & k_2 & \rightarrow & \lambda\nu\, k_2 \end{array}$$

Symmetry Reduction [Mansfield 01]

- Determine the symmetry
- Compute a generating set of invariants
- Rewrite the system in terms of these invariants

This talk : Scaling Symmetry Reduction

- Determine the scaling symmetry
- Compute a generating set of monomial invariants
- Rewrite the system in terms of these invariants

It's all linear algebra in the case of scalings!

Notes

Facts: In each example :

- One has a scaling symmetry.

$$(x_1, \ldots, x_n) \mapsto (s_1 \cdot x_1, \cdots, s_n \cdot x_n)$$

  where $s_j = \lambda_1^{a_{1j}} \cdots \lambda_r^{a_{rj}}$.

- Reduced system given in terms of invariants of symmetry.
- The reduced system is easier

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Questions:

- How to determine scaling symmetry?
- How to build reduced system
- How to reverse the process?

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Questions:

- How to determine scaling symmetry? (left nullspace)
- How to build reduced system (Invariants + Rewrite Rules)
- How to reverse the process? (Rational Section)

Scaling Symmetry

- Starting point : Matrix A of exponents giving scaling

$$\begin{array}{llllllllll} t & \rightarrow & \alpha^{-1}\, t, & r & \rightarrow & \alpha\, r, & h & \rightarrow & \nu\, h, \\ n & \rightarrow & \mu\, n, & s & \rightarrow & \alpha\, s, & k_1 & \rightarrow & \mu\, k_1, \\ p & \rightarrow & \mu\nu^{-1}\, p, & e & \rightarrow & \mu\, e, & k_2 & \rightarrow & \alpha\nu\, k_2 \end{array}$$

	s	r	e	h	k_1	k_2	n	p	t
ν	0	0	0	1	0	1	0	−1	0
μ	0	0	1	0	1	0	1	1	0
α	1	1	0	0	0	1	0	0	−1

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$$A = \left| \begin{array}{ccccccccc} 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{array} \right|$$

Scaling Matrix Notation :

$$(\nu, \mu, \alpha)^{A} \star (s, r, e, h, k_1, k_2, n, p, t) =$$

$$(\alpha\, s, \alpha\, r, \mu\, e, \nu\, h, \mu\, k_1, \alpha\nu\, k_2, \mu\, n, \mu\nu^{-1}\, p, \alpha^{-1}\, t)$$

Hermite Normal Forms

$H \in \mathbb{Z}^{r \times n}$, rank $r < n$ in (column) Hermite normal form if

$$H = (H_i, 0)$$

- H_i upper triangular
- all elements in H_i positive
- diagonal elements in H_i largest in each row

Can always transform any integer matrix A to one in column Hermite form via integer column operations. Same as:

- There exists a unimodular V such that $A \cdot V = (H, 0)$

Unimodular Multiplier

There exists a unimodular V such that $A \cdot V = (H, 0)$

- Unimodular $\equiv \det(V) = \pm 1$.
  - Thus have $W = V^{-1} \in \mathbb{Z}^{n \times n}$
- Useful to write : $A \cdot V = A \cdot (V_i, V_n) = (H, 0)$.
- Useful to write : $V \cdot W = [V_i, V_n] \begin{bmatrix} W_u \\ W_d \end{bmatrix} = I$.
- V not unique. Can normalize this.
- Note : Integer linear algebra not expensive.

Example

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}$$

- Hermite Normal Form : there exists $V \in \mathbb{Z}^{n\times n}$ unimodular

$$A\,V = (H, 0)$$

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$$A \begin{pmatrix} V_i & V_n \end{pmatrix} = \begin{pmatrix} H & 0 \end{pmatrix}$$

Example

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$$V = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Invariants

F is invariant under scaling $x \mapsto \lambda^A \star x$ if

$$F(\lambda^A \star x) = F(x).$$

Example : $A\,(V_i, V_n) = (H, 0)$. $F(x) = x^{V_n}$

- x^{V_n} is an invariant of the scaling

$$(\lambda^A \star x)^{V_n} = \lambda^{AV_n} \star x^{V_n} = x^{V_n}$$

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$$(\lambda^A \star x)^{V_n} = \lambda^{AV_n} \star x^{V_n} = x^{V_n}$$

- Important : x^{V_n} generates *all* scaling invariants!

