Numerical Solution of Hamilton-Jacobi-Bellman Equations in Finance

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Joint work with P.A. Forsyth (Waterloo).

This talk basically reports from :

(i) P.A.Forsyth and G. Labahn, Numerical Methods for Controlled Hamilton-Jacobi-Bellman PDEs in Finance, Journal of Computational Finance, 11(2) (2008) 1-44.

(ii) Y. Huang, P.A. Forsyth and G. Labahn, Combined Fixed Point and Policy Iteration for HJB Equations in Finance, Submitted to : SIAM Journal of Numerical Analysis (2010)

Outline

Motivation

Guaranteed Minimum Withdrawal Benefit (GMWB)

General Form

Viscosity Solutions

Solving Algebraic Equations

Policy Iteration

Example (GMWB continued)

Singular Control Version of GMWB

Conclusion

Outline

Variable Annuities

Quote from Washington Post, May, 2004

> *”They have stumbled onto a killer app for the financial needs of today’s boomers. It’s called a GMWB. The deal is that for a half-percentage point per year, you can invest with a guarantee that your entire principal will be returned to you, provided you do not withdraw at a rate greater than 7% annually.”*

The problem?

Globe and Mail article (December 2, 2008, “Manulife, in red, raises new equity,”), one of the large Canadian insurance companies, Manulife, posted a large mark-to-market writedown to account for losses associated with GMWB guarantees.

From the Globe and Mail Streetwise Blog, November 7, 2008

> *“Concerns that the market selloff will translate into massive future losses at Canada’s largest insurer sent Manulife shares reeling last month. Those concerns were a result of Manulife’s strategy of not fully hedging products such as annuities which promise investors income no matter what markets do.”*

The Retirement Risk Zone

Consider an investor with a retirement account, which is invested in the stock market

Over the long run (before retirement), it does not matter if

- the market first drops by 10% per year over several years and then goes up by 20% per year for several years; or
- the market first goes up by 20% per year and then drops by 10% per year

$$(.9)(.9)...(1.2)(1.2)... = (1.2)(1.2)...(.9)(.9)...$$

The Retirement Risk Zone II

This is not the case once the investor retires, and begins to make withdrawals from the retirement account

The outcomes will be very different in the cases:

- in the first few years after retirement, the market has losses, and the account is further depleted by withdrawals, followed by some years of good market returns; compared to
- a few years of good market returns, after retirement (including withdrawals), followed by some years of losses

Losses in the early years of retirement can be devastating in the long run!

Early bad returns can cause complete depletion of the account!

A Typical GMWB Example

Investor pays \$100 to an insurance company, which is invested in a risky asset.

Denote amount in risky asset sub-account by $W = 100$.

The investor also has a virtual guarantee account $A = 100$.

Suppose that the contract runs for 10 years, and the guaranteed withdrawal rate is \$10 per year.

A Typical GMWB Example II

At the end of each year, the investor can choose to withdraw up to \$10 from the account. If $\$\gamma \in [0, 10]$ is withdrawn, then

$$\begin{aligned} W_{new} &= \max(W_{old} - \gamma, 0) \quad ; \quad \textit{Actual investment} \\ A_{new} &= A_{old} - \gamma \quad ; \quad \textit{Virtual account} \end{aligned}$$

This continues for 10 years. At the end of 10 years, the investor can withdraw anything left, i.e. $\max(W_{new}, A_{new})$.

Note: the investor can continue to withdraw cash as long as $A > 0$, *even if* $W = 0$ (recall that W is invested in a risky asset).

Example: Order of Random Returns

Good Returns at Start

Time	Return (%)	Balance ($)	Withdrawal ($)
1	41.65	141.65	10
2	31.12	172.62	10
3	20.15	195.39	10
4	-30.25	129.31	10
5	18.05	140.85	10
6	16.82	152.86	10
7	20.12	171.60	10
8	7.44	173.62	10
9	-40.90	96.70	10
10	-7.5	80.20	10
Total Withdrawal Amount ($)			170.20
Ten year balance if no withdrawal ($)			151.37

