Rational invariants of scalings from Hermite normal forms

Evelyne Hubert [1]   George Labahn [2]

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[1] INRIA Méditerranée, France.
[2] University of Waterloo, Canada.

References

Report on papers

- E. Hubert and G. Labahn, Rational invariants of scalings from Hermite normal forms, *Proceedings of ISSAC 2012*, Grenoble, France, July 22-25, 2012.
- E. Hubert and G. Labahn, Scaling Invariants and Symmetry Reduction of Dynamical Systems, 2012. http://hal.inria.fr/inria-0066888

Dimensional Analysis:

Predator-prey model [Murray, Mathematical Biology (2002)]

$$\begin{aligned} \frac{dn}{dt} &= n\left(r\left(1-\frac{n}{K}\right)-k\frac{p}{n+d}\right), \\ \frac{dp}{dt} &= s\,p\left(1-h\,\frac{p}{n}\right) \end{aligned}$$

r, s, h, k, K, d parameters.

Initial variables are invariant under 3 parameter group action:

$$\begin{array}{lll} t \rightarrow \eta\, t, & r \rightarrow \eta^{-1}\, r, & s \rightarrow \eta^{-1}\, s, \\ n \rightarrow \mu\, n, & h \rightarrow \mu\,\nu^{-1}\, h & k \rightarrow \eta^{-1}\mu\nu^{-1}\, k, \\ p \rightarrow \nu\, p, & K \rightarrow \mu\, K, & d \rightarrow \mu\, d. \end{array}$$

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$$\left\{\begin{aligned} \frac{d\mathfrak{n}}{d\mathfrak{t}} &= \mathfrak{n}\,(1-\mathfrak{n})-\mathfrak{k}\frac{\mathfrak{n}\,\mathfrak{p}}{\mathfrak{n}+\mathfrak{d}}, \\ \frac{d\mathfrak{p}}{d\mathfrak{t}} &= \mathfrak{s}\,\mathfrak{p}\left(1-\frac{\mathfrak{p}}{\mathfrak{n}}\right) \end{aligned}\right.$$

r, s, h, k, K, d parameters.

$\mathfrak{s}, \mathfrak{h}, \mathfrak{k}$ parameters

$$\mathfrak{s}=\frac{s}{r},\ \mathfrak{k}=\frac{k}{rh},\ \mathfrak{d}=\frac{d}{K},\quad \mathfrak{t}=r\,t,\ \mathfrak{n}=\frac{n}{K},\ \mathfrak{p}=\frac{h}{K}\,p$$

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This talk : Scaling Symmetries

- Scaling Symmetries : $(\mathbb{K}^*)^r \times \mathbb{K}^n \to \mathbb{K}^n$

$$(z_1, z_2, z_3, z_4, z_5) \to \left(\alpha^6 z_1, \beta^3 z_2, \frac{\beta}{\alpha^4} z_3, \frac{\alpha}{\beta^4} z_4, \alpha^3\beta^3 z_5\right).$$

- Important constructions
  - rational invariants, rewrite rules, rational section

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- Important constructions
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- Our Contributions
  - Integer linear algebra solves all scaling problems
    - gives simple, complete and elegant description of above
  - Scalings of Dynamical systems
  - Reduction of Polynomial Systems given scaling

Other work

- Previous work
  - F. Lemaire and A. Ürgüplü (ISSAC 2010)
  - E. Hubert and A. Sedoglavic (2006), A. Sedoglavic (2007)

- Scaling symmetries are used in
  - reducing dynamical systems
  - reducing polynomial systems
  - dimensional analysis (e.g. Buckingham π theorem)
  - $\dots$

Integer Linear Algebra I

- Scaling matrix
- Matrix notation and its properties

Integer Linear Algebra I : Scaling Matrix

Scaling :

$$(z_1,\, z_2,\, z_3,\, z_4,\, z_5) \to \left(\alpha^6\, z_1,\, \beta^3\, z_2,\, \frac{\beta}{\alpha^4}\, z_3,\, \frac{\alpha}{\beta^4}\, z_4,\, \alpha^3\beta^3\, z_5\right).$$

