Sigma Bases, Kernel Bases and applications of fast computation for polynomial matrices

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Outline

1. Sigma Bases
2. Minimal Kernel Bases
3. Applications for fast Polynomial Matrix arithmetic
   - Application 1 : Polynomial Matrix Inversion
   - Application 2 : Determinant and Matrix Triangulation
   - Application 3 : Hermite Normal Form

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Padé Approximants

$$A(z) \cdot V(z) - U(z) = z^{m+n+1} W(z)$$

Solved via associated linear system

$$\begin{bmatrix} a_{m-n+1} & \cdots & \cdots & a_{n-1} & a_n \\ a_{m-n+2} & \cdots & \cdots & a_n & a_{n+1} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ a_{n-1} & \cdots & \cdots & & a_{m+n-2} \\ a_n & \cdots & \cdots & a_{m+n-2} & a_{m+n-1} \end{bmatrix} \cdot \begin{bmatrix} v_n \\ v_{n-1} \\ \vdots \\ \vdots \\ v_2 \\ v_1 \end{bmatrix} = -v_0 \begin{bmatrix} a_{m+1} \\ a_{m+2} \\ \vdots \\ \vdots \\ a_{m+n-1} \\ a_{m+n} \end{bmatrix}$$

- Nice when coefficient matrix is nonsingular.
- Matrix Padé $\equiv a_i$ square matrices.

Additional Information

Special structure of Hankel matrices implies:

- All Padé approximants known in scalar case
  - Padé table in scalar case has a type of block structure
- Algorithms take advantage of structure of Hankel matrices
  - in $n \times n$ case cost $O(n^2)$ or $O(n \log^2 n)$

However:

- Nothing known about structure of matrix Padé case
- Same issue with Hermite-Padé :
  Given $\mathbf{A}(z) = [A_1(z), \ldots, A_m(z)]$ and deg bounds $\{n_i\}$:

  Find $\mathbf{P}(z) = [P_1(z), ..., P_m(z)]^T$ with deg $P_i(z) \leq n_i$ and

$$\mathbf{A}(z) \cdot \mathbf{P}(z) = O(z^{n_1+\cdots+n_m+m})$$

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Sigma (or Order) Bases

Idea : look at order condition independently of degree bounds,

$$R_\sigma = \{\mathbf{Q}(z) \in \mathbb{K}^{(m)}[z] \mid \mathbf{A}(z) \cdot \mathbf{Q}(z) = O(z^\sigma)\}$$

Find basis of R_σ as a *module* over $\mathbb{K}[z]$.

Basis always has m elements : $\mathbf{M}_1(z), \ldots, \mathbf{M}_m(z)$.

- write as columns of an $m \times m$ matrix polynomial $\mathbf{M}(z)$.

Matrix Padé, Hermite-Padé Approximants, etc

- can be represented as vector order problems

Sigma Bases

Degree bounds? Given $\vec{n} = (n_1, \ldots, n_m)$:

Then

$$\mathbf{Q}(z) \quad = \quad \alpha_1(z)\mathbf{M}_1(z) + \cdots + \alpha_m(z)\mathbf{M}_m(z)$$

with

$$\deg \alpha_i(z) \leq \text{ defect } \mathbf{Q}(z) - \text{ defect } \mathbf{M}_i(z)$$

Here : defect $(\mathbf{Q}(z)) = \min_i\{ n_i + 1 - \deg Q_i(z) \}$

$\mathbf{M}(z)$ gives all solutions of $\mathbf{A}(z)\mathbf{Q}(z) = O(z^\sigma)$ having deg bounds .

Sigma Basis Algorithm [SIMAX - BL 1994]

1. Start with Order basis $\mathbf{M}(z) = \mathbf{I}$ and order $= 0$.
2. Of columns $\mathbf{M}_1(z), \ldots, \mathbf{M}_m(z)$ needing order increased:
   - pick one with minimal defect.
   - use to eliminate other columns needing order increase.
3. Multiply pivot column by z. Continue.
4. Quadratic complexity.
5. Double order everytime : obtain superfast version.