Same Random Returns: Different Order

No GMWB

Time	Return (%)	Balance ($)	Withdrawal ($)
1	-30.25	69.75	10
2	-40.90	35.31	10
3	16.82	29.57	10
4	7.44	21.03	10
5	41.65	15.62	10
6	20.12	6.75	6.75
7	31.12	0	0
8	18.05	0	0
9	20.15	0	0
10	-7.5	0	0
Total Withdrawal Amount ($)			56.75
Ten year balance if no withdrawal ($)			151.37

Unlucky Order of Returns: With GMWB

GMWB Protection

Time	Return (%)	Balance ($)	Withdrawal ($)
1	-30.25	69.75	10
2	-40.90	35.31	10
3	16.82	29.57	10
4	7.44	21.03	10
5	41.65	15.62	10
6	20.12	6.75	10
7	31.12	0	10
8	18.05	0	10
9	20.15	0	10
10	-7.5	0	10
Total Withdrawal Amount ($)			100
Ten year balance if no withdrawal ($)			151.37

Why is this useful?

The investor can participate in market gains, but still has a guaranteed cash flow, in the case of market losses.

This insulates pensioners from losses in the early years of retirement.

This protection is paid for by deducting a yearly fee α from the amount in the risky account W each year.

The simple form of GMWB described has many variants in practice: Guaranteed Lifetime Withdrawal Benefit (GLWB), ratchet increase of virtual account A if no withdrawals, etc.

We will keep things simple here, and look at the basic GMWB.

Most variable annuities sold in North America have some type of market guarantee.

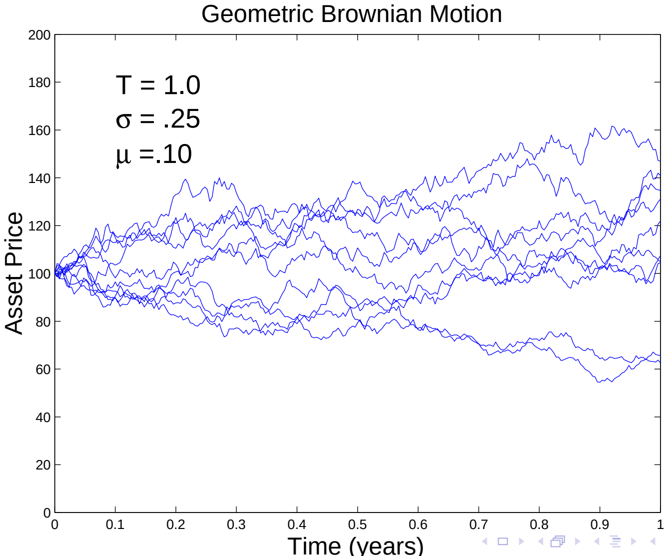
Outline

Stochastic Process for GMWB

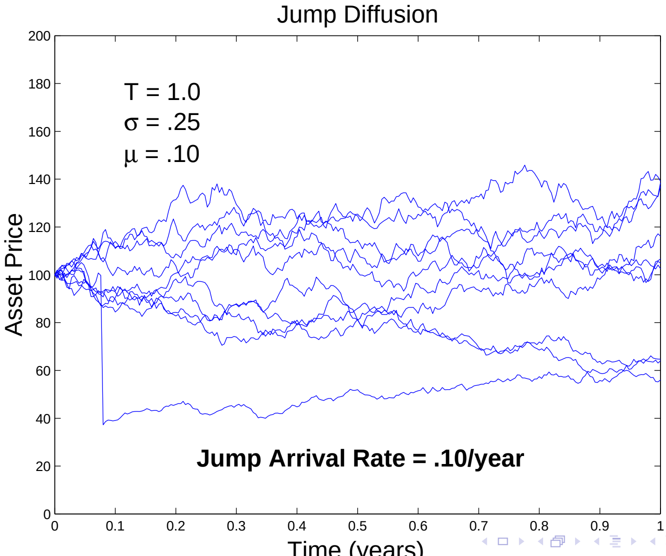
Let S denote the value of the risky asset, we assume that the risk neutral process followed by S is

$$
\begin{aligned}
dS &= rSdt + \sigma SdZ \\
&\quad r = \text{ risk free rate}; \quad \sigma = \text{ volatility} \\
&\quad dZ = \phi\sqrt{dt} \;\; ; \;\; \phi \sim \mathcal{N}(0,1)
\end{aligned}
$$