Scaling matrix:

$$A := \begin{bmatrix} 6 & 0 & -4 & 1 & 3 \\ 0 & 3 & 1 & -4 & 3 \end{bmatrix}$$

Exponent notation:

$$(\,\alpha,\, \beta\,)^A = \left(\alpha^6,\, \beta^3,\, \frac{\beta}{\alpha^4},\, \frac{\alpha}{\beta^4},\, \alpha^3\beta^3\right)$$

Scaling Symmetry using Matrix Notation

- Scaling symmetry : $(\ \lambda,\ \mathbf{z}\ ) \rightarrow \lambda^{A} \star \mathbf{z}$

  Star operator $\star$ is pointwise multiplication

- Some Properties :

$$
\begin{aligned}
(\lambda \star \mu)^{A} &= \lambda^{A} \star \mu^{A} \\
\lambda^{A+B} &= \lambda^{A} \star \lambda^{B} \\
\lambda^{AB} &= (\lambda^{A})^{B} \\
\lambda^{[\ A\ ,\ B\ ]} &= [\ \lambda^{A}\ ,\ \lambda^{B}\ ]
\end{aligned}
$$

Rational Invariants $\mathbb{K}(z)^A$

Definition

$F(z)$ is *invariant* under scaling $\mathbf{z} \mapsto \lambda^A \star \mathbf{z}$ if $F(\lambda^A \star \mathbf{z}) = F(\mathbf{z})$

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Invariant Laurent monomials:

$$\mathbf{z}^v = z_1^{v_1} \cdots z_n^{v_n}, \quad v \in \mathbb{Z}^n$$

$$(\lambda^A \star \mathbf{z})^v = \lambda^{Av}\mathbf{z}^v = \mathbf{z}^v \quad \Leftrightarrow \quad A \cdot v = 0$$

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Lemma

Rational Invariants: $F(z) \in \mathbb{K}(z)^A$*:*

$$F(z) = \frac{\sum_{v \in \ker A \cap \mathbb{Z}^n} a_v z^v}{\sum_{v \in \ker A \cap \mathbb{Z}^n} b_v z^v}$$

Integer Linear Algebra II

- Hermite normal form
- Unimodular multiplier
- Normalized unimodular multiplier

Hermite Normal Forms

$A \in \mathbb{Z}^{r \times n}$, rank $r < n$ in (column) Hermite normal form if

$$A = \begin{bmatrix} 7 & 5 & 4 & 3 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Zero elements in right column.

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Upper triangular in left columns with nonnegative entries.

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Diagonal entries in left columns largest in each row.

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Can always transform any integer matrix A to one in column Hermite form via integer column operations.

Same as: there exists a unimodular V such that $A \cdot V = [H, 0]$

Unimodular Multiplier

There exists a unimodular V such that $A \cdot V = [H, 0]$

- Unimodular means $W = V^{-1} \in \mathbb{Z}^{n \times n}$
- Useful to write : $A \cdot V = A \cdot [V_i, V_n] = [H, 0]$.
- Columns of V_n form a basis for lattice of $\ker A \cap \mathbb{Z}^n$
- V not unique.

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- Columns of V_n form a basis for lattice of $\ker A \cap \mathbb{Z}^n$
- V not unique.
  - However can normalize it : put V_n in HNF, then reduce V_i
- Cost of HNF with normalized unimodular multiplier : $O^{\sim}\left(n^4 s\right)$.