Example

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}$$

$$A \begin{pmatrix} V_i & V_n \end{pmatrix} = \begin{pmatrix} H & 0 \end{pmatrix}$$

$$V = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Invariants

$$\begin{pmatrix} s & r & e & h & k_1 & k_2 & n & p & t \end{pmatrix}^{V_n} = \begin{pmatrix} \frac{s}{r} & \frac{k_1}{e} & \frac{k_2}{r\,h} & \frac{n}{e} & \frac{h\,p}{e} & r\,t \end{pmatrix}$$

Rewrite Rules

$$A\left(V_i, V_n\right) = \left(H, 0\right)$$

Know : $y = x^{V_n}$ generates all invariants for scaling.

- Question : How to rewrite any invariant in terms of y?

Rewrite Rules

$$A\,(V_i, V_n) = (H, 0) \qquad V^{-1} = \begin{pmatrix} W_u \\ W_d \end{pmatrix}$$

Know : $y = x^{V_n}$ generates all invariants for scaling.

- Question : How to rewrite any invariant in terms of y?
- Answer : Replace : $x = y^{W_d}$

Call these *replacement rewrite rules*.

Example

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$$s \rightarrow \mathfrak{s}, r \rightarrow 1, e \rightarrow 1, h \rightarrow 1, k_1 \rightarrow \mathfrak{k}, k_2 \rightarrow \mathfrak{h}, n \rightarrow \mathfrak{n}, p \rightarrow \mathfrak{p}, t \rightarrow \mathfrak{t}$$

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How to solve problems $F(x_1, \ldots, x_n) = 0$?

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  - Get invariants $y = x^{V_n}$ and rewrite rules $x = y^{W_d}$.

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  - Roughly : Variety defined by $x^{V_i} = 1$.
  - Provides point of intersection of orbits to invariants.

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- Question: How to reverse this to solve in terms of x?
- Answer : Rational Section.
  - Roughly : Variety defined by $x^{V_i} = 1$.
  - Provides point of intersection of orbits to invariants.
  - Scaling parameterizes solutions along orbits.

Example : Recall Polynomial System

$$\left\{\begin{array}{rcl} z_1^4 z_3^6 - 5 z_1^2 z_2 z_3^3 + 6 z_2^2 & = & 0 \\ z_1^2 z_2^5 z_3^4 z_4^4 - 2 z_1^3 z_2^2 z_3^5 z_4^2 - z_1^2 z_3^3 + z_2 & = & 0 \\ z_1 z_2^3 z_3^4 z_4^3 z_5 - z_1^2 z_3^3 - z_2 & = & 0. \end{array}\right. \qquad \left\{\begin{array}{rcl} y_1^2 - 5 y_1 + 6 & = & 0 \\ y_2^2 - 2 y_1 y_2 + y_1 + 1 & = & 0 \\ y_2 y_3 - y_1 - 1 & = & 0. \end{array}\right.$$

new coordinates are :

$$y_1 = \frac{z_1^2 z_3^3}{z_2}, \; y_2 = z_1 z_2^2 z_3^2 z_4^2, \; y_3 = z_3 z_4 z_5,$$

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Solving Polynomial System II

- Solution set reduced system:

$$(2, 1, 3), (2, 3, 1), (3, 3 + \sqrt{5}, 3 - \sqrt{5}), (3, 3 - \sqrt{5}, 3 + \sqrt{5}).$$

- Rational section is variety of $(z_1 z_2 z_3 z_4 - 1, z_2 z_4 - 1)$.

Solving Polynomial System II

- Solution set reduced system:

$$(2,1,3),(2,3,1),(3,3+\sqrt{5},3-\sqrt{5}),(3,3-\sqrt{5},3+\sqrt{5}).$$

- Rational section is variety of $(z_1 z_2 z_3 z_4 - 1, z_2 z_4 - 1)$.
- Intersection of solution set of original system are

$$(2,1,3)^{W_d} = (1,\frac{1}{2},1,2,\frac{3}{2}), \quad (2,3,1)^{W_d} = (\frac{1}{3},3,3,\frac{1}{3},\frac{1}{2}),$$

$$(3,3+\sqrt{5},3-\sqrt{5})^{W_d}, \quad (3,3-\sqrt{5},3+\sqrt{5})^{W_d}$$

Solving Polynomial System II

- Solution set reduced system:

$$(2,1,3),(2,3,1),(3,3+\sqrt{5},3-\sqrt{5}),(3,3-\sqrt{5},3+\sqrt{5}).$$

- Rational section is variety of $(z_1z_2z_3z_4-1, z_2z_4-1)$.
- Intersection of solution set of original system are

$$(2,1,3)^{W_d}=(1,\frac{1}{2},1,2,\frac{3}{2}),\quad (2,3,1)^{W_d}=(\frac{1}{3},3,3,\frac{1}{3},\frac{1}{2}),$$

$$(3,3+\sqrt{5},3-\sqrt{5})^{W_d},\quad (3,3-\sqrt{5},3+\sqrt{5})^{W_d}$$

- Elements in orbits of these points solves the original system.