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Other Algorithms

- Stable numeric algorithms
  - (1) Padé approximate : (C-M SIMAX 1992)
  - (2) Hermite-Padé, Simultaneous-Padé : (C-J-L SIMAX 1994)
- Fraction-free algorithms
  - (1) Sigma bases : (B-L SIMAX 2000)
  - (2) General case : linear functional with special element

    - e.g. interpolation bases, M-Padé, etc

Superfast Computation

If $\mathbf{F} \in \mathbb{K}[z]^{m \times n}$, $m \leq n$, order σ

- σ-Basis : $O^{\sim}(n^2 m\sigma + nm^2\sigma)$
- MBasis : $O(n^{\omega}\sigma^{1+\epsilon})$ - in case of matrix input (GJV - 2003)
- Generating set : $O^{\sim}(n^{\omega}\lceil m\sigma/n \rceil)$ (Storjohann. 2006)
- Order basis : $O^{\sim}(n^{\omega}\lceil m\sigma/n \rceil)$ Zhou-L (2009)
- Interpolation basis: Neiger-J-S-V (2015) (see Wed talk)

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Minimal Kernel Bases

Given $\mathbf{F} \in \mathbb{K}[z]^{m \times n}, \quad m \leq n$:

A *Kernel Basis* for $\mathbf{F}$ is a $\mathbb{K}[z]$ module basis for

$$\{ \mathbf{p} \in \mathbb{K}[x]^n \mid \mathbf{F} \cdot \mathbf{p} = 0 \}$$

Can represent basis as matrix $\mathbf{M} \in \mathbb{K}[z]^{n \times *}$.

Minimal Kernel Basis if $\mathbf{M}$ is column reduced.

Shifted $\vec{s}$-Minimal Kernel Basis if $z^{\vec{s}} \cdot \mathbf{M}$ is column reduced.

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Classical Uses of Kernel Bases

Main application is use of polynomial matrix formalism in systems theory

- reduced transfer functions
- fault diagnostics

Specific algebraic problems

- right and left co-prime transfer function representations
- linear solving
- column reduction
- matrix normal forms