(c.f. Storjohann Thesis)

Rational Invariants and Rewrite Rules

Theorem

$$A \in \mathbb{Z}^{r \times n}, \qquad A \cdot V = A \cdot [V_i, V_n] = [H, 0], \qquad W = \begin{bmatrix} W_u \\ W_d \end{bmatrix}.$$

(a) *The* $n - r$ *components of* $g = [z_1, \ldots, z_n]^{V_n}$ *form a generating set of rational invariants.*

(b) $F \in \mathbb{K}(z)^A \implies F(z) = F(g^{W_d})$

Why?

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Why? $I = V_i W_u + V_n W_d$ and any $v \in \ker A \subset \ker W_u$:

$$\begin{aligned} z^v &= z^{(V_i W_u + V_n W_d)v} \\ &= z^{V_i W_u v}\, z^{V_n W_d v} \\ &= z^{V_n W_u v} = (z^{V_n})^{W_d v} = (g^{W_d})^v \end{aligned}$$

Then use Lemma.

Example

- Scaling matrix : $A = \begin{bmatrix} 6 & 0 & -4 & 1 & 3 \\ 0 & 3 & 1 & -4 & 3 \end{bmatrix}$

$$V = \begin{bmatrix} 1 & 1 & 2 & 1 & 0 \\ 1 & 0 & -1 & 2 & 0 \\ 1 & 1 & 3 & 2 & 1 \\ 1 & 0 & & 2 & 1 \\ 0 & 0 & & & 1 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 2 & -2 & -2 & 3 & -1 \\ 0 & 3 & 1 & -4 & 3 \\ 0 & -1 & 0 & 1 & -1 \\ -1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- Invariants:

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} = \begin{bmatrix} z_1 & z_2 & z_3 & z_4 & z_5 \end{bmatrix}^{V_n} = \begin{bmatrix} \frac{z_1^2 z_3^3}{z_2} & z_1 z_2^2 z_3^2 z_4^2 & z_3 z_4 z_5 \end{bmatrix}$$

- Rewrite rule:

$$\begin{bmatrix} z_1 & z_2 & z_3 & z_4 & z_5 \end{bmatrix} \rightarrow \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^{W_d} = \begin{bmatrix} \frac{1}{y_2} & \frac{y_2}{y_1} & y_2 & \frac{y_1}{y_2} & \frac{y_3}{y_1} \end{bmatrix}$$

- e.g. Rational Invariant : $z_1 z_2^2 z_3^3 z_4^3 z_5 - \frac{z_1^2 z_3^3}{z_2} - 1$

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Example : Predator-Prey Model

$$\frac{dn}{dt} = n\left(r\left(1-\frac{n}{K}\right) - k\frac{p}{n+d}\right),$$
$$\frac{dp}{dt} = s\,p\left(1-h\,\frac{p}{n}\right)$$

Scaling Matrix:

$$A = \begin{bmatrix} -1 & 0 & 0 & -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Invariants:

$$\mathfrak{s} = \frac{s}{r},\ \mathfrak{k} = \frac{k}{rh},\ \mathfrak{d} = \frac{d}{K},\quad \mathfrak{t} = r\,t,\ \mathfrak{n} = \frac{n}{K},\ \mathfrak{p} = \frac{h}{K}\,p$$

Rewrite rule:

$r \mapsto 1,\ h \mapsto 1,\ K \mapsto 1,\ s \mapsto \mathfrak{s},\ k \mapsto \mathfrak{k},\ d \mapsto \mathfrak{d},\ t \mapsto \mathfrak{t},\ n \mapsto \mathfrak{n},\ p \mapsto \mathfrak{p}.$

Rational Section

Roughly : irreducible variety $\mathcal{P}$ in $\mathbb{K}^n$, $\mathcal{P} \cap \mathcal{O}_z = \{point\}$ for all z.

Theorem

$$A \in \mathbb{Z}^{r\times n}, \qquad A \cdot V = A \cdot [V_i, V_n] = [H, 0], \qquad W = \begin{bmatrix} W_u \\ W_d \end{bmatrix}.$$

(a) $\mathcal{P} : z^{V_i} = 1_r$ *is a rational section on* $(\mathbb{K}^*)^n$

(b) *For any* $z \in (\mathbb{K}^*)^n$, $\mathcal{P} \cap \mathcal{O}_z = \{z^{V_n W_d}\}$.

