# Pick & Mix Your Icy Six

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The Sue-Ann Staff Estate Winery had a promotion on icewine in January 2024 that they called "Pick & Mix Your Icy Six". They asked how many ways there are to choose six bottles of wine from four possible types.

In their weekly newsletter, they said:

> "Any mathematicians out there? We think there are 360 possible combinations you could create out of this special. Let us know if we're wrong, and remember to show your work ;)"

I'm not sure where they got the answer 360. A standard mathematical formula gives the (correct) answer as the *binomial coefficient*

$$\binom{6+4-1}{4-1} = \binom{9}{3} = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = \frac{504}{6} = 84.$$

Unfortunately, it requires a background in university level mathematics to understand why this formula is correct. However, it is not too difficult to list all 84 combinations if we consider the possible "distributions" of bottles. This is more work, but no advanced mathematics is required.

Let's denote the four types of wine by $A, B, C$ and $D$. To begin, suppose we choose 6 bottles of the same type. There are obviously four ways to do this:

AAAAAA BBBBBB CCCCCC DDDDDD.

We say that this is a $(6, 0, 0, 0)$ distribution.

Next, we consider distributions where two types of wine are chosen. Suppose we have a $(5, 1, 0, 0)$ distribution. This means we choose five bottles of one type and one bottle of a different type. There are 12 ways to do this:

| | | |
|---|---|---|
| AAAAAB | AAAAAC | AAAAAD |
| BBBBBA | BBBBBC | BBBBBD |
| CCCCCA | CCCCCB | CCCCCD |
| DDDDDA | DDDDDB | DDDDDC. |

There are also 12 ways to do a $(4, 2, 0, 0)$ distribution (four bottles of one type and two bottles of a different type):

| | | |
|---|---|---|
| AAAABB | AAAACC | AAAADD |
| BBBBAA | BBBBCC | BBBBDD |
| CCCCAA | CCCCBB | CCCCDD |
| DDDDAA | DDDDBB | DDDDCC. |

And there are six ways to do a $(3, 3, 0, 0)$ distribution (three bottles of each of two types):

| | | |
|---|---|---|
| AAABBB | AAACCC | AAADDD |
| BBBCCC | BBBDDD | CCCDDD. |

So far, we have 4 + 12 + 12 + 6 = 34 combinations in which one or two types of wine are chosen.

We next consider distributions where three types of wine are chosen. There are 12 ways to do a $(4, 1, 1, 0)$ distribution:

| | | |
|---|---|---|
| AAAABC | AAAABD | AAAACD |
| BBBBAC | BBBBAD | BBBBCD |
| CCCCAB | CCCCAD | CCCCBD |
| DDDDAB | DDDDAC | DDDDBC. |

There are 24 ways to do a $(3, 2, 1, 0)$ distribution:

| | | | | | |
|---|---|---|---|---|---|
| AAABBC | AAABBD | AAACCB | AAACCD | AAADDB | AAADDC |
| BBBAAC | BBBAAD | BBBCCA | BBBCCD | BBBDDA | BBBDDC |
| CCCAAB | CCCAAD | CCCBBA | CCCBBD | CCCDDA | CCCDDB |
| DDDAAB | DDDAAC | DDDBBA | DDDBBC | DDDCCA | DDDCCB. |

And there are four ways to do a $(2, 2, 2, 0)$ distribution:

AABBCC AABBDD AACCDD BBCCDD.

So there are 12 + 24 + 4 = 40 combinations in which three types of wine are chosen.

Finally, we consider distributions where all four types of wine are chosen. There are six ways to do a $(2, 2, 1, 1)$ distribution:

| | | |
|---|---|---|
| AABBCD | AACCBD | AADDBC |
| BBCCAD | BBDDAC | CCDDAB. |

And there are four ways to do a $(3, 1, 1, 1)$ distribution:

AAABCD BBBACD CCCABD DDDABC.

So there are 6 + 4 = 10 combinations in which all four types of wine are chosen.

The total number of combinations is 34 + 40 + 10 = 84, which agrees with the mathematical formula.

I sent the above explanation to the winery, but I did not hear back form them.