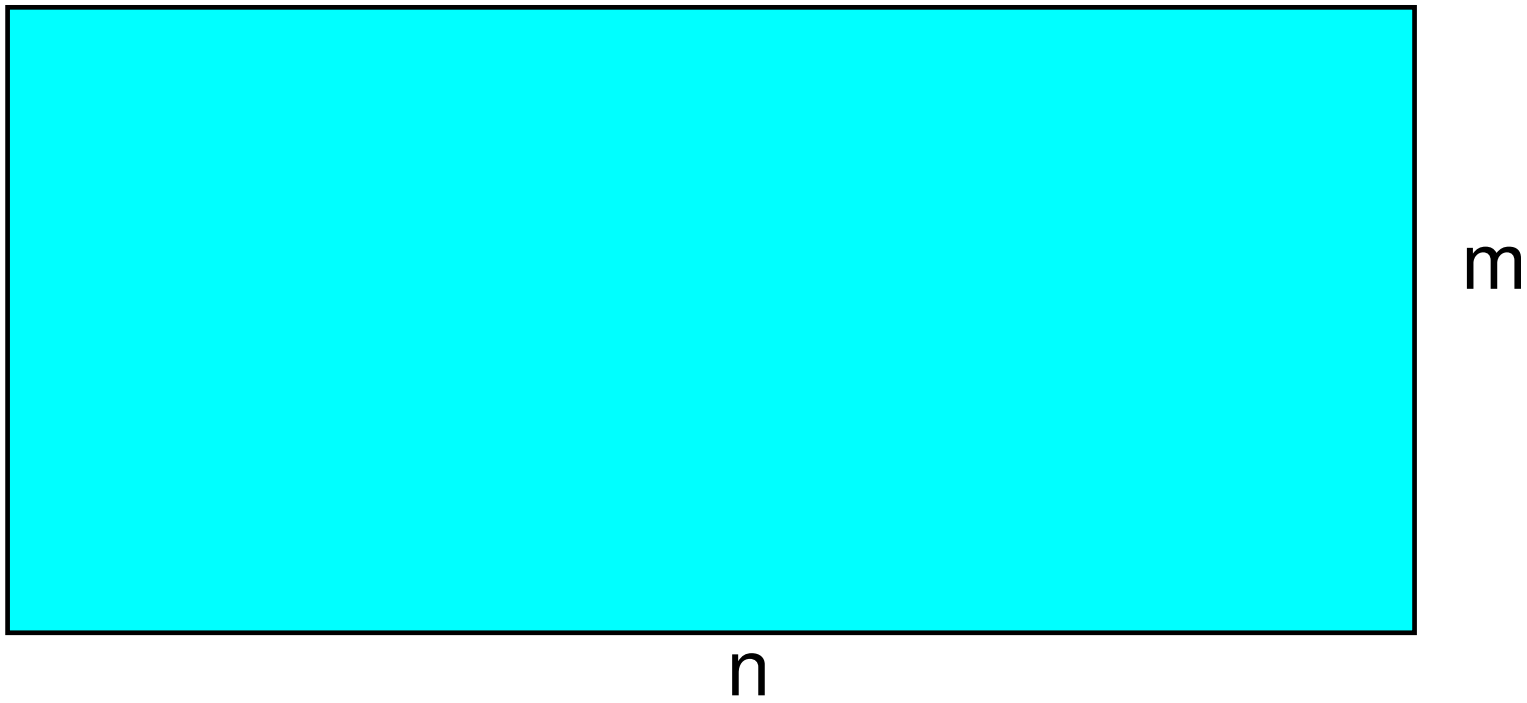
Algorithms

- Matrix pencil methods
  - convert problem to larger matrix size but degree 1.
  - Use Kronecker Normal Forms
  - Beelen, Dooren, Misra, etc.
  - Cost $O(m^2nd^3)$
- Resultant methods
  - convert to structured linear system (e.g. block Toeplitz)
  - use high dimension, get high complexity
- Elimination methods
  - Mulders and Storjohann (2003) Cost : $O(mnrd^2)$
  - Storjohann and Villard (2005) Cost : $O^{\sim}(mnr^{\omega-2}d)$

Key Ideas from Zhou, L, Storjohann [ISSAC 2012]

- Order basis computation results in a subset of kernel basis and reduces the column dimension and the degree.
- Row dimension can be reduced by computing the kernel basis of a subset of rows of the input matrix.
- We can use the initial column degrees (or shift) to produce a bound for the degrees of all intermediate subproblems.
- The more general problem involving degree shift can be easier to tackle than the problem without shift
- Can efficiently multiply matrices with unbalanced degrees