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Why? For any $z \in (\mathbb{K}^*)^n$:

$$\tilde{z} = \lambda^A \star z \in \mathcal{P} \quad \rightarrow \quad 1_r = (\lambda^A \star z)^{V_i} = \lambda^{A V_i} \star z^{V_i} \text{ so } \lambda^H = z^{-V_i}.$$

Thus

$$\begin{aligned} \tilde{z} &= (\lambda^A \star z)^{V_i W_u + V_n W_d} \\ &= z^{-V_i W_u} \star z^{V_i W_u + V_n W_d} = z^{V_n W_d} \end{aligned}$$

A geometric picture

$$A = \begin{bmatrix} 1 & 3 \end{bmatrix}$$

$$A \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

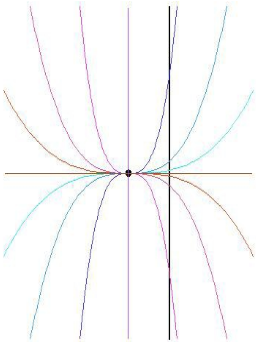
Scaling: $(\lambda z_1, \lambda^3 z_2)$

Invariant: $g = \frac{z_2}{{z_1}^3}$

Rewriting: $z_1 \to 1, \; z_2 \to g$

Section: $\mathcal{P} : z_1 = 1,$

$$\mathcal{O}_z \cap \mathcal{P} = \left\{ \left(1, \frac{z_2}{{z_1}^3} \right) \right\}$$



Reducing Polynomial System

How to reduce system $F(z_1, \ldots, z_n) = 0$ given a scaling matrix A?

1. Compute Hermite normal form, multiplier V and inverse W.

   Obtain invariants $\mathbf{y} = \mathbf{z}^{V_n}$ and rewrite rules $\mathbf{z} \to \mathbf{y}^{W_d}$.

2. Rewrite equations in terms of invariants and solve new system

3. All solutions $\mathbf{z}$ then of the form :

$$\mathbf{z} = \lambda^A \star \mathbf{y}^{W_d}$$

   for solutions $\mathbf{y}$ of reduced system.

Example : Polynomial System

$$\begin{array}{rcl} z_1^4 z_3^6 - 5 z_1^2 z_2 z_3^3 + 6 z_2^2 & = & 0 \\ z_1^2 z_2^5 z_3^4 z_4^4 - 2 z_1^3 z_2^2 z_3^5 z_4^2 + z_1^2 z_3^3 + z_2 & = & 0 \\ z_1 z_2^3 z_3^3 z_4^3 z_5 - z_1^2 z_3^3 - z_2 & = & 0. \end{array} \qquad \left\{ \begin{array}{rcl} y_1^2 - 5 y_1 + 6 & = & 0 \\ y_2^2 - 2 y_1 y_2 + y_1 + 1 & = & 0 \\ y_2 y_3 - y_1 - 1 & = & 0. \end{array} \right.$$

- Solution set of reduced system (in terms of $\mathbf{y}$):

$$(2,1,3), \quad (2,3,1), \quad (3, 3+\sqrt{5}, 3-\sqrt{5}), \quad (3, 3-\sqrt{5}, 3+\sqrt{5}).$$

- Gives four parameterized two dimensional solution subsets.

e.g.
$$\begin{array}{rcl} \lambda^A \star (2,\ 1,\ 3)^{W_d} & = & \lambda^A \star \left(1, \frac{1}{2}, 1, 2, \frac{3}{2}\right) \\ & = & \left(\alpha^6, \frac{\beta^3}{2}, \frac{\beta}{\alpha^4}, 2\frac{\beta}{\alpha^4}, \frac{3}{2}\alpha^3\beta^3\right) \end{array}$$