Stochastic Processes



Stochastic Processes



HJB Equations: A General Form

Many HJB problems encountered in finance can be written as

$$
\begin{aligned}
\mathcal{L}^Q V &\equiv a(x,\tau,Q)V_{xx} + b(x,\tau,Q)V_x - c(x,\tau,Q)V \\
& Q = \text{ control} \\
& c(x,\tau,Q) \geq 0
\end{aligned}
$$

$$
\begin{aligned}
V_\tau &= \sup_{Q \in \hat{Q}} \left\{ \mathcal{L}^Q V + d(x,\tau,Q) \right\} \\
& \hat{Q} = \text{ set of admissible controls} \quad .
\end{aligned}
$$

$+$ boundary and initial conditions (payoff, etc)

Examples

- Uncertain volatility : ($V(S,\tau)$ value of a contingent claim)

Short Position: $$V_\tau = \sup_{Q\in\hat{Q}} \left\{ \frac{Q^2 S^2}{2} V_{SS} + S V_S - rV \right\}$$

Long Position: $$V_\tau = \inf_{Q\in\hat{Q}} \left\{ \frac{Q^2 S^2}{2} V_{SS} + S V_S - rV \right\}$$

$$\hat{Q} = [\sigma_{min}, \sigma_{max}]$$

- Passport options
- Stock Borrowing Fees, Unequal Borrowing Lending Rates
- American options (penalty method)
- GMWB

Issues in Solving

- The HJB equation may not have smooth solutions. Need to consider viscosity solutions.
- Numerical solution not unique.
- Standard approach: determine optimal $Q(t)$ by differentiating $\{\mathcal{L}^Q V + d(Q)\}$ w.r.t. Q and setting to zero.

  From a computational perspective: this is a bad idea.
- Minimize truncation error of discretization

Example : HJB Equation for a Pension plan

HJB equation is

$$V_\tau \quad = \quad \sup_{p \in \hat{P}} \left\{ \frac{1}{2} (\sigma_X^p)^2 V_{xx} + \ \mu_X^p V_x \right\} \ ; \ \ x \in [0, \infty) \ ,$$

where

$$\begin{aligned} V(x, \tau = 0) \quad & = \gamma^{-1} x^\gamma \ \ ; \ \ \gamma < 0 \ \ , \\ & \mu_X^p = \pi + x(-\mu_Y + p\sigma_1(\xi_1 - \sigma_{Y_1}) + \sigma_{Y_0}^2 + \sigma_{Y_1}^2) \ \ , \\ & (\sigma_X^p)^2 = x^2(\sigma_{Y_0}^2 + (p\sigma_1 - \sigma_{Y_1})^2). \\ \hat{P} \quad & = \text{set of admissible allocation strategies} \end{aligned}$$

A Nice Example made Messy

Differentiating w.r.t. *p* and setting to zero in the Pension Plan example gives

$$\begin{aligned} V_\tau &= (\pi + \delta x) V_x + \frac{\sigma_{Y0}^2 x^2}{2} V_{xx} - \frac{(\xi_1 - \sigma_{Y1})^2}{2} \left(\frac{V_x^2}{V_{xx}} \right) \\ & \delta = -\mu_Y + \sigma_{Y0}^2 + \sigma_{Y1}\xi_1 \end{aligned} \tag{1}$$

We have taken a nice HJB equation, and made a mess of it!

Instead we do:

- First discretize
- Then, optimize

Find max/min of discrete equations not analytic approximations to discrete equations.

Viscosity Solution

What does it mean to solve a differential equation when the *solution* is not differentiable?

We can write our general HJB equation in the form

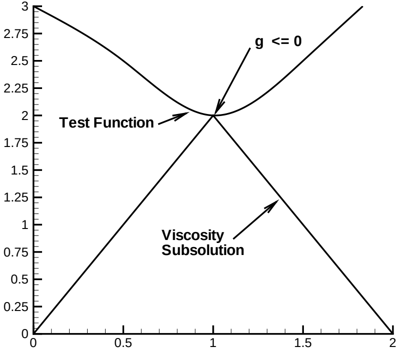
$$
\begin{aligned}
g(V_{xx}, V_x, V_\tau, V, x, \tau) &= V_\tau - \sup_{Q \in \hat{Q}} \left\{ \mathcal{L}^Q V + d(x, \tau, Q) \right\} \\
&= 0
\end{aligned}
$$

Suppose we have a $C^{2,1}$ test function ϕ such that $\phi \geq V$, and ϕ touches V at a single point (x_0, τ_0).