Dimension of Input



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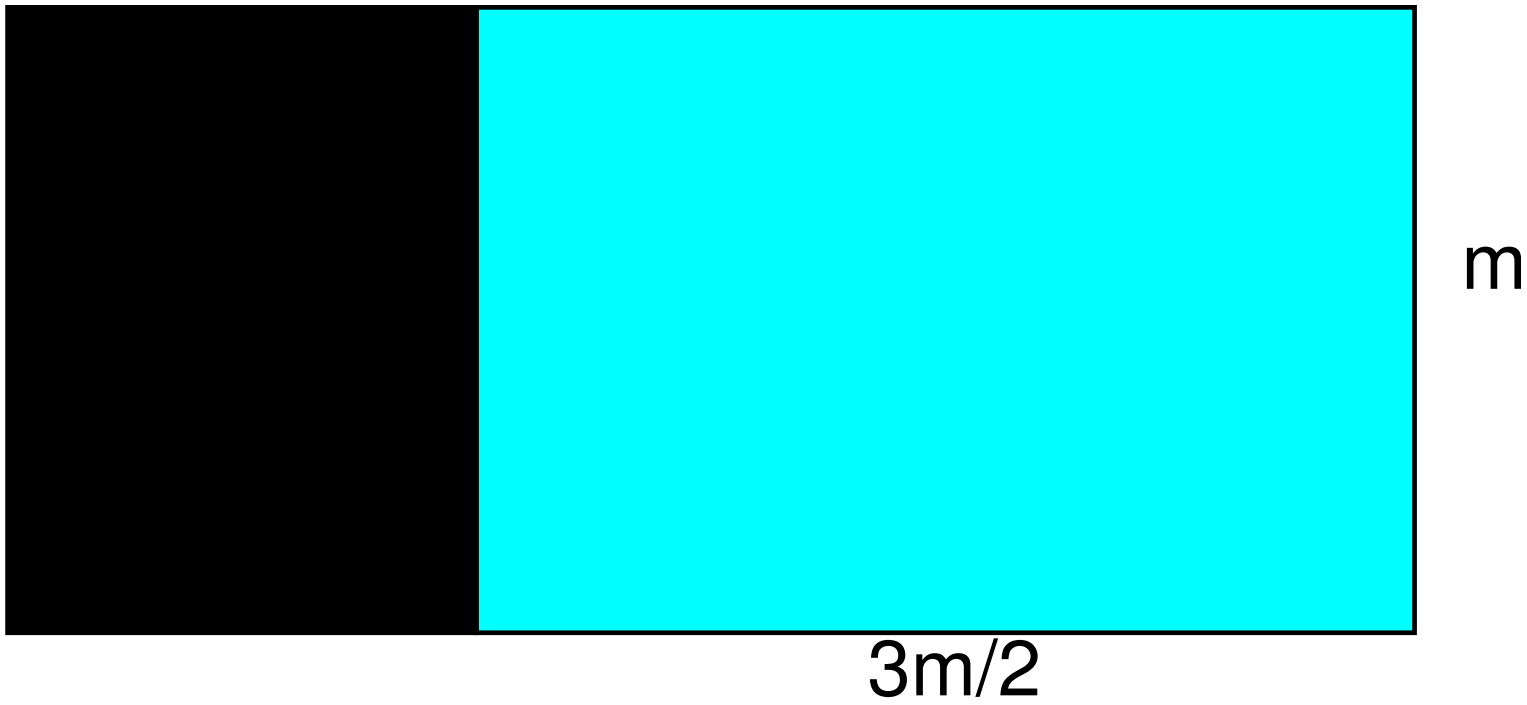

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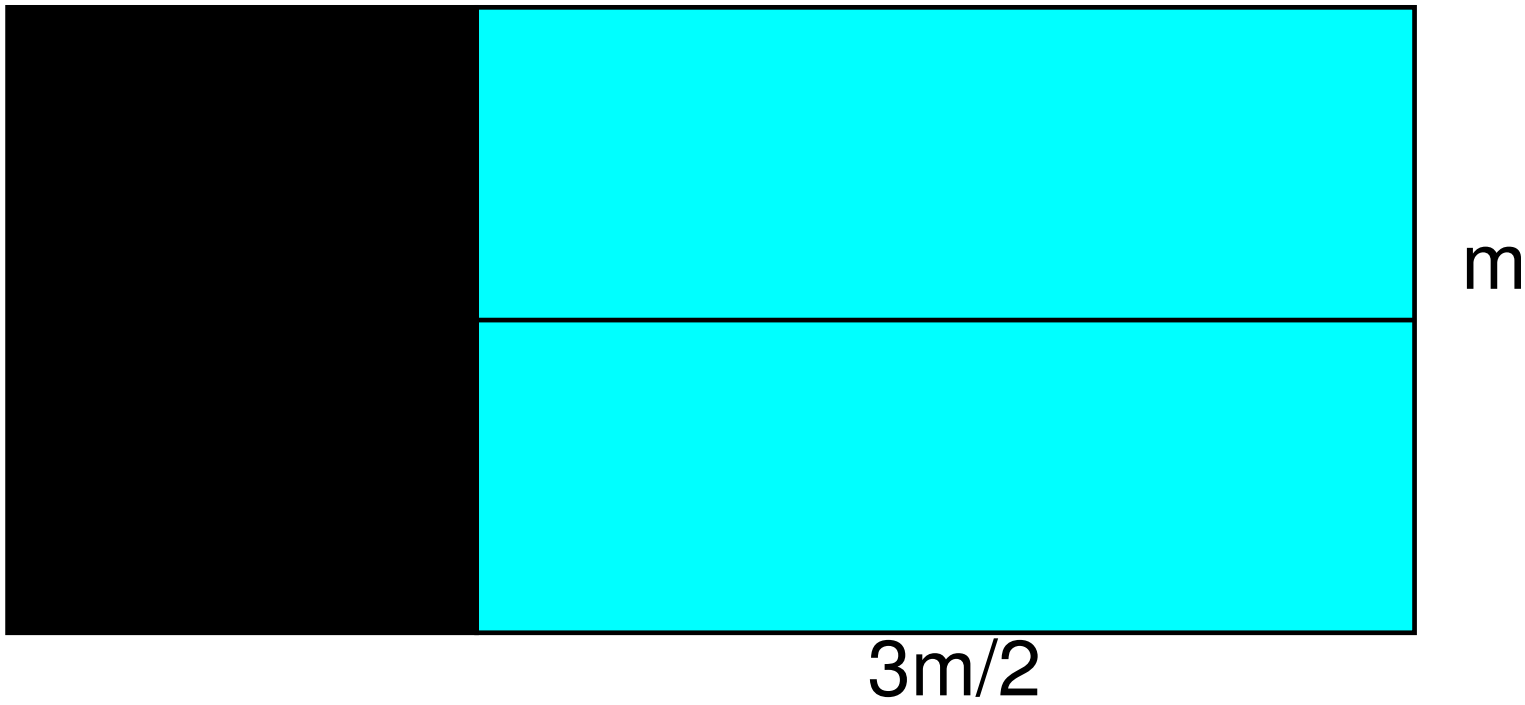