  Gives four parameterized two dimensional solution subsets.

  e.g. $\lambda^A \star (2,1,3)^{W_d} = (\mu^6, \frac{\nu^3}{2}, \frac{\nu}{\mu^4}, \frac{2\nu}{\mu^4}, \frac{3\mu^3\nu^3}{2})$

  is a parameterized two-dimensional subset of solutions.

A geometric picture

$$A = \begin{pmatrix} 1 & 3 \end{pmatrix}$$

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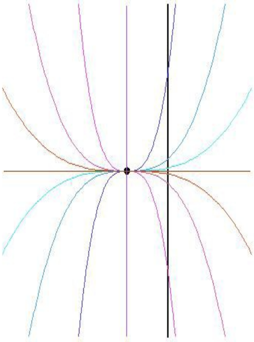

Scaling: $X = \lambda x, \; Y = \lambda^3 y$

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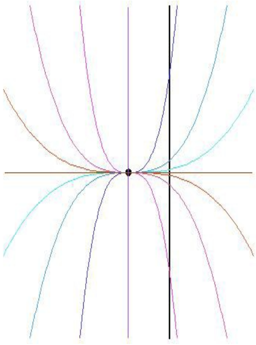
Invariant: $r = \frac{y}{x^3}$

Rewriting: $x \rightarrow 1,\ y \rightarrow r$



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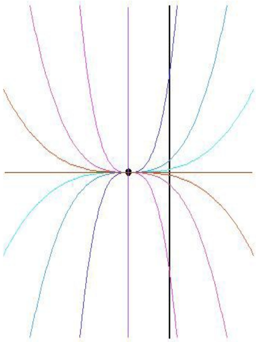

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Symmetry reduction

$\simeq$

Projection on the cross-section along the orbits

Dynamic System

$$
\begin{aligned}
n'(t) &= n(t)\left(r\left(1-\frac{n(t)}{K}\right)-k\frac{p(t)}{n(t)+d}\right),\\
p'(t) &= s\,p(t)\left(1-\frac{hp(t)}{n(t)}\right)
\end{aligned}
$$

- The reduced system

$$
\begin{aligned}
\mathfrak{n}'(\mathfrak{t}) &= \mathfrak{n}(\mathfrak{t})\left((1-\mathfrak{n}(\mathfrak{t}))-\mathfrak{k}\frac{\mathfrak{p}(\mathfrak{t})}{\mathfrak{n}(\mathfrak{t})+\mathfrak{d}}\right),\\
\mathfrak{p}'(\mathfrak{t}) &= \mathfrak{s}\,\mathfrak{p}(\mathfrak{t})\left(1-\frac{\mathfrak{p}(\mathfrak{t})}{\mathfrak{n}(\mathfrak{t})}\right).
\end{aligned}
$$

$$r \mapsto 1,\ h \mapsto 1,\ K \mapsto 1,\ s \mapsto \mathfrak{s},\ k \mapsto \mathfrak{k},\ d \mapsto \mathfrak{d},\ t \mapsto \mathfrak{t},\ n \mapsto \mathfrak{n},\ p \mapsto \mathfrak{p}.$$

- If $(\mathfrak{s}, \mathfrak{k}, \mathfrak{d}, \mathfrak{n}(\mathfrak{t}), \mathfrak{p}(\mathfrak{t}))$ solves reduced system, $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})$ any constants

$$\left(\frac{1}{\mathfrak{a}},\ \frac{\mathfrak{b}}{\mathfrak{c}},\ \mathfrak{b},\ \frac{\mathfrak{s}}{\mathfrak{a}},\ \frac{\mathfrak{b}}{\mathfrak{a}\,\mathfrak{c}}\,\mathfrak{k},\ \mathfrak{b}\,\mathfrak{d},\ \mathfrak{b}\,\mathfrak{n}\left(\frac{t}{\mathfrak{a}}\right),\ \mathfrak{c}\,\mathfrak{p}\left(\frac{t}{\mathfrak{a}}\right)\right)$$

is a solution of the original system.

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- Differential systems of higher order