For simplicity, assume that V is continuous.

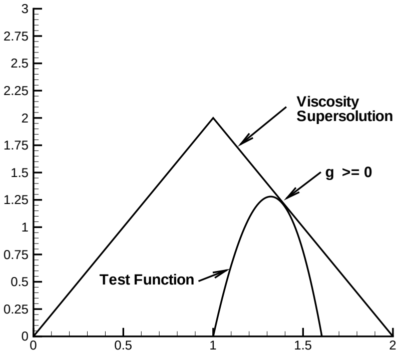
Subsolution

Figure: If, for any point (x_0, τ_0), for any test function $\phi \geq V$, where ϕ touches V at the single point (x_0, τ_0), $g(\phi_{xx}, \phi_x, \phi_\tau, \phi, x_0, \tau_0) \leq 0$, then V is a viscosity subsolution.



Supersolution

Figure: If, for any point (x_0, τ_0), for any test function $\phi \le V$, where ϕ touches V at the single point (x_0, τ_0), $g(\phi_{xx}, \phi_x, \phi_\tau, \phi, x_0, \tau_0) \ge 0$, then V is a viscosity supersolution.



Viscosity Solution

Any solution which is both a subsolution and a supersolution is a *viscosity solution*

Numerical issues:

- We want to ensure that our numerical scheme converges to the viscosity solution
- Usually need to assume the *Strong Comparison Principle* to ensure existence of a viscosity solution.
- For examples of cases where seemingly reasonable discretizations converge to non-viscosity solutions, see *Pooley, Forsyth, Vetzal*, IMA J. Num. Anal. (2003)

Viscosity Solution

Theorem (Barles, Souganidis (1993))

Any numerical scheme which is consistent, l_∞ stable, and monotone, converges to the viscosity solution.

Consistent Truncation error of scheme applied to smooth test functions goes to zero as mesh, timestep $\rightarrow 0$.

Stability (l_∞) Solution bounded in l_∞ as mesh, timestep $\rightarrow 0$.

Monotonicity

Let $(\mathcal{L}_h^Q V^n)_i$ denote the discrete form of the differential operator $\mathcal{L}^Q V$ at node (x_i, τ^n). Use central, forward, backward differencing to obtain

$$\begin{aligned}(\mathcal{L}_h^Q V^{n+1})_i &= \alpha(Q)_i^{n+1} V_{i-1}^{n+1} + \beta(Q)_i^{n+1} V_{i+1}^{n+1} \\ &\quad -(\alpha(Q)_i^{n+1} + \beta(Q)_i^{n+1} + c(Q)_i^{n+1}) V_i^{n+1}\end{aligned}$$

The discrete form of the HJB equation is then (fully implicit timestepping)

$$\frac{V_i^{n+1} - V_i^n}{\Delta\tau} = \max_{Q^{n+1} \in \hat{Q}} \left\{ (\mathcal{L}_h^{Q^{n+1}} V^{n+1})_i + d_i^{n+1} \right\} .$$

Monotonicity: What does it mean?

Let $V(x_i, \tau^n) = V_i^n$, $W(x_i, \tau^n) = W_i^n$ be two discrete solutions to the same HJB equation.

Lemma (Discrete Arbitrage Inequality)

If V_i^n, W_i^n *are generated using a monotone scheme, and* $\forall i, W_i^0 \geq V_i^0$, *then*

$$W_i^n \geq V_i^n \ \ ; \ \ \forall i; \forall n$$

In other words, if the payoff of W is everywhere at least as large as the payoff of V, then this must hold at all earlier times, for the discrete solution, regardless of the timestep or meshsize.

Monotonicity

Theorem (Monotonicity Condition)

The discrete scheme is monotone if it satisfies the positive coefficient condition

$$\alpha(Q)_i^{n+1} \geq 0 \;\; ; \;\; \beta(Q)_i^{n+1} \geq 0 \;\; ; \;\; c(Q)_i^{n+1} \geq 0 \tag{2}$$

Remark (Nonlinearity)

Note that in order to satisfy (2), the choice of central, forward, backward differencing will depend on the control Q_i^{n+1}*, which is determined by maximizing the discrete equations.*

We should use central differencing as much as possible in order to minimize truncation error (but still have a positive coefficient scheme).