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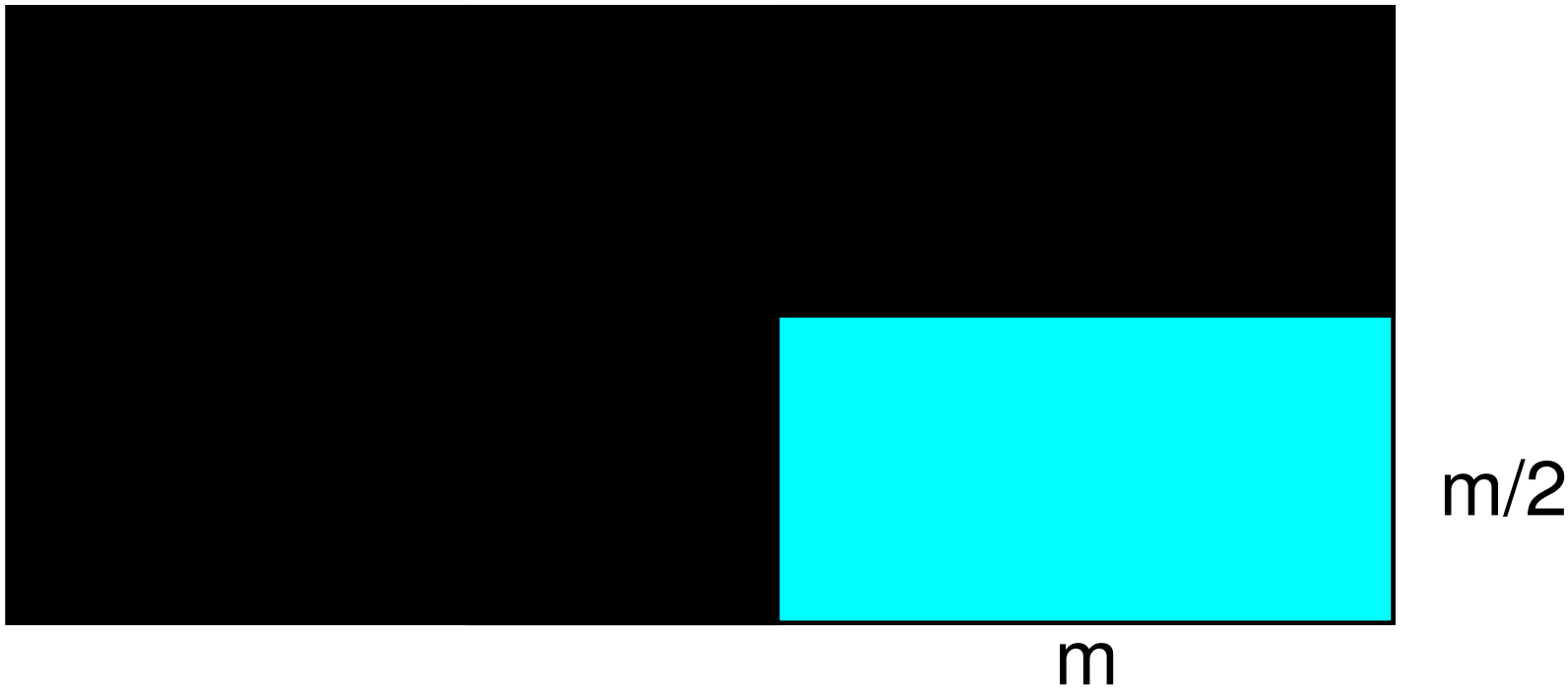

Partial Nullspace Basis using Order Bases

To compute a $\vec{s}$-minimal kernel basis of $\mathbf{F}$ we do:

(1) Compute an $(\mathbf{F}, \sigma, \vec{s})$-basis $\mathbf{P}$ of $\mathbf{F}$

(2) Let $[\mathbf{P}_1, \mathbf{P}_2] = \mathbf{P}$ such that $\mathbf{F}\mathbf{P}_1 = 0$

i.e. $\mathbf{P}_1$ is a part of a $\vec{s}$-minimal kernel basis

(3) Use residual $\mathbf{F}\mathbf{P}_2$ to compute the remaining kernel basis

- Let $\vec{b}_2$ be the $\vec{s}$-column degrees of $\mathbf{P}_2$
- Then $\mathbf{Q}$ is a $\vec{b}_2$-minimal kernel basis of $\mathbf{F}\mathbf{P}_2$, implies $[\mathbf{P}_1, \mathbf{P}_2\mathbf{Q}]$ is $\vec{s}$-minimal kernel basis of $\mathbf{F}$.

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Some Important Properties (ZLS ISSAC 2012)

$\mathbf{F} \in \mathbb{K}[x]^{m \times n}$, $\vec{s} \in \mathbb{Z}^n$ bounds column degrees, $\sum \vec{s} \leq \xi$

Theorem

(i) $\mathbf{A} \in \mathbb{K}[x]^{m \times n}$, $\vec{s} \in \mathbb{Z}^n$ *bounding column degrees of* $\mathbf{A}$

(ii) $\mathbf{B} \in \mathbb{K}[x]^{n \times k}$ *with* $k \in O(m)$, $\sum \operatorname{cdeg}_{\vec{s}} \mathbf{B} \leq \sum \vec{s} \in O(\xi)$

Multiply $\mathbf{A}$ *and* $\mathbf{B}$ *:* $O^{\sim}(n^2 m^{\omega-2} s) \subset O^{\sim}(n^{\omega} s)$, $s = \xi/n$.

Theorem

For $\mathbf{M}$ *a* $\vec{s}$*-minimal kernel basis of* $\mathbf{F}$*:* $\sum \operatorname{cdeg}_{\vec{s}} \mathbf{M} \leq \sum \vec{s}$

Algorithms

$\mathbf{F} \in \mathbb{K}[x]^{m \times n}$, $\vec{s} \in \mathbb{Z}^n$ bounds column degrees, $\sum \vec{s} \leq \xi$

Theorem

(Z-L-S ISSAC 2012)
$\vec{s}$-Minimal kernel basis computation costs $O^{\sim}(n^{\omega} s)$.

Also : Column basis : $\mathbf{F} \cdot \mathbf{U} = \mathbf{C}$.

Combining Minimal Kernel and Order Bases computation gives:

Theorem

(Z-L ISSAC 2013)
Column basis computation costs $O^{\sim}(m^{\omega-1} ns)$.