This makes the discrete equations highly nonlinear.

The other conditions?

Lemma (Stability)

The fully implicit, positive coefficient scheme is unconditionally stable.

Proof: Follows from a straightforward maximum analysis.

Lemma (Consistency)

The fully implicit positive coefficient scheme is consistent.

Proof: Taylor series applied to smooth ϕ.

- Good: Convergence guaranteed.
- Bad: Need to solve a set of nasty nonlinear discretized algebraic equations at each timestep!

Outline

Matrix form

Let $V^n = [V_0^n, V_1^n, ...,]^t$. Let

$$(\mathcal{L}_h^Q V^n)_i \quad = \quad [A^n V^n]_i$$

then we can write the discretized equations in matrix form

$$\left[I - \Delta\tau A(Q^{n+1})^{n+1}\right] V^{n+1} = V^n + \Delta\tau D(Q^{n+1})^{n+1}$$

where

$$Q_i^{n+1} \quad = \quad \arg\max_{Q \in \hat{Q}} \left\{ \left[A^{n+1}(Q) V^{n+1} + D^{n+1}(Q)\right]_i \right\}$$

Policy Iteration

```
Let $\hat{V}^k = (V)^k$ with $\hat{V}^0 =$ Initial solution
For $k = 0, 1, 2, \ldots$ until convergence
    Determine
        $Q_i^k = \underset{Q \in \hat{Q}}{\arg\max} \left\{ \left[ A^k(Q)\hat{V}^k + D^k(Q) \right]_i \right\}$
    Solve the linear system:
        $\left[ I - \Delta\tau A(Q^k) \right] \hat{V}^{k+1} = V^n + \Delta\tau D^k(Q^k)$
EndFor
```

Convergence of Iteration

Theorem (Convergence of Policy Iteration (Wang, Forsyth(2006)))

The policy iteration algorithm is globally convergent even for discontinuous local objective functions.

- Using central differencing as much as possible should be more accurate.
- But we have no guarantee on how many iterations required to solve the nonlinear equations.
- We have no guarantee on the order of convergence as the mesh is refined
    - Forward/backward differencing only, can be *at most* first order
    - Central weighting as much as possible *may* be second order

Outline

Examples

Recall that the investor pays no extra up-front fee for the guarantee (only the initial premium w_0).

The insurance company deducts an annual fee α from the balance in the sub-account W.

Problem: let $V(\alpha, W, A, \tau)$ be the value of the GMWB contract, for given yearly guarantee fee α.

Assume that the investor pays an initial premium w_0 at $t = 0$ $(\tau = T)$.

Find the no-arbitrage fee α such that $V(\alpha, w_0, w_0, T) = w_0$ (we do this by a Newton iteration).

Stochastic Process for GMWB

Let S denote the value of the risky asset, we assume that the risk neutral process followed by S is

$$\begin{aligned} dS &= rSdt + \sigma SdZ \\ & r = \text{ risk free rate}; \quad \sigma = \text{ volatility} \\ & dZ = \phi\sqrt{dt} \ ; \ \ \phi \sim \mathcal{N}(0,1) \end{aligned}$$

The risk neutral process followed by W is then (including withdrawals dA).

$$\begin{aligned} dW &= (r-\alpha)Wdt + \sigma WdZ + dA, \quad \text{if } W > 0 \\ dW &= 0, \quad \text{if } W = 0 \\ & \alpha = \text{ fee paid for guarantee} \ ; \ \ A = \text{ guarantee account} \end{aligned}$$