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Inversion : Reduce $\mathbf{F}$ to Diagonal Form

Partition and reduce $\mathbf{F}$ via

$$\mathbf{F} \cdot \mathbf{N} = \left[\begin{array}{c} \mathbf{F}_u \\ \hline \mathbf{F}_d \end{array} \right] \cdot [\mathbf{N}_\ell, \mathbf{N}_r] = \left[\begin{array}{cc} \mathbf{F}_u\mathbf{N}_\ell & \mathbf{F}_u\mathbf{N}_r \\ \mathbf{F}_d\mathbf{N}_\ell & \mathbf{F}_d\mathbf{N}_r \end{array} \right] = \left[\begin{array}{cc} \mathbf{R}_u & 0 \\ 0 & \mathbf{R}_d \end{array} \right]$$

Notice:

- $\mathbf{N}_\ell$ a kernel basis for $\mathbf{F}_d$
- $\mathbf{N}_r$ a kernel basis for $\mathbf{F}_u$

Recurse on $\mathbf{R}_u$ and $\mathbf{R}_d$ to get diagonal $\mathbf{B}$. Cost is $O^\sim(n^\omega s)$

Modeled on approach of Jeannerod and Villard (2003). $O^\sim(n^\omega d)$

Previous to Jeannerod/Villard fastest algorithms $O^\sim\left(n^{\omega+1}d\right)$

Measuring Size

$\mathbf{F}$ and $\vec{s}$ bound on column degree.

(i) Partition

$$\mathbf{F} \cdot \mathbf{N} = \left[\begin{array}{c} \mathbf{F}_u \\ \hline \mathbf{F}_d \end{array}\right] \cdot [\mathbf{N}_\ell, \mathbf{N}_r] = \left[\begin{array}{cc} \mathbf{F}_u\mathbf{N}_\ell & \mathbf{F}_u\mathbf{N}_r \\ \mathbf{F}_d\mathbf{N}_\ell & \mathbf{F}_d\mathbf{N}_r \end{array}\right] = \left[\begin{array}{cc} \mathbf{R}_u & 0 \\ 0 & \mathbf{R}_d \end{array}\right]$$

(ii) Size control: $\mathbf{M}$ a $\vec{s}$-minimal kernel basis

$$\sum \operatorname{cdeg}_{\vec{s}} \mathbf{M} \leq \sum \vec{s}$$

Implies : $\sum \operatorname{cdeg}_{\vec{s}} \mathbf{N}_\ell \leq \sum \vec{s}$ and $\sum \operatorname{cdeg}_{\vec{s}} \mathbf{N}_r \leq \sum \vec{s}$

Complexity

Theorem

Inversion of $\mathbf{F} \in \mathbb{K}[x]^{n \times n}$ *costs* $O^{\sim}(n^{\omega} s)$ *field operations.*

Proof.

If cost : $g(n)$ then recurrence relation:

$$\begin{aligned} g(n) &\in O^{\sim}(n^{\omega} s) + g(\lceil n/2 \rceil) + g(\lfloor n/2 \rfloor) \\ &\in O^{\sim}(n^{\omega} s) + 2g(\lceil n/2 \rceil) \\ &\in O^{\sim}(n^{\omega} s). \end{aligned}$$

□

Also compute largest invariant factor : $\operatorname{lcm}(b_{11}(x), \ldots, b_{nn}(x))$.

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Triangulating

Partition and reduce $\mathbf{F}$ via

$$\mathbf{F} \cdot \mathbf{U} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{F}_d \end{bmatrix} \begin{bmatrix} \mathbf{U}_\ell & \mathbf{U}_r \end{bmatrix} = \begin{bmatrix} \mathbf{G}_1 & 0 \\ * & \mathbf{G}_2 \end{bmatrix}.$$

(i) $\mathbf{G}_1$ is nonsingular and a column basis of $\mathbf{F}_u$.

(ii) $\mathbf{U}_r$ a right kernel basis of $\mathbf{F}_u$

(iii) $\mathbf{G}_2 = \mathbf{F}_d \cdot \mathbf{U}_r$,

Recurse on $\mathbf{G}_1$ and $\mathbf{G}_2$ to get diagonal elements.

Theorem

$\mathbf{F} \in \mathbb{K}[x]^{n \times n}$. Diagonals costs $O^{\sim}(n^\omega s)$ with $s = \frac{\sum \operatorname{cdeg} \mathbf{F}}{n}$.

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Determinants

Previous algorithms by:

Storjohann $O^{\sim}(n^{\omega+1}d)$, Mulders and Storjohann $O(n^3d^2)$, E-G-V $O^{\sim}(n^{2+\omega/2}d)$; Storj; G-J-V $O^{\sim}(n^{\omega}d)$; Kaltofen; Kaltofen and Villard

$$\det \mathbf{F} = \frac{\det \mathbf{G}_1 \cdot \det \mathbf{G}_2}{\det \mathbf{U}}$$

For $\det \mathbf{U} = \det [\, \mathbf{U}_\ell \; \mathbf{U}_r]$ we do:

1. $\det \mathbf{U} = \det \mathbf{U} \bmod z = \det U = \det [U_\ell, U_r]$
2. $\mathbf{V} = \mathbf{U}^{-1} = \begin{bmatrix} \mathbf{V}_u \\ \mathbf{V}_d \end{bmatrix}$
3. $\mathbf{U}_r$ and $\mathbf{V}_u$ determined in column bases computation
4. Find $U_\ell{}^*$ such that $U^* = [U_\ell{}^*, U_r]$ is unimodular
5. Let $V_u = \mathbf{V}_u \bmod z$. Then $\det \mathbf{U} = \frac{\det U^*}{\det V_u U_\ell{}^*}$

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Finding Hermite Normal Form

Problem : Given nonsingular $\mathbf{F} \in \mathbb{K}[x]^{n \times n}$. Compute $\mathbf{U}$ and $\mathbf{H}$:

(i) $\mathbf{U}$ unimodular,

(ii) $\mathbf{H}$ in (column) Hermite form

(iii) $\mathbf{F} \cdot \mathbf{U} = \mathbf{H}$

Results :

(i) Deterministic algorithm

(ii) Complexity : $O^{\sim}(n^{\omega} d)$ where $d = \text{degree } \mathbf{H}$

Finding Hermite Normal Form

Problem : Given nonsingular $\mathbf{F} \in \mathbb{K}[x]^{n \times n}$. Compute $\mathbf{U}$ and $\mathbf{H}$:

(i) $\mathbf{U}$ unimodular,

(ii) $\mathbf{H}$ in (column) Hermite form

(iii) $\mathbf{F} \cdot \mathbf{U} = \mathbf{H}$

Results :

(i) Deterministic algorithm

(ii) Complexity : $O^{\sim}(n^{\omega} d)$ where $d =$ degree $\mathbf{H}$

Hermite Form: Finding Rest of $\mathbf{H}$

Use method of Gupta and Storjohann (2012) to get rest of $\mathbf{H}$.

(i) Convert HNF to shifted $\vec{s}$-minimal kernel basis problem

$$\mathbf{FU} = \mathbf{H} \quad \text{same as} \quad [\mathbf{F} \quad -\mathbf{I}] \begin{bmatrix} \mathbf{U} \\ \mathbf{H} \end{bmatrix} = \mathbf{0}.$$

Hermite Form: Finding Rest of $\mathbf{H}$

Use method of Gupta and Storjohann (2012) to get rest of $\mathbf{H}$.

(ii) Adjust to alternate $\vec{s}'$-minimal kernel basis problem

$$[\mathbf{F} \quad -\mathbf{E}] \begin{bmatrix} \mathbf{U} \\ \mathbf{H}' \end{bmatrix} = \mathbf{0}.$$

Easy to construct $\mathbf{E}$. Easy to get $\mathbf{H}$ from $\mathbf{H}'$

Hermite Form: Finding Rest of $\mathbf{H}$

Use method of Gupta and Storjohann (2012) to get rest of $\mathbf{H}$.

(iii) Find $\mathbf{Q}$ and $\mathbf{R}$ such that $\mathbf{E} = \mathbf{FQ} + \mathbf{R}$. Solve via HOL.

Then repeat (ii) but with $\mathbf{E}$ replaced by $\mathbf{R}$.

Future Work

- Shifted kernel basis algorithm depending on rank
- Popov from Column basis
- Alternate domains (noncommutative)