GMWB: Singular Control Formulation

Set $V = V(W, A, \tau)$ where $\tau = T - t$.
Dai et al pose problem as a singular control

$$\min \left[V_\tau - \mathcal{L}V - G \max(\mathcal{F}V, 0), \kappa - \mathcal{F}V \right] = 0$$

$$\mathcal{L}V = \frac{1}{2}\sigma^2 W^2 V_{WW} + (r - \alpha) W V_W - rV$$

$$\mathcal{F}V = 1 - V_W - V_A$$

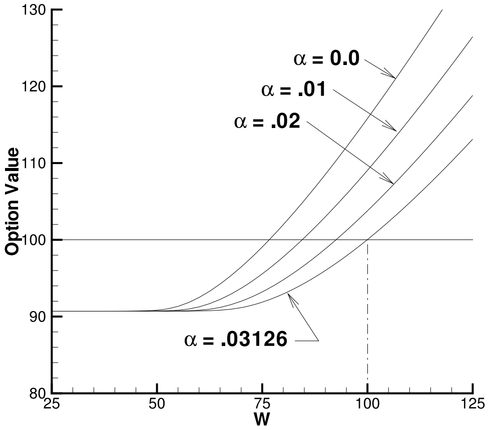
This singular control formulation can be solved numerically using penalty method (viewed as a particular type of control)

$$V_\tau^\varepsilon = \mathcal{L}V^\varepsilon + \lambda \mathcal{J} V^\varepsilon + \max_{\substack{\varphi \in \{0,1\}, \psi \in \{0,1\} \\ \varphi\psi = 0}} \left[\varphi G \mathcal{F} V^\varepsilon + \psi \left(\frac{(\mathcal{F}V^\varepsilon - \kappa)}{\varepsilon} + \kappa G \right) \right] .$$

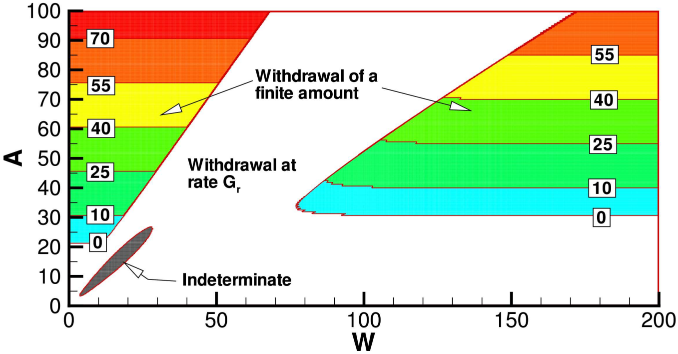
Examples

Parameter	Value
Expiry time T	10.0 years
Interest rate r	.05
Maximum withdrawal rate G_r	10/year
Withdrawal penalty κ	.10
Volatility σ	.30
Initial Lump-sum premium w_0	100
Initial guarantee account balance	100
Initial sub-account value	100
Continuous Withdrawal	Yes

No-arbitrage Fee ($t = 0$, $A = 100$)



Optimal Withdrawal Strategy



No-arbitrage fee charged ($V(\alpha, w_0, w_0, T) = w_0$).

$t = 0$, fair fee charged for $w_0 = 100$. Indeterminate region: appears to converge to optimal withdrawal rate $\hat{\gamma} = 0$?

No-arbitrage Fee

- $\sigma = .15 \rightarrow \alpha = .007$ (70 bps)
- $\sigma = .20 \rightarrow \alpha = .014$ (140 bps)
- $\sigma = .30 \rightarrow \alpha = .03$ (300 bps)
- $\sigma = .15$ + jumps $\rightarrow \alpha = .035$ (350 bps)
- Typical fees charged: $\alpha = .005$ (50 bps). May be too low for high volatility investments (Milevsky and Salisbury, 2006).
- Insurance companies seem to be charging fees based on marketing considerations, not hedging costs.
- Situation even worse if we take into account fees paid to manager of underlying mutual fund, other contract options

Outline

Summary

- Many problems in finance can be formulated in terms of optimal stochastic control $\rightarrow$ nonlinear HJB equation/variational inequality.
- Solutions are in general non-differentiable $\rightarrow$ viscosity solution.
- There are precise rules to follow which will guarantee convergence of a numerical scheme to the viscosity solution
- It is a bad idea (numerically) to analytically determine the optimal control (i.e. by differentiating and setting equal to zero), and then to discretize the PDE
- A better approach: first discretize and then optimize the discrete equations.

HJB Equations: Conclusions

- Discretize first, then maximize (NOT the other way around)
    - Maximize the discrete equations directly → ensures monotonicity, position constraints easily handled
- Positive Coefficient Discretization
    - Guarantees convergence to viscosity solution
    - Guarantees convergence of policy iteration
- Central weighting as much as possible
    - Not usually done
    - Higher convergence rate at little cost